On exact optimal sampling designs for processes with a product covariance structure

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Outline of the talk

- The model with a one-dimensional parameter
- Product covariance functions
- Exact regular optimal designs
- The information about the parameter
- The Wiener equivalence theorem
- Optimal designs for the case of the Wiener covariance function
- Optimal designs for the parabolic, square-root, and hyperbolic classes of regression functions
- Nested optimal designs
- Ultimate efficiency
- Notes on models with a multidimensional parameter

The model and basic assumptions

Assume a random process $(Y(t) : t \in (A, B))$, $-\infty \leq A < B \leq +\infty$ defined by

$$Y(t) = \beta f(t) + \epsilon(t).$$

- $\beta \in \mathbb{R}$ is an unknown parameter, which is to be estimated from observations of the process in a finite number of "times" t chosen in an optimal way within a given experimental domain.
- $f : (A, B) \to \mathbb{R}$ is a known continuously differentiable "regression" function with at most two roots on (A, B).
- $(\epsilon(t) : t \in (A, B))$ is a Gaussian random process of errors with the zero mean value function and the covariance function $R(s, t)$, $A < s, t < B$.

We will assume that $R(s, t)$ is the so-called product covariance function, i.e., there exist functions $u, v : (A, B) \to \mathbb{R}$ such that

$$R(s, t) = u(s)v(t) \text{ for all } A < s \leq t < B.$$

This model will be denoted by (f, u, v).

The product covariance functions

The product covariance functions have been used in the context of optimal design problems in, e.g:

- Sacks J, Ylvisaker D (1966): *Designs for Regression Problems with Correlated Errors*, The Annals of Mathematical Statistics 37, pp. 66-89
- Mukherjee B (2003): *Exactly Optimal Sampling Designs for Processes with a Product Covariance Structure*, The Canadian Journal of Statistics / La Revue Canadienne de Statistique 31, pp. 69-87.
- Harman R, Štulajter F (2009): *Optimality of equidistant sampling designs for a nonstationary Ornstein-Uhlenbeck process*, In: Proceedings of the 6th St.Petersburg Workshop on Simulation, S.M.Ermakov, V.S.Melas, A.N.Pepelyshev (eds.), St. Petersburg 2009, pp. 1097-1101

The product covariance functions

Assumptions:

- The functions u, v are positive on (A, B).
- The function $r = u/v$ is increasing on (A, B).
- $\lim_{s \to A_+} r(s) = 0$, $\lim_{t \to B_-} r(t) = +\infty$.

Lemma

Let Σ be a symmetric matrix of type $n \times n$ with entries $\Sigma_{ij} = u_i v_j$, for $i \leq j$, $u_i > 0, v_j > 0$, and let $r_i = u_i/v_i$ satisfy $r_1 < r_2 < \cdots < r_n$. Then

$$\det(\Sigma) = r_1 \prod_{i=1}^{n-1} [r_{i+1} - r_i] \prod_{i=1}^{n} v_i^2 > 0.$$

The lemma implies that if u, v are positive on (A, B), then $R(s, t) = u(s)v(t)$ for $A < s \leq t < B$ is positive semidefinite, i.e., it is a covariance function, if $r = u/v$ is nondecreasing on (A, B).

The correlation function

Note that

$$\mathrm{cor}(Y(s), Y(t)) = \sqrt{\frac{r(s)}{r(t)}};\ \text{for } A < s \leq t < B.$$

Our assumptions imply that

- For a fixed $t \in (A, B)$, the correlation $\mathrm{cor}(Y(s), Y(t))$ is increasing as a function of $s \in (A, t)$ with

$$\lim_{s \to t_-} \mathrm{cor}(Y(s), Y(t)) = 1, \quad \lim_{s \to A_+} \mathrm{cor}(Y(s), Y(t)) = 0.$$

- For a fixed $s \in (A, B)$, the correlation $\mathrm{cor}(Y(s), Y(t))$ is decreasing as a function of $t \in (s, B)$ with

$$\lim_{t \to s_+} \mathrm{cor}(Y(s), Y(t)) = 1, \quad \lim_{t \to B_-} \mathrm{cor}(Y(s), Y(t)) = 0$$

