Universal plane curve and moduli spaces of 1-dimensional coherent sheaves

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Plan

Simpson moduli problem, some questions

Universal plane curve

- Definition
- Parameter space
- Universal singular locus
- Universal curve as a space of sheaves

New objects

- Construction
- Classification, main statement

$\mathfrak{X}$ projective variety, $P \in \mathbb{Q}[m] \implies$

Theorem (Carlos Simpson)

There is a moduli space $M_P(\mathfrak{X})$ of semi-stable sheaves with Hilbert polynomial P.

- $M = M_P(\mathfrak{X})$ projective.
- M contains sheaves that are not locally free on their support (**singular sheaves**).
- $M' \subseteq M$ subvariety of singular sheaves. Not empty in general! $\implies$
- $M \setminus M'$ space of vector bundles (on the support), **non-compact**
- M is a compactification of $M \setminus M'$.
- $\operatorname{codim}_M M' > 1$ (not a divisor) $\implies M$ is not "maximal"

Questions

- Find a maximal compactification of $M \setminus M'$
- Find a compactification of $M \setminus M'$ by vector bundles

For $\mathfrak{X} = \mathbb{P}_2$, $P(m) = 3m + 1$:

- M is the universal plane cubic curve.
- M' is the universal singular locus, $\operatorname{codim}_M M' = 2$.
- $\widetilde{M} = \operatorname{Bl}_{M'} M$ is an answer to both questions.

Question.

- Is there a similar interpretation for the universal curve of any degree d?

Universal plane curve of degree d $\quad (d \geqslant 3)$

V v. sp. over $\Bbbk$, $\dim_{\Bbbk} V = 3 \implies \mathbb{P}_2 = \mathbb{P}V$ projective plane

$V^* = \mathrm{Hom}(V, \Bbbk)$ dual space; $S^d V^*$ d-th symmetric power
$\implies$
given a basis $\{x_0, x_1, x_2\}$ of V^* $\implies$
$S^d V^*$ homogeneous polynomials in x_0, x_1, x_2 of degree d.

$\mathbb{P}_N = \mathbb{P}(S^d V^*)$ space of plane curves of degree d
$(\dim = N = \frac{(d+2)(d+1)}{2} - 1)$

Definition (Universal plane curve of degree d)

$M = \{(C, p) \mid p \in C\} = \{(\langle f \rangle, \langle x \rangle) \in \mathbb{P}_N \times \mathbb{P}_2 \mid f(x) = 0\}$.

Remark

M *is smooth, projective,* $\dim M = N + 1 = \frac{(d+2)(d+1)}{2}$.

Parameter space

X space of matrices $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$, where

z_1, z_2 linear forms; q_1, q_2 forms of degree $d-1$ such that:

- z_1 und z_2 linear independent;
- $\det A \neq 0$

$\Longrightarrow X \subseteq \mathbb{k}^{d^2+d+6}$ (open).

$$X \xrightarrow{\nu} M, \quad \left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right) \mapsto (\langle z_1 q_2 - z_2 q_1 \rangle, z_1 \wedge z_2),$$

$z_1 \wedge z_2 \in \mathbb{P}_2$ common zero of z_1 and z_2.

Lemma

- *ν is surjective.*
- *$\nu(A_1) = \nu(A_2) \iff g A_1 h = A_2$ for $g \in \mathrm{GL}_2(\mathbb{k})$, $h = \begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \in \mathbb{k}^*$, q homogeneous of degree $d-2$.*

Geometric quotient

Algebraic group

$$H = \{ \begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{k}^*, q \text{ homogeneous of degree } d-2 \}$$

$\Longrightarrow$ algebraic group $G = \mathrm{GL}_2(\mathbb{k}) \times H \Longrightarrow$ Group action

$$G \times X \to X, \quad (g, h) \cdot A = gAh^{-1}.$$

Lemma

M is the orbit space of this action:
points of $M \overset{1:1}{\longleftrightarrow}$ orbits.

$\forall A \in X$

$$St = \{ (\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, (\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix})) \mid \lambda \in \mathbb{k}^* \}.$$

$\mathbb{P}G := G/St \Longrightarrow \mathbb{P}G$ acts freely on X.
Locally over M $\exists$ section $M \supseteq U \to X$ of $X \overset{\nu}{\to} M$
Main theorem of Zariski $\Longrightarrow X$ is a $\mathbb{P}G$-principal bundle over M
$\Longrightarrow M$ geometric quotient.

Definition (Universal singular locus)

$M' \subseteq M$, $\quad M' = \{(C, p) \mid p \in \mathrm{Sing}(C)\}$.

- M' is smooth,
- $\mathrm{codim}_M M' = 2$.

