Derived categories and rationality of conic bundles

joint work with M.Bernardara

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Fano 3-folds with $\kappa(V) = -\infty$

V smooth irreducible projective 3-fold over $\mathbb{C}$.
V is birational (Reid-Mori, Miyaoka) to X with at most terminal singularities of the following types :

- X is a minimal $\mathbb{Q}$-Fano variety
- X is a minimal Del Pezzo fibration
- X is a minimal conic bundle $\pi : X \to S$, S normal.

Up to birational equivalence X and S are nonsingular and projective.

Rationality of conic bundles

$\pi : X \to S$ standard conic bundle over S smooth, rational, projective surface. $C \subset S$ degeneracy curve, $\tilde{C} \to C$ associated double cover and $J(X)$ the intermediate Jacobian of X.

Necessary : S rational, C connected [A-M] and $J(X)$ direct sum of Jacobians of smooth projective curves [C-G].

Theorem (Beauville)

$J(X)$ is isomorphic to the Prym variety $P(\tilde{C}/C)$.

Theorem (Shokurov, Beauville)

If S is minimal, X is rational if and only if $J(X)$ splits.

Rationality of conic bundles

Possible cases :

- $S = \mathbb{P}^2$ and C is a cubic or a quartic.
- $S = \mathbb{P}^2$, C is a quintic and $\tilde{C} \to C$ is induced by an even θ-characteristic.
- $S \to \mathbb{P}^1$ is ruled, C is trigonal or hyperelliptic and the g^1_r is induced by the ruling.

Derived categories and rationality

Question : Can we relate the derived category $\mathrm{D^b}(X)$ and the rationality of X ?

The most promising way is by looking at semi-orthogonal decompositions of $\mathrm{D^b}(X)$.

If $\mathbf{T}$ is a linear triangulated category, a semiorthogonal decomposition

$$\mathbf{T} = \langle \mathbf{T}_1, \dots, \mathbf{T}_n \rangle$$

is an ordered collection of orthogonal (from right to left, $Hom_{\mathbf{T}}(\mathbf{T}_{>i}, \mathbf{T}_i) = 0$) subcategories generating the whole category.

Semi-orthogonal decompositions

Example : [BMMS] If X is a smooth cubic threefold

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{T}, \mathcal{O}_X, \mathcal{O}_X(1) \rangle,$$

and the equivalence class of the category $\mathbf{T}$ corresponds to the isomorphism class of $J(X)$. Both $\mathbf{T}$ and $J(X)$ are indecomposable.

Idea : The rationality of conic bundles may be equivalent to admitting a certain semi-orthogonal decomposition.

Theorem (Kuznetsov)

If $\pi : X \to S$ is a conic bundle, let $\mathcal{B}_0$ be the sheaf of even parts of the Clifford algebra associated to it and $\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$ the derived category of $\mathcal{B}_0$-algebras.

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \Phi \mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0), \pi^* \mathrm{D}^{\mathrm{b}}(S) \rangle,$$

where Φ and π^* are fully faithful.

If S is rational, $\mathrm{D}^{\mathrm{b}}(S)$ is generated by exceptional objects and then :

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \Phi \mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0), E_1, \ldots, E_s \rangle,$$

Main Theorems

Theorem (-,Bernardara)

If there are smooth projective curves Γ_i with fully faithful functors $\Psi_i : \mathrm{D}^{\mathrm{b}}(\Gamma_i) \to \mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$ and a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0) = \langle \Psi_1 \mathrm{D}^{\mathrm{b}}(\Gamma_1), \ldots \Psi_k \mathrm{D}^{\mathrm{b}}(\Gamma_k), E_1, \ldots, E_l \rangle; \qquad (1)$$

with E_j exceptional objects in $\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$, then $J(X) \cong \bigoplus J(\Gamma_i)$.

Theorem (-,Bernardara)

If S is minimal, then X is rational and $J(X) \cong \bigoplus J(\Gamma_i)$ if and only if $\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$ decomposes like (1).

From semiorthogonal decomposition to rationality

FIRST PART : Assume that

$$\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0) = \langle \Psi_1 \mathrm{D}^{\mathrm{b}}(\Gamma_1), \ldots \Psi_k \mathrm{D}^{\mathrm{b}}(\Gamma_k), E_1, \ldots, E_l \rangle,$$

then

$$J(X) \cong \bigoplus J(\Gamma_i).$$

From semiorthogonal decomposition to rationality

We suppose that there exists $\Psi : \mathrm{D^b}(\Gamma) \to \mathrm{D^b}(S, \mathcal{B}_0) \subset \mathrm{D^b}(X)$ and study the map that Ψ induces on the motive $h^1(\Gamma)$, for Γ a smooth, projective curve and $g(\Gamma) > 0$.

