Correlation testing for affine invariant properties on $\mathbb{F}_p^n$

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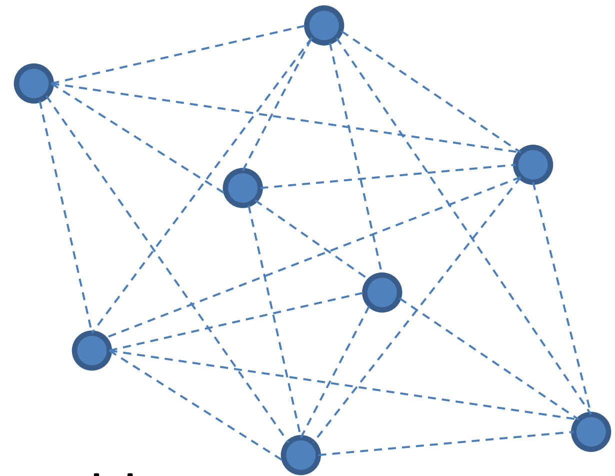
Property testing

- Math: infer global structure from local samples
- CS: Super-fast (randomized) algorithms for approximate decision problems
- Decide if large object approximately has property, while testing only a tiny fraction of it

Graph properties: 3-colorability



- Input: graph G
- Is G 3-colorable?

- Local test:
  - Sample $(1/\varepsilon)^{O(1)}$ vertices
  - Accept if induced subgraph is 3-colorable

- Analysis:
  - Test always accepts 3-colorable graphs
  - Test rejects (w.h.p) graphs ε-far from 3-colorable

[Goldreich-Goldwasser-Ron'96]

Algebraic properties: linearity

0	1	3	0	2	5	1	2

- Input: function $f : \mathrm{F}_p^n \to \mathrm{F}_p$
- Is f linear?

- Local test:
  - Sample $x, y \in \mathrm{F}_p^n$
  - Check if $f(x+y) = f(x) + f(y)$
  - Repeat $1/\varepsilon^{O(1)}$ times

- Analysis:
  - Test always accepts linear functions
  - Test rejects (w.h.p) functions ε-far from linear

[Blum-Luby-Rubinfeld'90]

Codes: locally testable codes

- Code: $C \subset F_p^n$
  distinct elements have large distance
- Input: word $w \in F_p^n$
- C is locally testable if there exists a (randomized) test which queries a few coordinates and
  - Always accepts codewords
  - Rejects (w.h.p) if w is far from all codewords
- The "mathematical core" of the PCP theorem
- Open: can C have constant rate, distance and testability?

Proofs: Probabilistic Checkable Proofs

- PCP Theorem: robust proof system

- Encoding of theorems +
  randomized local test (queries few bits of proof)
  - Test always accepts legal proofs of theorems
  - Test rejects (w.h.p) proofs of false theorems

- Major tool to prove hardness of approximation

Property testing: general framework

- Universe: set of objects (e.g. graphs)
- Property: subset of objects (e.g. 3-col graphs)
- Test: randomized small sample (e.g. small subgraph)

- Property is testable if local consistency implies approximate global structure

Which properties are testable?

- **Graph properties:** well understood
- **Algebraic properties:** partially understood
- **Locally testable codes:** major open problems
- **PCP / hardness of approximation**: whole field

Correlation testing

- Property testing: **strong** global structure
  - Is the object close to having property
- Correlation testing: **weak** global structure
  - Is the object slightly related to having property
- Motivation: possible generalizations of the inverse Gowers ~~conjecture~~ theorem

Correlation testing

Linearity correlation testing

- Function $f : \mathbb{F}_p^n \to \mathbb{C}$

- Correlation of *f,g*: $< f, g > = \mathbb{E}_{x \in \mathbb{F}_p^n}[f(x)\overline{g(x)}]$

