Deligne–Hodge polynomial of character varieties for genus 1 and 2

joint work with
V. Muñoz and P. Newstead
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Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM)

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Introduction

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- $\mathcal{M}(G) \leftrightarrow G$-local systems on X, hence defining flat bundles on X.
- For deg $\neq 0$ the G-local systems correspond to representations $\pi_1(X \backslash p_0)$, p_0 fixed point and $\rho(\gamma) = -\frac{d}{n}\,\mathrm{Id}$, giving rise to the twisted character varieties

$$\mathcal{M}^d(G) = \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = e^{2pid/n}\,\mathrm{Id}\,\}//G$$

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- For $\{p_1, \dots, p_n\}$ marked points on X and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset G$ fixed conjugacy classes, the corresponding character variety is $\mathcal{PM}^{\mathcal{C}_1,\dots,\mathcal{C}_n}(G) :=$

$$\{(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_n) \in G^{2g+n} \ \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{n} C_j = \mathrm{Id}\,,$$

$$C_j \in \mathcal{C}_j\} // G$$

- $\mathcal{PM}^{\mathcal{C}_1,\dots,\mathcal{C}_n}(G) \simeq \mathcal{PH}^{d+\alpha}(G)$, i.e. homeomorphic to the moduli space of Higgs bundles with parabolic structures at $p_1, \dots, p_n$.

Known results

Poincaré polynomials:

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- $\mathcal{M}^d(G)$ and $\mathcal{M}^d(G^L)$?

$\mathrm{SL}(2,\mathbb{C})$-*Character varieties*

GOAL: give a method to compute the Deligne–Hodge polynomials for the following character varieties.

$$\begin{aligned} M_\xi &:= \{(A_1,B_1,\dots,A_g,B_g)\in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^{g}[A_i,B_i]=\xi\}//\,\mathrm{Stab}(\xi) \\ &= \{(A_1,B_1,\dots,A_g,B_g)\in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^{g}[A_i,B_i]\in C_\xi\}//\,\mathrm{SL}(2,\mathbb{C}) \end{aligned}$$

where ξ is one of the following

$$\mathrm{Id} \qquad -\mathrm{Id} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

and $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where $\lambda \neq 0, \pm 1$.

Mixed Hodge Structures, Deligne-Hodge polynomials.

- *Pure Hodge structure* of weight k: H finite dimensional complex vector space with a real structure, and a decomposition

$$H = \bigoplus_{k=p+q} H^{p,q} \qquad s.t. \qquad H^{q,p} = \overline{H^{p,q}}$$

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- The Hodge filtration: a descending filtration $F^p = \bigoplus_{s \geq p} H^{s,k-s}$. We have that $\mathrm{Gr}_F^p(H) = F^p / F^{p+1} = H^{p,k-p}$.

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- A *mixed Hodge structure*: H, an ascending (weight) filtration

$$\ldots \subset W_{k-1} \subset W_k \subset \ldots \subset H$$

  and a descending (Hodge) filtration F s.t. F induces a pure Hodge structure of weight k on each $\mathrm{Gr}_k^W(H) = W_k/W_{k-1}$.

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- Denote $H^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(H)$, and $h^{p,q} := \dim H^{p,q}$ *Hodge numbers*.

Let Z be any quasi-projective algebraic variety.

- $H^k(Z)$ and $H_c^k(Z)$ have mixed Hodge structures, which are pure of weight k if Z is smooth and projective (Deligne).

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  where $\chi_c^{p,q}(Z) = \sum_k (-1)^k h_c^{k,p,q}(Z)$.
- $h^{k,p,q} = 0$ for $p \neq q$, then $e(Z)(q)$ where $q := uv$ and we call them of *unmixed type.*
- $H_c(Z)(u,v,t) := \sum_{p,q,k} h_c^{p,q,k} u^p v^q t^k$, which encodes all the Hodge numbers, is known as the *mixed Hodge polynomial*

Basic properties of the Deligne–Hodge polynomials

Let Z be an algebraic variety over an algebraically closed field.

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- if $G \to P \to B$ is a principal fiber bundle, and G is a connected complex Lie group, then $e(P) = e(G)e(B)$,
- B is connected and $\pi : Z \to B$ is an algebraic fibre bundle with fibre F (not necessarily locally trivial in the Zariski topology) and that the action of $\pi_1(B)$ on $H_c^*(F)$ is trivial. Then $e(Z) = e(F)e(B)$.