If $\Psi : \mathrm{D^b}(\Gamma) \to \mathrm{D^b}(S, \mathcal{B}_0)$ is fully faithful, then it is a FM. Ψ admits a right adjoint Ψ_R, which is also FM. In $\mathrm{D^b}(\Gamma \times X)$ we have

$$\mathcal{E} := Ker(\Psi), \quad (1)$$

$$\mathcal{F} := Ker(\Psi_R), \quad (2)$$

$$\Psi \circ \Psi_R = Id_{\mathrm{D^b}(\Gamma)}. \quad (3)$$

Define $e := ch(\mathcal{E}).Td(\Gamma)$ and $f := ch(\mathcal{F}).Td(X)$, mixed cycles in $CH^*_{\mathbb{Q}}(X \times \Gamma)$.

From semiorthogonal decomposition to rationality

$e := \mathrm{ch}([\mathcal{E}]).\mathrm{Td}(\Gamma) \in CH^*_{\mathbb{Q}}(X \times \Gamma)$

$$\begin{array}{ccc} \mathrm{D^b}(\Gamma) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathrm{D^b}(X) \\ \downarrow & & \downarrow \\ CH^*_{\mathbb{Q}}(\Gamma) & \xrightarrow{e} & CH^*_{\mathbb{Q}}(X), \end{array} \tag{4}$$

The vertical arrows are obtained by taking the Chern character. This (plus $\langle E \rangle = \mathrm{D^b}(pt)$) induces the decomposition :

$$CH^*_{\mathbb{Q}}(X) = \bigoplus_{i=1}^{k} CH^*_{\mathbb{Q}}(\Gamma_i) \bigoplus \mathbb{Q}^r.$$

The motive of a projective curve

If Γ is a smooth projective curve

$$h(\Gamma) = h^0(\Gamma) \oplus h^1(\gamma) \oplus h^2(\Gamma),$$

where $h^0(\Gamma) = \mathbb{Q}$, $h^2(\Gamma) = \mathbb{Q}(-1)$ and $h^1(\Gamma)$ corresponds to $J(\Gamma)$ up to isogenies, i.e.

$$Hom(h^1(\Gamma), h^1(\Gamma')) = Hom(J(\Gamma)_{\mathbb{Q}}, J(\Gamma')_{\mathbb{Q}}).$$

No nontrivial map $h^1(\Gamma) \to h^1(\Gamma)$ factors through $\mathbb{Q}(-j)$.

The cycles e and f induce $e_i : h(\Gamma) \to h(X)(i-3)$ and $f_i : h(X)(i-3) \to h(\Gamma)$. By GRR we have $f \cdot e = Id_{h(\Gamma)}$.

From semiorthogonal decomposition to rationality

Proposition (Nagel,Saito)

$\pi : X \to S$ standard conic bundle. there is a submotive $Prym \subset h^1(\tilde{C})$, corresponding to the Prym variety, and $Prym(-1) \subset h^3(X)(-1)$. If S is rational, $h(X)$ is the direct sum of $Prym(-1)$ and a finite number of copies of $\mathbb{Q}(-j)$ (with different twists).

Hence $h^1(\Gamma)$ is a direct summand of $h(X)(-1)$, notably of $Prym(-1)$, and we have an isogeny $\psi_{\mathbb{Q}}$ between $J(\Gamma)$ and a subvariety of $J(X)$.

Now : $f \cdot e = \oplus_i (f_i \cdot e_{4-i})$. By the decomposition of $h(X)$, $(f_i \cdot e_{4-i})_{|h^1(\Gamma)}$ is zero unless $i = 2$. The isogeny $\psi_{\mathbb{Q}}$ is the algebraic morphism $\psi : J(\Gamma) \to J(X)$ given by the cycle $ch_2(\mathcal{E})$.

From semiorthogonal decomposition to rationality

$P(\tilde{C}/C) \cong J(X)$ is the algebraic representative of the algebraically trivial part $A^2(X) \subset CH^2(X)$. The polarization θ_P is the incidence polarization with respect to X. In particular $\psi^*\theta_{J(X)} = \theta_{J(\Gamma)}$ and then ψ is an isomorphism between $J(\Gamma)$ and a principally polarized abelian subvariety of $J(X)$.

