Vortices on Riemann Surfaces

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1. $U(1)$ Vortices on a Surface

- A $U(1)$ (Abelian Higgs) Vortex is a Topological Soliton in two space dimensions, stabilised by its magnetic flux. It is a localised object of fixed positive size.
- The fields are locally a complex scalar field ϕ and a $U(1)$ gauge potential A_j $(j = 1, 2)$. Time-dependent fields also require a potential A_t.
- We avoid boundary conditions by considering vortices on a closed Riemann surface M, of genus g.
- To get a net magnetic flux through a surface, one needs the fields to be a section and connection on a non-trivial $U(1)$ bundle over the surface. The magnetic flux is 2π times the first Chern number.

- We need a metric ds^2 on the surface M, compatible with the complex structure. Using a (local) complex coordinate $z = x_1 + ix_2$ we express the metric as

$$ds^2 = \Omega(z, \bar{z})\, dz\, d\bar{z}\,. \tag{1}$$

- The effect of the curvature of the surface on a vortex is interesting.
- We assume vortices have no back-reaction on the metric. They are not gravitating. Some vortices – cosmic strings – have a gravitational effect.

2. Energy and Bogomolny Equations

- For static fields there is a potential energy

$$V = \frac{1}{2}\int_M \left(B^2 + \frac{1}{\Omega}|D_j\phi|^2 + \frac{\lambda}{4}(1-|\phi|^2)^2 \right) \Omega \, d^2x \qquad (2)$$

where $B = \frac{1}{\Omega}(\partial_1 A_2 - \partial_2 A_1)$ and $D_j\phi = \partial_j\phi - iA_j\phi$.

- The total magnetic flux is

$$\int_M B \, \Omega \, d^2x = 2\pi N \qquad (3)$$

for some (positive) integer N.

- The vortex centres are where ϕ vanishes. Here the magnetic flux is concentrated.
- Mathematical progress is possible at critical coupling $\lambda = 1$. Here there are many static vortex solutions, as the force between vortices vanishes.
- There are static solutions of the full (second-order) field equations even if $\lambda \neq 1$, e.g. the Abrikosov vortex lattice, but the vortices are at special locations.

- When $\lambda = 1$, the potential energy V can be re-expressed as [E.B. Bogomolny]

$$V = \pi N + \frac{1}{2}\int_M \left\{ \left(B - \frac{1}{2}(1-|\phi|^2)\right)^2 + \frac{1}{\Omega}\left|D_1\phi + iD_2\phi\right|^2 \right\} \Omega \, d^2x \tag{4}$$

where we have dropped a total derivative term that integrates to zero.

- The minimum energy fields, for given N, occur if

$$D_1\phi + iD_2\phi = 0 \,, \tag{5}$$

$$B = \frac{1}{2}(1-|\phi|^2) \,. \tag{6}$$

These are the Bogomolny equations. The first says that ϕ is a holomorphic section. The second has an interpretation in terms of a moment map.

- The Bogomolny equations have a solution, unique up to gauge transformations, with vortex centres at any N specified locations, provided the area of the surface

$$A = \int_M \Omega \, d^2x \tag{7}$$

  is large enough [Taubes, Bradlow, Garcia-Prada].
  Call the vortex locations $Z_1, Z_2, \ldots, Z_N$ (or Z if there is only one vortex). ϕ vanishes at these points.
- These solutions are almost never explicitly known, not even the solution for one vortex in a plane.

Constraint on the Area

- Integrating the second Bogomolny equation gives

$$2\pi N = \int_M B\,\Omega\,d^2x = \frac{1}{2}A - \frac{1}{2}\int_M |\phi|^2\,\Omega\,d^2x\,. \qquad (8)$$

The last integral is non-negative, and we do not want it to vanish, so

$$A > 4\pi N\,. \qquad (9)$$

Vortex solutions of the (combined) Bogomolny equations exist if and only if this inequality holds [Bradlow].
When $A \to 4\pi N$, ϕ vanishes and $B \to \frac{1}{2}$. The vortices dissolve.

- The integrated second Bogomolny equation gives the normalisation of $|\phi|$

$$\int_M |\phi|^2 \, \Omega \, d^2x = A - 4\pi N \, . \qquad (10)$$

- The N-vortex moduli space, $\mathcal{M}_N$, is the space of static N-vortex solutions. It is the Nth symmetrised power of M, i.e. M^N modulo permutations of the N points. This is a smooth complex manifold of complex dimension N (not an orbifold).
- The product metric on M^N, induced by $\Omega \, d^2x$ on M, does not descend to a smooth metric on $\mathcal{M}_N$. Nevertheless, the dynamical field theory gives a natural smooth metric on $\mathcal{M}_N$.

