Compactifications of Reductive Groups as Moduli Stacks of Bundles

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joint work with Michael Thaddeus (Columbia University)

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Let G be (connected) split reductive group over a field (i.e. over $\mathbb{C}$, $G = K_{\mathbb{C}}$ with K compact Lie group)

e.g. G =semi-simple, $GL(n, \mathbb{C}), (\mathbb{C}^*)^n, \mathrm{Spin}^c_{\mathbb{C}}, \dots$

Question

What are 'good' compactifications $\overline{G}$ of G?

Here 'good' should mean

- $G \times G$-equivariant
- smooth, with all orbit closures smooth
- boundary $\overline{G} \setminus G$ is a *smooth normal crossing divisor*
- nice enumeration of orbits
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Ideally want some conceptual understanding of boundary $\overline{G} \setminus G$

modular compactification?

For two extreme cases question is well understood:

- $ZG = G$, i.e. G is torus $(\mathbb{G}_m)^n \Rightarrow$ toric varieties
- $ZG = \{1_G\}$, i.e. G is adjoint $\Rightarrow$ wonderful compactification

Notation:
Choose maximal torus T_G

Have co-character lattice

$$\Lambda_G = \mathrm{Hom}(\mathbb{G}_m, T_G)$$

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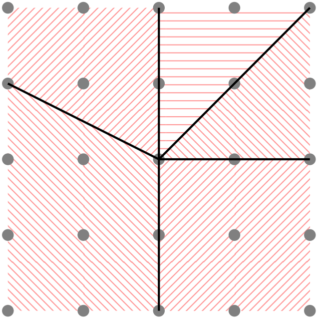
Toric varieties & fans

Toric varieties $\overline{T}$ are normal T-equivariant varieties with open dense orbit

Determined by fans: collection of strongly convex, rational cones in $\Lambda_T \otimes_{\mathbb{Z}} \mathbb{Q}$



- every cone simplicial $\Rightarrow$ at worst finite quotient singularities
- for every top dim cone minimal lattice elements along rays generate $\Lambda \Rightarrow \overline{T}$ smooth
- fan complete $\Rightarrow \overline{T}$ compact

Toric orbifolds

Definition

An orbifold is a smooth separated Deligne-Mumford stack of finite type with trivial generic stabilizer

locally: $\mathbb{C}^n/\Gamma$, for Γ finite group

A (torsion-free) *stacky fan* is a simplicial fan + choice of integral element on each ray (*ray-vector*)

Theorem (Fantechi-Mann-Nironi, I, B-C-S, L-T)

Toric orbifolds are classified by stacky fans

Take coarse moduli scheme = forget stacky structure, keep fan

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Wonderful compactification of adjoint groups

G adjoint, i.e. $ZG = \{1\}$
e.g. $PGL(n)$, $SO(2n+1, \mathbb{C})$, E_8, F_4, G_2

λ regular dominant weight
highest weight representation V_λ

have

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, hook] & End(V_\lambda) \arrow[d, dashed] \\ & \mathbb{P}(End(V_\lambda)) \end{tikzcd}$$

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Wonderful compactification

Definition (De Concini - Procesi)

The wonderful compactification $\overline{G}^w$ of an adjoint group is the closure in $\mathbb{P}(\mathrm{End}(V_\lambda))$

Independent of choice of λ

T_G maximal torus in G, take closure in $\overline{G}^w$
$\Rightarrow$ get toric variety $\overline{T_G}$

Fan of $\overline{T_G}$ = Weyl chambers + Λ_G

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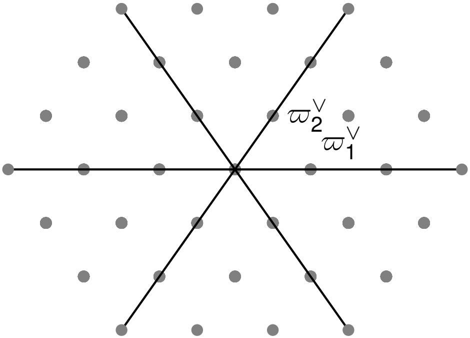

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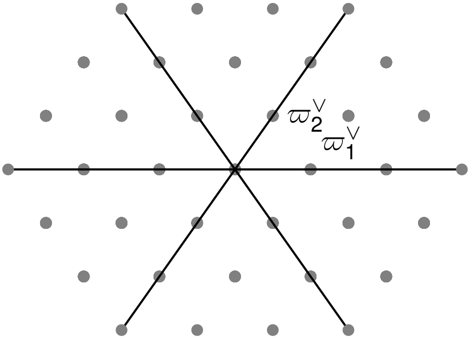

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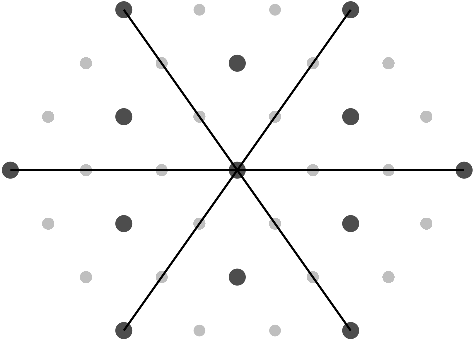
e.g. $SL(3,\mathbb{C})$



Corresponding toric variety no longer smooth!