Consider the semiorthogonal decomposition of $\mathrm{D^b}(S, \mathcal{B}_0)$.

$$\mathrm{D^b}(X) = \langle \Psi_1 \mathrm{D^b}(\Gamma_1), \ldots \Psi_k \mathrm{D^b}(\Gamma_k), E_1, \ldots, E_r \rangle.$$

Each Ψ_i gives a morphism $\psi_i : J(\Gamma_i) \to J(X)$. Let $\psi = \oplus \psi_i$.

$$CH^*_{\mathbb{Q}}(X) = \bigoplus_{i=1}^{k} CH^*_{\mathbb{Q}}(\Gamma_i) \oplus \mathbb{Q}^r = \bigoplus_{i=1}^{k} Pic_{\mathbb{Q}}(\Gamma_i) \oplus \mathbb{Q}^{r+k}.$$

From semiorthogonal decomposition to rationality

$$\begin{array}{ccccc}
0 & \longrightarrow & \bigoplus_{i=0}^{k} Pic^0_{\mathbb{Q}}(\Gamma_i) & \longrightarrow & \bigoplus_{i=0}^{k} Pic_{\mathbb{Q}}(\Gamma) \oplus \mathbb{Q}^{k+r} \\
 & & \Big\downarrow{\scriptstyle \psi_{\mathbb{Q}}} & & \Big\downarrow \\
0 & \longrightarrow & A^2_{\mathbb{Q}}(X) & \longrightarrow & CH^2_{\mathbb{Q}}(X)
\end{array}$$

The coker of $\psi_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space.
$\psi : \oplus J(\Gamma_i) \to J(X)$ is a morphism of abelian varieties. Hence coker is trivial. Then ψ is an isomorphism. ■

Corollary

If S is minimal and $\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$ admits a decomposition like (1), then X is rational and $J(X) \cong \oplus J(\Gamma_i)$.

From rationality to semiorthogonal decomposition

SECOND PART : Assume X is rational (i.e. $S = \mathbb{P}^2$ and C cubic, quartic, quintic etc...), then

$$\mathrm{D^b}(S, \mathcal{B}_0) = \langle \Psi_1 \mathrm{D^b}(\Gamma_1), \dots \Psi_k \mathrm{D^b}(\Gamma_k), E_1, \dots, E_l \rangle.$$

From rationality to semiorthogonal decomposition

$\pi : X \to S$ a rational, standard CB and S a minimal rational surface.

$$0 \longrightarrow Br(S) \longrightarrow Br(K(S)) \xrightarrow{\alpha} \bigoplus_{D \subset S} H^1_{et}(K(D), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \bigoplus_{x \in S} \mu^{-1},$$

$Br(S)$ is trivial, and $\mathcal{B}_0$ is a 2-torsion point of $Br(K(S))$, hence $\mathcal{B}_0$ and $\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$ are fixed once it is fixed the data of the discriminant curve C and its double cover $\tilde{C} \to C$.

From rationality to semiorthogonal decomposition

Given a degeneration curve and its double cover, we provide an explicit construction as follows :

$$\begin{array}{ccc} & X & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \chi} \\ S & \xleftarrow[\overline{\pi}]{} & Z. \end{array}$$

Z is a smooth projective rational threefold with known semiorthogonal decomposition, $\pi : X \to S$ is induced by an explicit linear system on Z, and χ is the blow up of the smooth curve Γ in the base locus.

From rationality to semiorthogonal decomposition

The decompositions are obtained comparing (via mutations) the decompositions induced respectively by the blow-up and by the conic bundle structure :

$$(A) \qquad \mathrm{D^b}(X) = \langle \Psi \mathrm{D^b}(\Gamma), \chi^* \mathrm{D^b}(Z) \rangle,$$

$$(B) \qquad \mathrm{D^b}(X) = \langle \Phi \mathrm{D^b}(S, \mathcal{B}_0), \pi^* \mathrm{D^b}(S) \rangle.$$

Here is a table summarizing the five different cases

$C \subset S$	Z	Γ	$\mathrm{D^b}(S, \mathcal{B}_0)$
quintic in $\mathbb{P}^2$	$\mathbb{P}^3$	genus 5	$\mathrm{D^b}(\Gamma)$, 1 exc.
quartic in $\mathbb{P}^2$	Quadric	genus 2	$\mathrm{D^b}(\Gamma)$, 1 exc.
cubic in $\mathbb{P}^2$	$\mathbb{P}^1$-bd. over $\mathbb{P}^2$	$\emptyset$	3 exc.
trigonal in $\mathbb{F}_n$	$\mathbb{P}^2$-bd. over $\mathbb{P}^1$	tetragonal	$\mathrm{D^b}(\Gamma)$, 2 exc.
hyperell. in $\mathbb{F}_n$	Quadr. bd. over $\mathbb{P}^1$	hyperell.	$\mathrm{D^b}(\Gamma), \mathrm{D^b}(\Gamma')$

THANK YOU !

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