3. N-Vortex Moduli Space and its Geometry

- We consider a dynamical theory, with kinetic energy T, and potential energy V as earlier. The Lagrangian $L = T - V$ is

$$L = \frac{1}{2} \int_M \left(\frac{1}{\Omega} E_j E_j + |D_t \phi|^2 \right) \Omega \, d^2x \; - \; V \tag{11}$$

  where $E_j = \partial_t A_j - \partial_j A_t$ and $D_t \phi = \partial_t \phi - i A_t \phi$. A_t is fixed so that $(E_1, E_2, D_t \phi)$ is orthogonal to infinitesimal gauge transformations (Gauss' law).

- A static vortex has potential energy π. In motion it has kinetic energy too. A vortex moving non-relativistically in a plane, with moving centre $Z(t)$, has kinetic energy

$$T = \frac{1}{2} \pi \frac{dZ}{dt} \frac{d\bar{Z}}{dt} \, . \tag{12}$$

  The vortex mass is π, because the dynamical theory is Lorentz invariant.

Strachan–Samols Metric Formula

- For one vortex in non-relativistic motion on the surface M, the kinetic energy is

$$T = \frac{1}{2}\pi\left(\Omega(Z,\bar{Z}) + 2\frac{\partial b}{\partial Z}\right)\frac{dZ}{dt}\frac{d\bar{Z}}{dt}\,. \tag{13}$$

  This implies a metric on $\mathcal{M}_1$

$$ds^2 = \left(\Omega(Z,\bar{Z}) + 2\frac{\partial b}{\partial Z}\right) dZd\bar{Z}\,. \tag{14}$$

- The $\Omega(Z,\bar{Z})$ factor is what is expected for a pointlike vortex. The second term allows for the finite size of the vortex and the curvature of the surface.
- b measures the deformation of the vortex field away from circular symmetry, near the vortex centre. It is defined as a coefficient in the expansion of $|\phi|^2$ about Z:

$$|\phi|^2 = C|z-Z|^2\left(1+\frac{1}{2}\bar{b}(z-Z)+\frac{1}{2}b(\bar{z}-\bar{Z})+O(|z-Z|^2)\right). \tag{15}$$

- For N vortices in non-relativistic motion, satisfying the Bogomolny equations but with time-varying centres $Z_1(t), Z_2(t), \ldots, Z_N(t)$, there is a similar formula for the kinetic energy, involving only local field data at and near each vortex:

$$T = \frac{1}{2}\pi \sum_{r=1}^{N} \sum_{s=1}^{N} \left(\Omega(Z_r, \bar{Z}_r)\delta_{rs} + 2\frac{\partial b_r}{\partial Z_s} \right) \frac{dZ_s}{dt}\frac{d\bar{Z}_r}{dt} \,. \qquad (16)$$

- Again, b_r is a coefficient in the expansion of $|\phi|^2$ about the vortex centred at Z_r:

$$|\phi|^2 = C_r|z-Z_r|^2\left(1+\frac{1}{2}\bar{b}_r(z-Z_r)+\frac{1}{2}b_r(\bar{z}-\bar{Z}_r)+O(|z-Z_r|^2)\right), \qquad (17)$$

and it depends on the curvature of M and on the locations of the other vortices.

- This result is derived using the Bogomolny equations (their lineariston) and the field kinetic energy. A_t is eliminated using Gauss' law. Localisation is achieved by integration by parts.
- The formula for T is interpreted as giving a metric on N-vortex moduli space $\mathcal{M}_N$,

$$ds^2 = \sum_{r=1}^{N} \sum_{s=1}^{N} \left(\Omega(Z_r, \bar{Z}_r) \delta_{rs} + 2 \frac{\partial b_r}{\partial Z_s} \right) dZ_s d\bar{Z}_r \,. \qquad (18)$$

N-Vortex Motion

- The natural motion (if any) of vortices is to follow a geodesic in the N-vortex moduli space $\mathcal{M}_N$. Even for one vortex this differs somewhat from a geodesic on the underlying surface M, except in the pointlike vortex limit.
- The vortices have a pseudo-gravitational effect, as vortex motion depends on the background geometry and on the locations of other vortices.
- In a head-on collision, two vortices scatter through 90° (like the pair $\pm\sqrt{a}$ as a moves from positive to negative).

4. Volume of N-Vortex Moduli Space

- Although we do not have explicit vortex solutions on most surfaces, the Strachan–Samols formula gives a Kähler metric on the vortex moduli space $\mathcal{M}_N$, and hence a volume form (the Nth power of the Kähler 2-form, divided by $N!$).
- The total volume just depends on the (real) cohomology class of the Kähler 2-form, which can be determined exactly [N.S.M. and S. Nasir, T. Perutz]. Its coefficients are determined from its integral over the two special surfaces in moduli space:
  (i) N coincident vortices ($Z_1 = Z_2 = \cdots = Z_N$),
  (ii) One vortex free to move, with the remainder fixed.