- Correlation with linear functions (characters):

$$\| \hat{f} \|_\infty = \max_{\ell : \mathbb{F}_p^n \to \mathbb{F}_p \ \text{linear}} |< f, \omega_p^{\ell} >|$$

$$(\omega_p = e^{2\pi i / p})$$

Linearity correlation testing

- Linear correlation: global property

  Witnessed by local average

$$\mathrm{E}_{x,y,z\in \mathrm{F}_p^n}[f(x+y+z)\overline{f(x+y)f(x+z)}f(x)]$$

$$=\sum_{\alpha}|\hat{f}(\alpha)|^4=\|f\|_{U^2}^4$$

- Identifies functions correlated with linear funcs:
  - f correlated to linear: $\|\hat{f}\|_\infty \geq \varepsilon \Rightarrow \|f\|_{U^2} \geq \varepsilon$
  - f is not correlated: $\|\hat{f}\|_\infty \leq \delta \Rightarrow \|f\|_{U^2} \leq \sqrt{\delta}$

Linearity correlation testing

- Discrete setting: $f : \mathbb{F}_p^n \to \mathbb{F}_p$
  Test queries 4 locations, accepts f if
  $$f(x+y+z) - f(x+y) - f(x+z) + f(x) = 0$$
- Acceptance probability:
  - ε-correlated with linear: prob. $\geq 1/p + \varepsilon^2$
  - negligible correlation: prob. $\leq 1/p + o(1)$
- Property testing: #queries depends on ε
- Here: #queries=4, acceptance prob. depends on ε

Testing correlation with polynomials

- Inverse Gowers Theorem (for finite fields):

  Global structure: correlation with low-degree polynomials (Higher-order Fourier coefs)

  Witnessed by local average

Testing correlation with polynomials

- Correlation with degree d polynomials:

$$\| f \|_{u(Poly_d)} = \max_{Q:\mathbb{F}_p^n \to \mathbb{F}_p \text{ polynomial degree} \le d} |\langle f, \omega_p^Q \rangle|$$

- Gowers norm: average over 2^{d+1} points

$$\| f \|_{U^{d+1}}^{2^{d+1}} = \mathrm{E}_{x, y_1, \ldots, y_{d+1} \in \mathbb{F}_p^n} \left[\prod_{I \subseteq [d+1]} \mathrm{C}^{d-|I|} f\left(x + \sum_{i \in I} y_i\right) \right]$$

$$\mathrm{C} = Conjugation$$

Testing correlation with polynomials

- Direct theorem [Gowers]

$$\| f \|_{u(Poly_d)} \geq \varepsilon \Rightarrow \| f \|_{U^{d+1}} \geq \varepsilon$$

- Inverse Theorem [Bergelson-Tao-Ziegler]

$$\| f \|_{U^{d+1}} \geq \varepsilon \Rightarrow \| f \|_{u(Poly_d)} \geq \delta(\varepsilon)$$

(if p<d then $Poly_d$ = non classical polynomials)

Main theorem

- Gowers norms: local averages which witness global correlation to low-degree polynomials

- Question: are there other such properties?
  - Correlation witnessed by local averages

- Theorem [today]: no
  (affine invariant properties, in large fields)

Correlation with property

- Property $P \subset \{g : \mathrm{F}_p^n \to \mathrm{F}_p\}$

  (can also consider $P \subset \{g : \mathrm{F}_p^n \to \square\}$)

- Function $f : \mathrm{F}_p^n \to \mathrm{F}_p$

- Correlation of *f* with property P:

$$\| f \|_{u(P)} = \max_{g \in P} |< \omega^f, \omega^g >|$$

Local test

- Local test (with q queries):
  - Distribution over $\{x_1,\ldots,x_q\} \subset \mathrm{F}_p^n$
  - Local test $T : \mathrm{F}_p^q \to \{0,1\}$

    $$T(f) = \mathrm{E}[T(f(x_1),\ldots,f(x_q))]$$

- T tests correlation with property P if

  $\forall \varepsilon \;\; \exists \delta \in (0,\varepsilon), \theta^- < \theta^+$ such that

  $$\| f \|_{u(P)} \geq \varepsilon \Rightarrow T(f) \geq \theta^+$$

  $$\| f \|_{u(P)} \leq \delta(\varepsilon) \Rightarrow T(f) \leq \theta^-$$

Affine invariant properties

- Property $P \subset \{f : \mathbb{F}_p^n \to \mathbb{F}_p\}$
- *P* is affine invariant if

$$f(x) \in P \Leftrightarrow g(x) = f(Ax+b) \in P$$

- Examples:
  - Linear functions; degree-d polynomials
  - Functions with sparse / low-dim. Fourier representation