Lemma

- $A \in X$, $\quad A = \left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)$, $\quad p = z_1 \wedge z_2 \implies$ $\nu(A) \in M' \iff q_1(p) = q_2(p) = 0 \implies$
- $X' \subseteq X$ *parameter space of* M'*: global complete intersection (two equations)*

Universal curve as a space of sheaves

$A \in X \iff$ injections $2\mathcal{O}_{\mathbb{P}_2}(-d+1) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} \implies$
X parameter space of sheaves with resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-d+1) \xrightarrow{\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0 \quad (1)$$

$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-d+1)) =$
$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-d+1)) = 0 \implies$

Morphisms of sheaves as in $(1) \overset{1:1}{\longleftrightarrow}$ morphisms of resolutions.

$$\begin{array}{ccccccccc} 0 \longrightarrow & 2\mathcal{O}_{\mathbb{P}_2}(-d+1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F}_2 & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & 2\mathcal{O}_{\mathbb{P}_2}(-d+1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F}_1 & \longrightarrow 0. \end{array}$$

$\implies$

Proposition

- *Points of $M \overset{1:1}{\longleftrightarrow}$ isomorphism classes of sheaves with (1).*
- *Points of $M' \overset{1:1}{\longleftrightarrow}$ isomorphism classes of singular sheaves with (1)*

Universal curve and Simpson moduli spaces

- If all sheaves from M are stable $\implies M$ subvariety of a Simpson moduli space.

Example

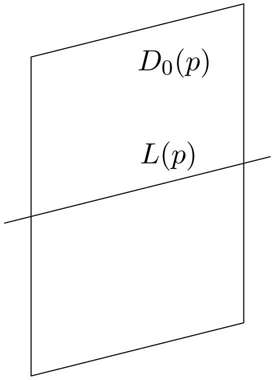
$d = 3$, $d = 4$, $d = 5 \implies M$ *consists of stable sheaves.*

- $d = 3 \implies M = M_{3m+1}(\mathbb{P}_2)$.
- $d = 4 \implies M \subseteq M_{4m-1}(\mathbb{P}_2)$ *closed subvariety of codimension* 2.
- $d = 5 \implies M \subseteq M_{5m-4}(\mathbb{P}_2)$ *closed subvariety of codimension* 5.

Surfaces $D(p)$ (subvarieties in $\mathbb{P}_2 \times \mathbb{P}_2$)

$p \in \mathbb{P}_2$

$D(p) = D_0(p) \cup D_1(p)$: $D_0(p) = \mathrm{Bl}_p(\mathbb{P}_2)$, $L(p)$ exceptional divisor, $D_1(p) = \mathbb{P}_2$, and $D_0 \cap D_1 = L$.



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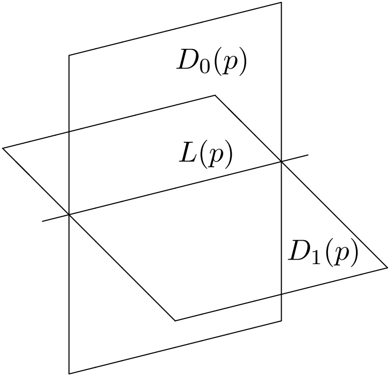


Picard-Group of $D(p)$

- $D(p) \subseteq \mathbb{P}_2 \times \mathbb{P}_2$, $\mathcal{O}_{\mathbb{P}_2}(1)$ on each $\mathbb{P}_2 \implies$
- $\mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1,0) = pr_1^*(\mathcal{O}_{\mathbb{P}_2}(1))$, $\mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(0,1) = pr_2^*(\mathcal{O}_{\mathbb{P}_2}(1))$
- restrictions to $D(p)$:

$$\mathcal{O}_{D(p)}(1,0) = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1,0)|_{D(p)}, \quad \mathcal{O}_{D(p)}(0,1) = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(0,1)|_{D(p)}.$$

- $\operatorname{Pic} D(p) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators:

$$\mathcal{O}_{D(p)}(1,0), \quad \mathcal{O}_{D(p)}(0,1).$$

New objects (R-bundles)

$A \in X'$, $B \in T_A X \implies$ sheaf $\mathcal{E} = \mathcal{E}(A,B)$ on $D(p)$ with resolution

$$0 \to 2\mathcal{O}_{D(p)}(-d{+}2,-1) \xrightarrow{\Phi(A,B)} \mathcal{O}_{D(p)}(-d{+}2,0) \oplus \mathcal{O}_{D(p)} \to \mathcal{E} \to 0.$$

$\mathcal{E}(A,B)$ is locally free on its support $\iff B \in T_A X \setminus T_A X'$.