- Using what we know about the pole-like behaviour of b_j when vortices approach coincidence gives the cohomology class of the Kähler 2-form, and hence the total volume of $\mathcal{M}_N$

$$\mathrm{Vol}_N = \sum_{n=0}^{g} \frac{(4\pi)^n (A - 4\pi N)^{N-n} g!}{n!(N-n)!(g-n)!} \,. \qquad (19)$$

  The sum is up to the top term $n = N$ if $N < g$.
- J.M. Baptista has extended these results on the cohomology of vortex moduli spaces to vortices with larger toric gauge groups and nonlinear Higgs fields.

5. A Vortex Gas

- Consider N large and A large. Assume the N vortices are on a (topological) sphere.
- The N-vortex configuration space is $\mathcal{M}_N$ and its volume is

$$\mathrm{Vol}_N = \frac{(A - 4\pi N)^N}{N!} . \tag{20}$$

  The vortices are interacting, because each vortex removes some area from the space available to the others.

- Notice the $N!$ factor required for a gas of unlabelled particles is automatically present. In the usual (Gibbs) treatment of a gas in classical statistical mechanics this factor is put in by hand.
- The partition function for the vortex gas is

$$\mathcal{Z} = \left(\frac{T}{2\hbar^2}\right)^N \frac{(A - 4\pi N)^N}{N!} \, . \tag{21}$$

Hence, from the free energy $F = -T \log \mathcal{Z}$, we find the pressure $P = -\partial F/\partial A$, and the Clausius equation of state

$$P(A - 4\pi N) = NT \, . \tag{22}$$

This is an exact result, but usually occurs as an approximation (van der Waals theory).

6. One Vortex on a Large Surface

- There is no exact formula for the metric on the 1-vortex moduli space $\mathcal{M}_1$, for a general surface M. An expansion is possible when M is large and its curvature small [N.S.M., M. Dunajski and N.S.M.]

$$ds^2 = \left(1 - \frac{1}{2}R + c\Delta R + \cdots\right)\Omega(Z, \bar{Z})dZd\bar{Z} \,. \qquad (23)$$

  Here R is the Ricci scalar curvature of M, and $\Delta = \frac{1}{\Omega}\partial_j\partial_j$ is the geometric Laplacian. The coefficient c is not yet known, but appears to be universal.

- This metric is a deformation of the underlying metric on M, rather analogous to the effect of Ricci flow on the metric.
- Ricci flow would give, after unit time, the metric (23) with $c = -1/8$.

- The area of the moduli space is known exactly (the $N = 1$ case of Vol_N). It is

$$\mathcal{A} = A + 4\pi(g - 1)\,. \tag{24}$$

This is consistent with (23), by the Gauss–Bonnet formula, and implies that no R^2 term is possible in (23).

7. Dissolving Vortices on a Small Surface

- The dissolving limit is where A approaches $4\pi N$ from above.
- For N dissolving vortices on a surface of genus 0 (a sphere of any shape) the N-vortex moduli space simplifies to complex projective space $\mathbb{CP}^N$ with its symmetric Fubini–Study metric, and radius going to zero as $A \to 4\pi N$ [J.M. Baptista and N.S.M.].

- For $N = 1$ and $g \geq 1$, N.S.M. and N. Romao have shown that the metric on $\mathcal{M}_1$ in the dissolving vortex limit is the Bergman metric on M. This is the restriction of the flat metric on the Jacobian of M to the Abel–Jacobi image of M in the Jacobian.
- The reason is that in the dissolving limit, the Higgs field ϕ is very small, the magnetic field becomes constant, and all that matters when a vortex moves is that there is a motion of the divisor defined by the vortex (the point where $\phi = 0$), and a corresponding motion in the space of line bundles over M.

8. Some Exact Hyperbolic Vortex Solutions

- The vortex equations are integrable if the surface M is the hyperbolic plane, with constant Ricci scalar curvature -1. This was first noticed by Witten in connection with $SU(2)$ instantons in flat $\mathbb{R}^4$ with $SO(3)$ symmetry. There are various elaborations of this, e.g. by A.D. Popov, and recently to more general hyperbolic surfaces by N.S.M. and N.A. Rink.
- Here we show how to construct a few solutions.

- Let's rewrite the Bogomolny equations on a general surface as

$$\begin{aligned} D_{\bar{z}}\phi &= 0\,, && (25) \\ F_{z\bar{z}} &= \frac{i}{4}\Omega(1-\phi\bar{\phi})\,, && (26) \end{aligned}$$

where $D_{\bar{z}}\phi = \partial_{\bar{z}}\phi - iA_{\bar{z}}\phi = \frac{1}{2}(D_1\phi + iD_2\phi)$, and $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$.