- Local tests for affine invariant properties are w.l.o.g local averages over linear forms

Local average over linear forms

- Variables $X=(X_1,\ldots,X_k)\in(\mathbb{F}_p^n)^k$
- Linear form $L(X)=\lambda_1 X_1+\ldots+\lambda_k X_k \qquad (\lambda_i\in\mathbb{F}_p)$
- System of linear forms $\mathrm{L}=\{L_1,\ldots,L_q\}$
  - E.g. $\mathrm{L}=\{X+Y+Z, X+Y, X+Z, X\}$

- Average over linear forms:

$$T_{\mathrm{L},\alpha}(f)=\mathrm{E}_{X\in(\mathbb{F}_p^n)^k}[\omega_p^{\alpha_1 f(L_1(X))+\ldots+\alpha_q f(L_q(X))}]$$

$(\alpha\in\mathbb{F}_p^q)$

Local tests: affine invariant properties

- Local tests for affine invariant properties are w.l.o.g averages over homogenous linear forms
  - $\mathrm{L} = \{L_1, \ldots, L_q\}$ homogenous if $L_i(X) = X_1 + \sum_{i=2}^{k} \lambda_i X_i$
- $\exists$ systems of linear forms L_i, α_i such that the sets

$$\{(T_{\mathrm{L}_1,\alpha_1}(f), \ldots, T_{\mathrm{L}_m,\alpha_m}(f)) \,|\, \|f\|_{u(P)} \geq \varepsilon\}$$

$$\{(T_{\mathrm{L}_1,\alpha_1}(f), \ldots, T_{\mathrm{L}_m,\alpha_m}(f)) \,|\, \|f\|_{u(P)} \leq \delta(\varepsilon)\}$$

are disjoint

Local tests: affine invariant properties

- Claim: any local test → local averages

- Proof: P affine invariant, so $\forall A, b$

$$\| f \|_{u(P)} = \| f(Ax+b) \|_{u(P)}$$

- Choosing A,b uniformly:
  - transform each query $(x_1, \ldots, x_q)$
  - to a homogeneous system $(Ax_1 + b, \ldots, Ax_q + b)$

Main theorem (1)

- Property $P = (P_n \subset \{g : \mathbb{F}_p^n \to \mathbb{F}_p\})_{n\in\mathbb{N}}$
  - Consistent $P_n \subset P_{n+1}$
  - Affine invariant
  - Sparse $|P_n| = p^{o(p^n)}$

- Thm: If P is locally testable with q queries (p>q) then $\exists d \le q$ such that for any sequence of functions $(f_n : \mathbb{F}_p^n \to \mathbb{F}_p)_{n\in\mathbb{N}}$ which are unbiased $\lim_{n\to\infty} \mathrm{E}\,\omega_p^{f_n} = 0$

$$\lim_{n\to\infty} \| f_n \|_{u(P)} = 0 \Leftrightarrow \lim_{n\to\infty} \| f_n \|_{U^d} = 0$$

Main theorem (2)

- Consistent property

$$P = (P_n \subset \{g : \mathbb{F}_p^n \to \mathbb{C}, \| g \|_\infty \le 1\})_{n \in \mathbb{N}}$$

- Thm: If P is testable by systems of q linear forms (p>q) then $\exists d \le q$, for any bounded functions $(f_n : \mathbb{F}_p^n \to \mathbb{C})_{n \in \mathbb{N}}$

$$\lim_{n \to \infty} \| f_n - \mathrm{E} f_n \|_{u(P)} = 0 \Leftrightarrow \lim_{n \to \infty} \| f_n - \mathrm{E} f_n \|_{U^d} = 0$$

- Q: Is this true for any norm defined by linear forms?

