

Compactifications of Reductive Groups as Moduli Stacks of Bundles

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joint work with Michael Thaddeus (Columbia University)

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Let G be (connected) split reductive group over a field
(i.e. over \mathbb{C} , $G = K_{\mathbb{C}}$ with K compact Lie group)

e.g. $G =$ semi-simple, $GL(n, \mathbb{C})$, $(\mathbb{C}^*)^n$, $\text{Spin}_{\mathbb{C}}^c, \dots$

Question

What are 'good' compactifications \overline{G} of G ?

Here 'good' should mean

- $G \times G$ -equivariant
- smooth, with all orbit closures smooth
- boundary $\overline{G} \setminus G$ is a *smooth normal crossing divisor*
- nice enumeration of orbits
- ...

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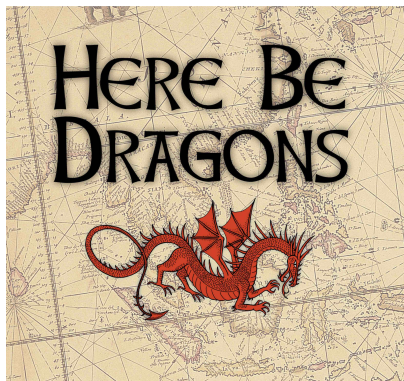
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Ideally want some conceptual
understanding of boundary $\overline{G} \setminus G$
modular compactification?



For two extreme cases question is well understood:

- $ZG = G$, i.e. G is torus $(\mathbb{G}_m)^n \Rightarrow$ toric varieties
- $ZG = \{1_G\}$, i.e. G is adjoint \Rightarrow wonderful compactification

Notation:

Choose maximal torus T_G

Have co-character lattice

$$\Lambda_G = \text{Hom}(\mathbb{G}_m, T_G)$$

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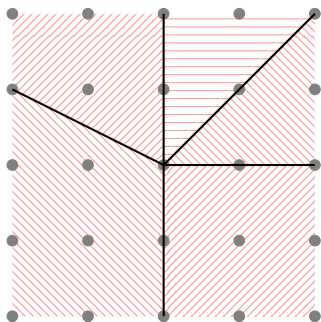
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Toric varieties & fans

Toric varieties \bar{T} are normal T -equivariant varieties with open dense orbit

Determined by fans: collection of strongly convex, rational cones in $\Lambda_T \otimes_{\mathbb{Z}} \mathbb{Q}$



- every cone simplicial \Rightarrow at worst finite quotient singularities
- for every top dim cone minimal lattice elements along rays generate $\Lambda \Rightarrow \bar{T}$ smooth
- fan complete $\Rightarrow \bar{T}$ compact

Toric orbifolds

Definition

An orbifold is a smooth separated Deligne-Mumford stack of finite type with trivial generic stabilizer

locally: \mathbb{C}^n/Γ , for Γ finite group

A (torsion-free) *stacky fan* is a simplicial fan + choice of integral element on each ray (*ray-vector*)

Theorem (Fantechi-Mann-Nironi, I, B-C-S, L-T)

Toric orbifolds are classified by stacky fans

Take coarse moduli scheme = forget stacky structure, keep fan

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Wonderful compactification of adjoint groups

G adjoint, i.e. $ZG = \{1\}$

e.g. $PGL(n)$, $SO(2n+1, \mathbb{C})$, E_8 , F_4 , G_2

λ regular dominant weight

highest weight representation V_λ

have

$$\begin{array}{ccc} G & \hookrightarrow & \text{End}(V_\lambda) \\ & \searrow & \downarrow \text{dashed} \\ & & \mathbb{P}(\text{End}(V_\lambda)) \end{array}$$

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Wonderful compactification

Definition (De Concini - Procesi)

The wonderful compactification \overline{G}^w of an adjoint group is the closure in $\mathbb{P}(\text{End}(V_\lambda))$

Independent of choice of λ

T_G maximal torus in G , take closure in \overline{G}^w
 \Rightarrow get toric variety \overline{T}_G

Fan of $\overline{T}_G =$ Weyl chambers + Λ_G

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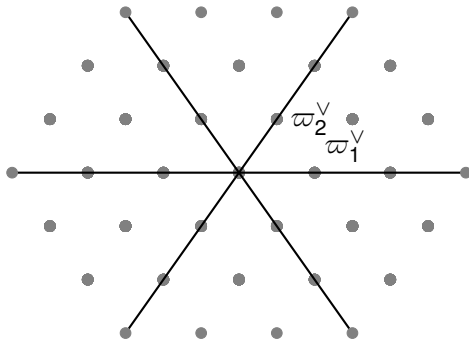
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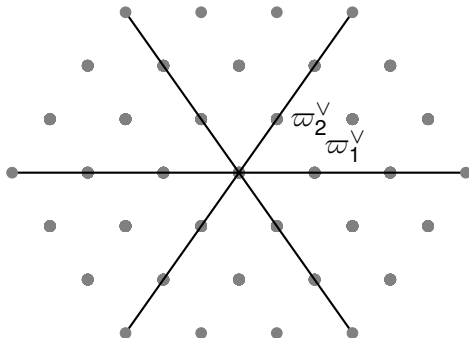
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e.g. $PGL(3)$:



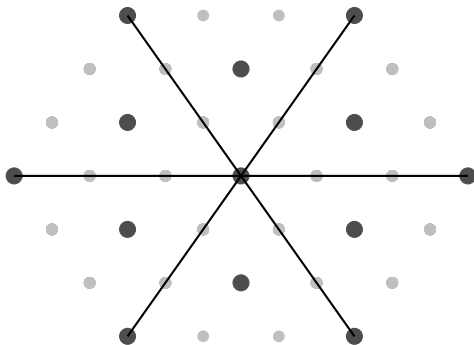
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e.g. $SL(3, \mathbb{C})$



Corresponding toric variety no longer smooth!