Definition

R-bundles $\overset{def}{\iff}$ *sheaves $\mathcal{E}(A,B)$ for $A \in X'$, $B \in T_A X \setminus T_A X'$.*

Remark

- *R-bundles are flat limits of non-singular sheaves*
- *Morphisms of sheaves $\overset{1:1}{\longleftrightarrow}$ morphisms of resolutions*

Construction

- $A \in X', \quad B \in T_A X \implies$ one parameter family over $S \subseteq \Bbbk^1$

$$0 \to 2\mathcal{O}_{S\times\mathbb{P}_2}(-d+1) \xrightarrow{A+tB} \mathcal{O}_{S\times\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{S\times\mathbb{P}_2} \to \boldsymbol{\mathcal{F}} \to 0,$$

- $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}, \quad p = z_1 \wedge z_2 \implies Z \xrightarrow{\sigma_p} U \times \mathbb{P}_2$ blow up at $0 \times p$.
- Pull back to Z and factor through the canonical section of the exceptional divisor:

$$\begin{array}{ccccccc} 0 \to 2\mathcal{O}_Z(-d+1,0) & \longrightarrow & \mathcal{O}_Z(-d+2,0) \oplus \mathcal{O}_Z & \longrightarrow & \sigma_p^* \boldsymbol{\mathcal{F}} \to 0 \\ \downarrow & & \| & & \downarrow \\ 0 \to 2\mathcal{O}_Z(-d+2,-1) & \longrightarrow & \mathcal{O}_Z(-d+2,0) \oplus \mathcal{O}_Z & \longrightarrow & \widetilde{\boldsymbol{\mathcal{F}}} \longrightarrow 0 \end{array}$$

- restrict to $D(p)$ (total transform of $\{0\} \times \mathbb{P}_2$) $\implies$

$$0 \to 2\mathcal{O}_{D(p)}(-d+2,-1) \xrightarrow{\Phi(A,B)} \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \to \mathcal{E} \to 0.$$

Properties

Support of an R-bundle: reducible curve $C_0 \cup C_1$

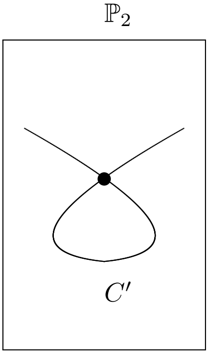
- C_0 curve in D_0
- C_1 curve in D_1 (conic section in $D_1 = \mathbb{P}_2$)

Restrictions of to the components of $D(p)$:

- restriction to D_0: structure sheaf of C_0;
- restriction to D_1: Bundle of degree 1 on C_1.

Illustration

Generic situation



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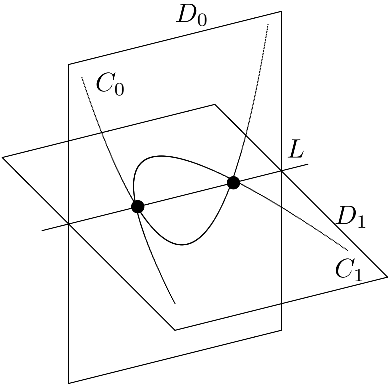

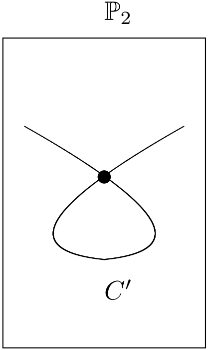

Classification

Definition

R-bunldes $\mathcal{E}_1$ and $\mathcal{E}_2$ on $D(p)$ constructed at $A \in X'$

$\mathcal{E}_1$ and $\mathcal{E}_2$ equivalent $\overset{def}{\Longleftrightarrow}$ $\exists$ automorphism ϕ of $D(p)$, identical on D_0, $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$.

Theorem

Equivalence classes of R-bundles at $A \in X'$ $\overset{1:1}{\longleftrightarrow}$ points of $\mathbb{P}N_A$.

Corollary

- *$\widetilde{M} = \mathrm{Bl}_{M'} M$ space of the isomorphism classes of non-singular sheaves $(M \setminus M')$ together with the equivalence classes of R-bundles.*
- *$\widetilde{X} = \mathrm{Bl}_{X'} X$ parameter space.*

Idea of the proof.

$\mathcal{E}_1 = \mathcal{E}_1(A, B_1)$, $\mathcal{E}_2 = \mathcal{E}_2(A, B_2)$ two R-bundles
$A \in X'$, $B_1, B_2 \in T_A X$.

- Equivalence $\mathcal{E}_1 \sim \mathcal{E}_2 \implies$ commutative diagram

$$\begin{array}{ccc} 2\mathcal{O}_{D(p)}(-d+2,-1) & \longrightarrow & \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \\ \downarrow & & \downarrow \\ 2\mathcal{O}_{D(p)}(-d+2,-1) & \longrightarrow & \mathcal{O}_{D(p)}(-d+2,0) \oplus \mathcal{O}_{D(p)} \end{array}$$

  arrows = matrices

- $B_1 - \lambda B_2$ satisfies the tangent equations of X' in A for some $\lambda \in \mathbb{k}^* \implies$
- $[\overline{B}_1] = [\overline{B}_2]$ in $\mathbb{P}N_A$
  ($\overline{B}_i \in N_A = T_A X / T_A X'$, $[\overline{B}_i] \in \mathbb{P}N_A$)

□