Proof

Main theorem

- $P=(P_n \subset \{g:\mathbb{F}_p^n \to \mathbb{C}, \|g\|_\infty \le 1\})_{n\in\mathbb{N}}$

- P testable by systems of q linear forms (q<p)

- Thm: u(P) norm equivalent to some U^d norm:

  if $\lim_{n\to\infty} \mathrm{E} f_n = 0$ then

$$\lim_{n\to\infty} \|f_n\|_{u(P)} = 0 \Leftrightarrow \lim_{n\to\infty} \|f_n\|_{U^d} = 0$$

Proof idea

- Dfn: S = {degrees d: $\forall$large n $\exists$degree-d poly Q_n
  1. Q_n correlated with property P
  2. Q_n has “high enough” rank}
- D=Max(S)
  - D is bounded (bound depends on the linear systems)
- Lemma 1: $\lim_{n\to\infty} \| f_n \|_{U^{D+1}} = 0 \Rightarrow \lim_{n\to\infty} \| f_n \|_{u(P)} = 0$
- Lemma 2: $\lim_{n\to\infty} \| f_n \|_{u(P)} = 0 \Rightarrow \lim_{n\to\infty} \| f_n \|_{U^{D+1}} = 0$

Polynomial rank

- Q – degree d polynomial

- Rank(Q) – minimal number of lower-degree polynomials $R_1,\ldots,R_c$ needed to compute Q
  - $Q(x) = \Gamma(R_1(x),\ldots,R_c(x))$

- Thm [Green-Tao, Kaufman-L.]

  If P has high enough rank, it has negligible correlation with lower degree polynomials

Polynomial factors

- Polynomial factor: $B = \{Q_1, \ldots, Q_C : \mathrm{F}_p^n \to \mathrm{F}_p\}$
  - Sigma-algebra defined by $Q_1, \ldots, Q_C$
  - $f : \mathrm{F}_p^n \to \mathbb{R}$: average over B, $\mathrm{E}[f \mid B]$

- Complexity(B): C = number of basis polys
- Degree(B): max degree of $Q_1, \ldots, Q_C$
- Rank(B): min. rank of linear comb. of $Q_1, \ldots, Q_C$
  - Large rank: $Q_1(x), \ldots, Q_C(x)$ are nearly independent

Decomposition theorems

- Fix d<p

- $f : \mathbb{F}_p^n \to \mathbb{C}$ can be decomposed as $f = f_1 + f_2$
  - $f_1 = \mathrm{E}[f \mid B]$

    B has degree d, high rank, bounded complexity

  - $\| f_2 \|_{U^{d+1}} \ll 1$

Complexity of linear systems

- Linear form: $L(X) = \lambda_1 X_1 + \ldots + \lambda_k X_k$
- Linear system: $\mathrm{L} = \{L_1, \ldots, L_q\}$
- Average: $T_{\mathrm{L}}(f) = \mathrm{E}_{\overline{X} \in (\mathrm{F}_p^n)^k} \prod_{i=1}^{q} f(L_i(X))$
- Complexity: min. d, if $f = f_1 + f_2$, $\| f_2 \|_{U^{d+1}} \ll 1$

  then $T_{\mathrm{L}}(f) \approx T_{\mathrm{L}}(f_1)$
- C-S complexity [Green-Tao]
- True complexity [Gowers-Wolf, Hatami-L.]

Proof idea

- Dfn: S = {degrees d: $\forall$large n $\exists$degree-d poly Q_n
  1. Q_n correlated with property P
  2. Q_n has "high enough" rank}
- D=Max(S)
  - D is bounded ($\leq$ complexity of linear systems)
- Lemma 1: $\lim_{n\to\infty} \| f_n \|_{U^{D+1}} = 0 \Rightarrow \lim_{n\to\infty} \| f_n \|_{u(P)} = 0$
- Lemma 2: $\lim_{n\to\infty} \| f_n \|_{u(P)} = 0 \Rightarrow \lim_{n\to\infty} \| f_n \|_{U^{D+1}} = 0$

Lemma 1: Small U^{D+1} → small u(P)