Moduli Problem

Moduli problem:

$\mathbb{G}_m$-equivariant G-principal bundles on chains of projective lines

Framed at north- and south-poles

Lenght of chain is arbitrary finite, can vary in families

Problem:

Too many objects: stack is not separated nor of finite type

Cure this by imposing a stability condition

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Which bundles are stable?

Theorem (Birkhoff-Grothendieck-Harder)

Every G-principal bundle on $\mathbb{P}^1$ reduces to the maximal torus and up to isomorphims is entirely determined by a co-character

$$\Lambda \ni \rho : \mathbb{G}_m \to G$$

unique up to W_G-action

Take two charts given by stereographic projection from s and n, use ρ as transition function

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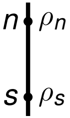

given by co-characters ρ_n and ρ_s, unique up to W_G.

Underlying non-equivariant bundle determined by $\rho_n - \rho_s$

Theorem (M.-Thaddeus)

Every $\mathbb{G}_m$-equivariant G-principal bundle on a chain-of-lines of length n reduces to the maximal torus T_G and is given up to isomorphism by an element of Λ^{n+1}/W_G

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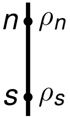

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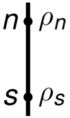

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Σ-stable bundles

Choose a (stacky) fan Σ for T_G, satisfying:

- Σ is simplicial
- Σ is Weyl-invariant
- Σ refines the Weyl-chambers

Choose ordering of integral elements

$$\rho_1, \ldots, \rho_j$$

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Definition

A bundle on chain of lines is Σ-*stable* if

- co-chars on extremal n and s are 0
- co-chars on nodes are ray-vectors of Σ:

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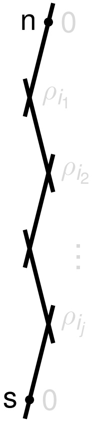

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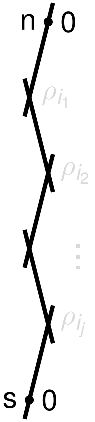

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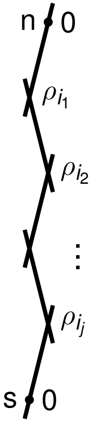
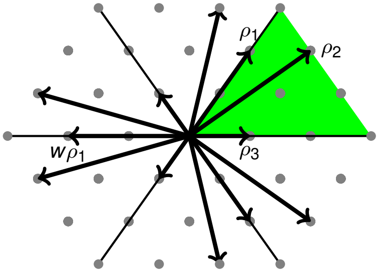


Example

e.g. $PGL(3)$



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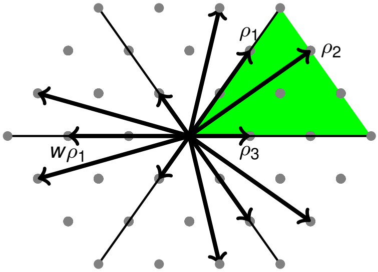

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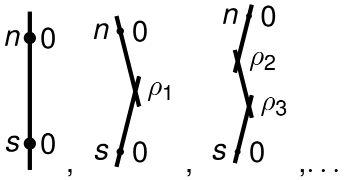

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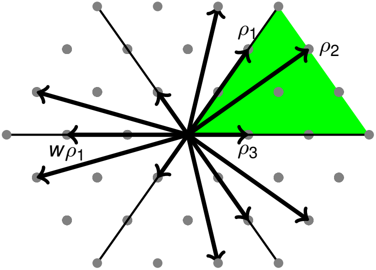

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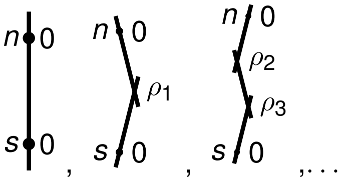

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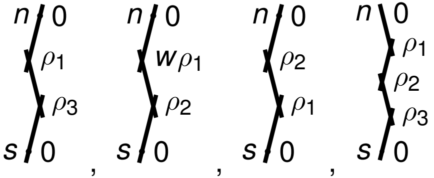

Main result

Theorem (M.-Thaddeus)

Given Σ, the corresponding moduli stack of Σ-stable bundles-on-chains $\mathcal{M}(\Sigma)$ is a (smooth) orbifold compactifying G

Fine print:

Moduli problem depends on ordering, but different orderings give canonically isomorphic stacks

Need to *rigidify* to remove $\mathbb{G}_m$ from automorphisms

If p=char$(k) > 0$, stack is *tame* in general, DM if ρ_i linearly independent over $\mathbb{F}_p$

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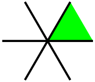

Weyl chambers strongly convex, no refinement necessary
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- G non-semi-simple reductive:

Weyl chamber not strongly convex,
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- $G = T$ torus

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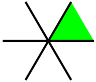

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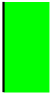

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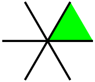

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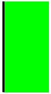

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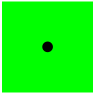

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