- Let $\phi = H^{1/2}\chi(z)$ where χ is holomorphic and H real. Then we can solve (25),

$$A_z = \frac{i}{2}\partial_z(\log H)\,, \quad A_{\bar{z}} = -\frac{i}{2}\partial_{\bar{z}}(\log H)\,, \tag{27}$$

and (26) simplifies to

$$\partial_z\partial_{\bar{z}}(\log H) = -\frac{\Omega}{4}\left(1 - H\chi(z)\overline{\chi(z)}\right)\,. \tag{28}$$

- As M is hyperbolic,

$$\partial_z \partial_{\bar{z}} (\log \Omega) = \frac{\Omega}{4} \,. \tag{29}$$

- A solution is now constructed using any holomorphic map f from M to another hyperbolic surface $\widetilde{M}$. Let $\widetilde{M}$ have complex coordinate w, and metric $\widetilde{\Omega}(w, \bar{w})$ satisfying

$$\partial_w \partial_{\bar{w}} (\log \widetilde{\Omega}) = \frac{\widetilde{\Omega}}{4} \,. \tag{30}$$

- The map f is given locally by $w = f(z)$. Set

$$\chi(z) = \frac{dw}{dz} = \frac{d}{dz} f(z) \tag{31}$$

i.e. χ is the derivative of the map. Set

$$H(z, \bar{z}) = \frac{\widetilde{\Omega}(f(z), \overline{f(z)})}{\Omega(z, \bar{z})}\,, \tag{32}$$

the ratio of the metrics at equivalent points. This solves (28) as

$$\begin{aligned}
\partial_z \partial_{\bar{z}} (\log H) &= -\partial_z \partial_{\bar{z}} (\log \Omega) + \partial_z \partial_{\bar{z}} (\log \widetilde{\Omega}) \\
&= -\partial_z \partial_{\bar{z}} (\log \Omega) + \partial_w \partial_{\bar{w}} (\log \widetilde{\Omega}) \frac{dw}{dz} \overline{\frac{dw}{dz}} \\
&= -\frac{\Omega}{4} + \frac{\widetilde{\Omega}}{4} \chi \overline{\chi} \\
&= -\frac{\Omega}{4} (1 - H \chi \overline{\chi})\,.
\end{aligned} \tag{33}$$

- The vortex locations are the points where $\frac{df}{dz}$ vanishes – the ramification points of the map f (where the inverse map has branch points).
- Example 1. Maps from a hyperbolic plane to a hyperbolic plane (mapping boundary to boundary) are the Blaschke rational functions – used by Witten:

$$f(z) = \prod_{m=1}^{N+1} \frac{z - a_m}{1 - \overline{a_m} z} \tag{34}$$

(with $|a_m| < 1 \; \forall m$), giving the scalar field

$$|\phi|^2 = \frac{(1 - z\bar{z})^2}{(1 - f(z)\overline{f(z)})^2} \frac{df}{dz} \overline{\frac{df}{dz}} \tag{35}$$

of N vortices in the hyperbolic plane.

- Example 2. On the compact surface M of genus $2p-1$ defined by the algebraic equation

$$y^2 = (z^2 - e_1)(z^2 - e_2)\dots(z^2 - e_{2p}) \tag{36}$$

(with $e_k \neq 0 \ \forall k$, and $p > 2$) we can use the map

$$w = f(z) = z^2 \tag{37}$$

to the compact surface of genus $p-1$ defined by

$$y^2 = (w - e_1)(w - e_2)\dots(w - e_{2p})\,. \tag{38}$$

There are four vortices on M where $df/dz = 0$, i.e. at $z = 0$ and $z = \infty$, on both sheets. In this example there is a unique hyperbolic metric on each surface, but not explicitly known.
A similar construction works for any Riemann surface with automorphisms, where we can map to the quotient surface.

- Example 3. There are two types of quotient of the hyperbolic plane by the additive group $\mathbb{Z}$, giving a hyperbolic trumpet, or a hyperbolic cylinder. On each of these surfaces we can construct explicit vortex solutions using generalised Blaschke functions (algebraic, trigonometric or elliptic). Some of these are related to calorons (instantons in $\mathbb{R}^4$ invariant under action of $\mathbb{Z}$ and $SO(3)$).

9. Open Problems

- Understand better the 1-vortex moduli space metric as a geometric flow of M. Generalise to the case of N vortices on a large surface.
- Find more explicit vortex solutions on compact hyperbolic surfaces using pairs of Fuchsian groups acting on the hyperbolic plane, with the surfaces represented as hyperbolic polygons with sides identified.
  Construct hyperbolic vortices with more moduli.
- Understand moduli space geometry and dynamics for non-Abelian vortices.