- D: max deg of high rank polys correlate with P
- Assume $\| f \|_{U^{D+1}} \ll 1$ but $\| f \|_{u(P)} \geq \varepsilon$

- Step 1: reduce to "structured function"
  - Linear system $\mathcal{L}$ of complexity S (S>D)
  - Decompose: $f = f_1 + f_2$

$$f_1 = \mathrm{E}[f \mid B], \| f_2 \|_{U^{S+1}} \ll 1$$

- Reduce to studying f_1 - func. of deg ≤S polys:
  - $\| f_1 \|_{U^{D+1}} \ll 1$
  - $T_{\mathcal{L}}(f) \approx T_{\mathcal{L}}(f_1) \Rightarrow \| f_1 \|_{u(P)} \geq \varepsilon'$

Lemma 1: Small U^{D+1} → small u(P)

- D: max deg of high rank polys correlate with P
- Structured function: $f_1 = \mathrm{E}[f \mid B], \quad \deg(B) \le S$
  - $\| f_1 \|_{U^{D+1}} \square\ 1$
  - $\| f_1 \|_{u(P)} \ge \varepsilon'$
- Will show: $\| f_1 \|_{u(P)} \approx 0$
- Use the structure: $f_1(x) = \sum \alpha_i \omega_p^{Q_i(x)}, \quad \deg Q_i \le S$
  - $\deg(Q_i) \le D \Rightarrow \alpha_i \approx 0$ because $\| f_1 \|_{U^{D+1}} \square\ 1$
  - $\deg(Q_i) > D \Rightarrow \| \omega_p^{Q_i} \|_{u(P)} \approx 0$ by def of D

Lemma 2: small u(P) → small U^{D+1}

- Key ingredient: invariance principle
  - High rank polynomials "look the same" to averages

- $\{Q_1,\dots,Q_c\},\{Q'_1,\dots,Q'_c\}$ high rank, $\deg(Q_i)=\deg(Q'_i)$

$$f(x)=\Gamma(Q_1(x),\dots,Q_c(x))$$
$$f'(x)=\Gamma(Q_1'(x),\dots,Q_c'(x))$$

Then local averages cannot distinguish f,f':

$$T_L(f)\approx T_L(f')$$

Part 2: small u(P) → small U^{D+1}

- D: max deg of high rank polys correlate with P
- Assume $\| f \|_{u(P)} \ll 1$ but $\| f \|_{U^{D+1}} \geq \varepsilon$
  - Reduce to structured function, $f_1 = \mathrm{E}[f \mid B]$

- f_1 correlated with high-rank Q of degree ≤D
  - Assume for now: deg(Q)=D

- Dfn of D: Exists high rank poly Q', deg(Q')=D,
  Q' correlated with some function g∈P

- Contradiction: Define $f'_1 = f_1$ with Q replaced by Q'
  - Invariance principle: $T_L(f_1) \approx T_L(f_1')$
  - f'_1 is correlated with g ∈P

Part 2: small u(P) → small U^{D+1}

- Problem: what if f_1' correlated with high rank poly of degree < D?
  - Solution: can find Q' correlated with property P for of all degrees ≤ D
  - Reason: systems of averages are robust

- Thm: for any family of linear systems, the set

$$\{(T_{L_1}(f),\ldots,T_{L_k}(f)) : f : \mathbb{F}_p^n \to \mathbb{C}, \|f\|_\infty \le 1\} \subset \mathbb{C}^k$$

has a non-empty interior for some finite n (unless not for trivial reasons)
  - analog of [Erdos-Lovasz-Spencer] for additive settings

Summary

- Property testing: witness strong structure by local samples
- Correlation test: witness weak structure

- Main result: any affine invariant property which is correlation testable, is essentially equivalent to low-degree polynomials

Open problems

- Which norms can be defined by local averages
  - Are always equivalent to some U^d norm?

- Testing in low characteristics

- Is it possible to test if a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is correlated with cubic polynomials?
  - U^4 norm doesn't work
  - Unknown even if #queries depends on correlation

THANK YOU!