Main idea: use stratifications

- For each g, consider the following $\mathrm{SL}(2,\mathbb{C})$ map,

$$\begin{aligned} f : \mathrm{SL}(2,\mathbb{C})^{2g} &\longrightarrow \mathrm{SL}(2,\mathbb{C}) \\ (A,B) &\mapsto \prod_{i=1}^{g} [A,B] = ABA^{-1}B^{-1} \end{aligned}$$

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- $e(\mathrm{SL}(2,\mathbb{C})) = e(\mathbb{C}^2 \setminus \{0\})e(\mathbb{C}) = q(q^2-1) = q^3 - q$,

$$\mathbb{C} \longrightarrow \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathbb{C}^2 \setminus \{0\},$$

since for given $(a,b) \neq (0,0)$, the equation $ad - bc = 1$ defines a line in the plane $\mathbb{C}^2$ with coordinates (c,d).

We decompose $\mathrm{SL}(2,\mathbb{C})$ into its orbits to get pieces whose Deligne–Hodge polynomials we are able to compute. (Notation use ξ instead of $\mathcal{C}_\xi$)

- Id := orbit through $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- $-\mathrm{Id}$:= orbit through $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
- J_+ := orbit through $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- J_- := orbit through $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.
- ξ_λ :=orbits through $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $\lambda \in \mathbb{C}\setminus\{0, \pm 1\}$.

We get,

$$e(\mathrm{Id}) + e(-\mathrm{Id}) + e(J_+) + e(J_-) + e(\xi_\lambda) = e(\mathrm{SL}(2,\mathbb{C})).$$

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Method for $e(Z/\mathbb{Z}_2)$

Denote by $H_c^*(Z)^{\pm}$ the $\pm$-invariant part of $H_c^*(Z)$, with a corresponding notation

$$e(Z)^{\pm} := e(H_c^*(Z)^{\pm}).$$

Note that

$$\begin{aligned} e(Z)^{+} &= e(Z/\mathbb{Z}_2), \\ e(Z)^{-} &= e(Z) - e(Z)^{+}. \end{aligned}$$

Example, if $Z = \mathbb{C}^*$ with the action $\lambda \mapsto \lambda^{-1}$, then $Z/\mathbb{Z}_2 \cong \mathbb{C}$, and

$$e(\mathbb{C}^*)^{+} = q, \quad e(\mathbb{C}^*)^{-} = -1. \tag{1}$$

Let

$$\begin{array}{ccc} F & \xrightarrow{=} & F \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z/\mathbb{Z}_2 \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{2:1} & B/\mathbb{Z}_2 \end{array}$$

be a diagram of fibrations, where B is irreducible, $\widetilde{\pi}$ and π are smooth morphisms, $\widetilde{\pi}$ is a locally trivial fibration in the usual topology and the monodromy action of $\pi_1(B)$ on $H_c^*(F)$ is trivial. Then

$$e(Z/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^-, \tag{2}$$

where $e(F)^\pm$ are defined by an action of $\mathbb{Z}_2$ on $H_c^*(F)$, which is compatible with the Hodge structure.

The stabiliser of ξ_λ is $\mathbb{C}^*$. Therefore we have a diagram

$$\begin{array}{ccc}
\mathrm{PGL}(2,\mathbb{C})/\mathbb{C}^* & \xrightarrow{=} & \mathrm{PGL}(2,\mathbb{C})/\mathbb{C}^* \\
\downarrow & & \downarrow \\
\widetilde{\xi_\lambda} & \longrightarrow & \xi_\lambda \\
\downarrow & & \downarrow \\
\mathbb{C}\setminus\{0,\pm 1\} & \xrightarrow{2:1} & \mathbb{C}\setminus\{\pm 2\}.
\end{array}$$

where $\widetilde{\xi_\lambda} := \{(\lambda, P) | \lambda \in \mathbb{C}\setminus\{0,\pm 1\}, P \in \mathrm{PGL}(2,\mathbb{C})/\mathbb{C}^*\}$.