Moduli Problem

Moduli problem:

\mathbb{G}_m -equivariant G -principal bundles
on chains of projective lines

Framed at north- and south-poles

Length of chain is arbitrary finite, can
vary in families

Problem:

Too many objects: stack is not
separated nor of finite type

Cure this by imposing a stability
condition



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Which bundles are stable?

Theorem (Birkhoff-Grothendieck-Harder)

Every G -principal bundle on \mathbb{P}^1 reduces to the maximal torus and up to isomorphisms is entirely determined by a co-character

$$\Lambda \ni \rho : \mathbb{G}_m \rightarrow G$$

unique up to W_G -action

Take two charts given by stereographic projection from s and n , use ρ as transition function

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Every \mathbb{G}_m -equivariant G -principal bundle on \mathbb{P}^1 is determined by action of \mathbb{G}_m on fibers over n and s ,

$$\begin{array}{c} n \bullet \rho_n \\ | \\ s \bullet \rho_s \end{array}$$

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Underlying non-equivariant bundle determined by $\rho_n - \rho_s$

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Σ -stable bundles

Choose a (stacky) fan Σ for T_G , satisfying:

- Σ is simplicial
- Σ is Weyl-invariant
- Σ refines the Weyl-chambers

Choose ordering of integral elements

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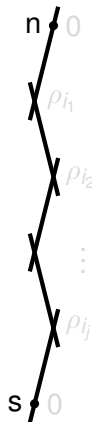
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A bundle on chain of lines is Σ -stable if

- co-chars on extremal n and s are 0
- co-chars on nodes are ray-vectors of Σ :

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in order & all rays of single cone



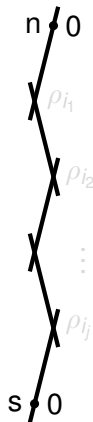
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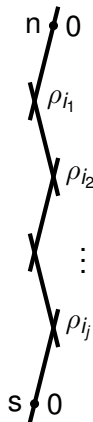
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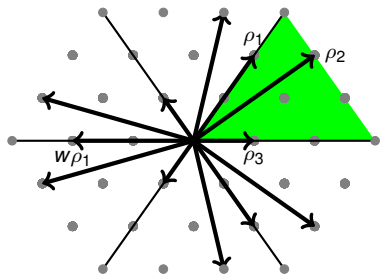
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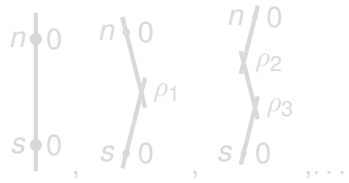


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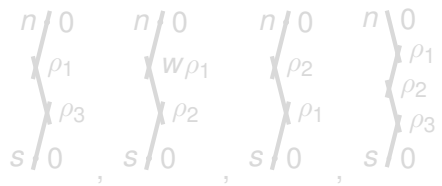
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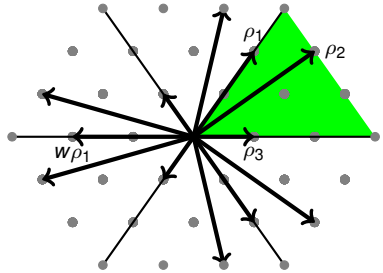


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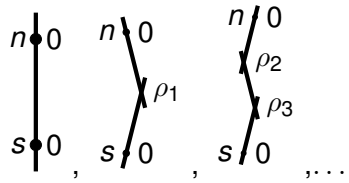


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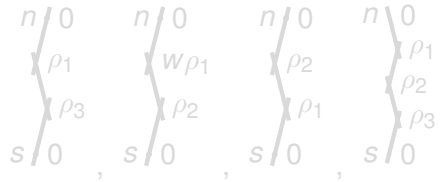
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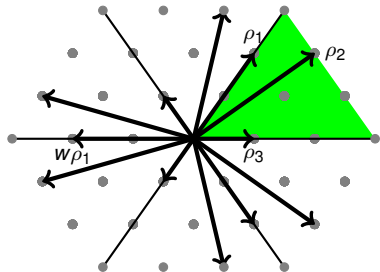


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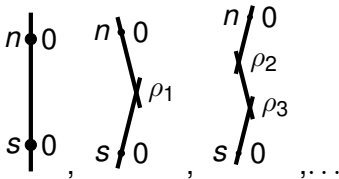


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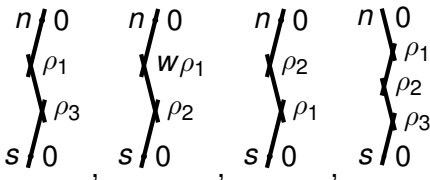
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Σ -stable:



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Main result

Theorem (M.-Thaddeus)

Given Σ , the corresponding moduli stack of Σ -stable bundles-on-chains $\mathcal{M}(\Sigma)$ is a (smooth) orbifold compactifying G

Fine print:

Moduli problem depends on ordering, but different orderings give canonically isomorphic stacks

Need to *rigidify* to remove G_m from automorphisms

If $p = \text{char}(k) > 0$, stack is *tame* in general, DM if ρ_i linearly independent over \mathbb{F}_p

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Weyl chambers strongly convex, no refinement necessary
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- G non-semi-simple reductive:



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