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$$e(\xi_\lambda) = e(F)^+e(B)^+ + e(F)^-e(B)^- = q^2(q-2) - q = q^3 - 2q^2 - q.$$

- $e(\mathrm{Id}\,) = e(-\mathrm{Id}\,) = e(\text{point}) = 1$
- $e(J_+) = e(J_-) = e(\mathrm{PGL}(2,\mathbb{C})/\mathbb{C}) = e(\mathbb{C}^*)e(\mathbb{P}^1) = q^2 - 1$
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$$e(\mathrm{Id}) + e(-\mathrm{Id}) + e(J_+) + e(J_-) = q^3 - q$$

Hodge polynomial of $\mathrm{SL}(2,\mathbb{C})^2$ *using orbit spaces*

Let

$$\begin{array}{rcl} f:\mathrm{SL}(2,\mathbb{C})^2 & \longrightarrow & \mathrm{SL}(2,\mathbb{C}) \\ (A,B) & \mapsto & [A,B]=ABA^{-1}B^{-1} \end{array}$$

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We stratify $X = \mathrm{SL}(2,\mathbb{C})^2$ as follows

- $X_0 = f^{-1}(\mathrm{Id})$,
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Hence $X = \sqcup_{i=0}^4 X_i$, so again we will test that

$$e(X) = e(\mathrm{SL}(2,\mathbb{C}))^2 = \sum_i e(X_i) = q^2(q+1)^2(q-1)^2 = q^6 - 2q^4 + q^2.$$

Genus 1

Theorem

Let X be a complex curve of genus $g = 1$. Then

$$
\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}) &= q^2 + 1\,, \\
e(\mathcal{M}_{-\mathrm{Id}}) &= 1\,, \\
e(\mathcal{M}_{J_+}) &= q^2 - 2q - 3\,, \\
e(\mathcal{M}_{J_-}) &= q^2 + 3q\,, \\
e(\mathcal{M}_{\xi_\lambda}) &= q^2 + 4q + 1\,.
\end{aligned}
$$

Genus 2

Theorem

Let X be a complex curve of genus $g = 2$. Then

$$\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}) &= q^6 + 17q^4 - 26q^3 + 67q^2 + 26q - 65\,. \\
e(\mathcal{M}_{-\mathrm{Id}}) &= q^6 - 2q^4 - 30q^3 - 2q^2 + 1\,, \\
e(\mathcal{M}_{J_+}) &= q^8 - 3q^6 - 4q^5 - 39q^4 - 4q^3 - 15q^2\,, \\
e(\mathcal{M}_{J_-}) &= q^8 - 3q^6 + 5q^5 - 30q^4 + 37q^3 + 36q^2 + 18q\,, \\
e(\mathcal{M}_{\xi_\lambda}) &= q^8 + q^7 - 2q^6 + 13q^5 - 26q^4 + 13q^3 - 2q^2 + q + 1\,.
\end{aligned}$$

Poincaré polynomials

Note that for Z smooth

$$P_t(Z) = \sum_k b^k(Z)t^k = \sum_k (\sum_{p+q} h_c^{p,q}(Z))t^k = e(Z)(t,t)$$

but the information obtained before is not enough to get all $h_c^{k,p,q}$. However in some cases we have a explicit description of the variety so, for instance,

- $h_c^{4,2,2}(\mathcal{M}_{J_+} = 1$, $h_c^{3,1,1}(\mathcal{M}_{J_+} = 2$,$h_c^{2,0,0}(\mathcal{M}_{J_+}) = 1$, $h_c^{1,0,0}(\mathcal{M}_{J_+}) = 4$
  $P_t^c(\mathcal{M}_{J+}) = t^4 + 2t^3 + t^2 + 4t$, $P_t(\mathcal{M}_{J+}) = 4t^3 + t^2 + 2t + 1$
- the explicit description give us the Betti numbers so
  $P_t^c(\mathcal{M}_{J\_}) = t^4 + t^3 + 5t^2 + t$, $P_t(\mathcal{M}_{J\_}) = t^3 + 5t^2 + t + 1$
  but this is not enough to get the Hodge numbers, they are either

$$h_c^{3,1,1}(\mathcal{M}_{J\_}) = 1, \quad h_c^{2,1,1}(\mathcal{M}_{J\_}) = 4, \quad h_c^{2,0,0}(\mathcal{M}_{J\_}) = 1$$

or

$$h_c^{3,0,0}(\mathcal{M}_{J\_}) = 1, \quad h_c^{2,1,1}(\mathcal{M}_{J\_}) = 3, \quad h_c^{2,0,0}(\mathcal{M}_{J\_}) = 2.$$