Examples of product covariance functions

There is an infinite number of product covariance functions, one for each couple of functions u, v positive on (A, B) such that $r = u/v$ is increasing on (A, B). Some of them correspond to well-know processes:

- Wiener process ($\sigma^2 > 0$):

$$\begin{aligned} R_W(s,t) &= \sigma^2 s; \;\; s \le t; s,t \in (A,B) = (0,\infty), \\ u_W(s) &= \sigma^2 s, \; v_W(t) \equiv 1, \; r_W(t) = \sigma^2 s. \end{aligned}$$

- Brownian Bridge ($\sigma^2 > 0$):

$$\begin{aligned} R_{BB}(s,t) &= \sigma^2 s(1-t); s \le t; s,t \in (A,B) = (0,1), \\ u_{BB}(s) &= \sigma^2 s, \; v_{BB}(t) = 1-t, \; r_{BB}(t) = \sigma^2 t/(1-t). \end{aligned}$$

- Ornstein-Uhlenbeck ($\sigma^2 > 0, \lambda > 0$):

$$\begin{aligned} R_{OU}(s,t) &= \sigma^2 \exp(-\lambda(t-s)); \;\; s \le t; s,t \in (A,B) \in (-\infty,\infty), \\ u_{OU}(s) &= \sigma^2 \exp(\lambda s), \; v_{OU}(t) = \exp(-\lambda t), \; r_{OU}(t) = \sigma^2 \exp(2\lambda t). \end{aligned}$$

Examples of product covariance functions

- Covariance function of a nonautonomous Ornstein-Uhlenbeck process defined by

$$\mathrm{d}X(t) = \kappa(\mu - X(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

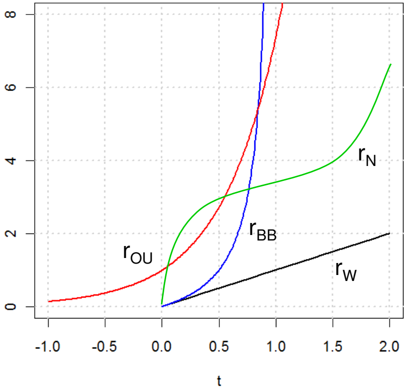
$\sigma^2 : (0, \infty) \to (0, \infty)$, $\kappa > 0$, $\mu \in \mathbb{R}$:

$$\begin{aligned} R_N(s,t) &= \exp(-\kappa(s+t)) \int_0^s \exp(2\kappa x)\sigma^2(x)\mathrm{d}x, \; 0 < s \le t < \infty \\ u_N(s) &= \exp(-\kappa s) \int_0^s \exp(2\kappa x)\sigma^2(x)\mathrm{d}x, \; v_N(t) = \exp(-\kappa t), \\ r_N(t) &= \int_0^t \exp(2\kappa x)\sigma^2(x)\mathrm{d}x. \end{aligned}$$

For details see the poster: Lacko V: *Planning of experiments for a nonautonomous Ornstein-Uhlenbeck process*.

The function $r = u/v$

Note that $r : (A, B) \to (0, \infty)$ is a bijection, i.e., there is a well defined inverse function $r^{-1} : (0, \infty) \to (A, B)$.



Regular sampling designs

Assume that we can observe the process $(Y(t) : t \in (A, B))$ in $n \geq 3$ design points, or sampling times, chosen in the experimental domain $[T_*, T^*] \subset (A, B)$.

A "regular" n-point sampling design on $[T_*, T^*]$ is any vector $\tau = (t_1, ..., t_n)'$, such that

$$T_* = t_1 < t_2 < \cdots < t_{n-1} < t_n = T^*.$$

The set of n-point regular designs on $[T_*, T^*]$ will be denoted by $\mathfrak{T}_n$.

- $\mathfrak{T}_n$ is *not* compact (unless $n = 2$). That is, even if our criterion that measures the "quality" of designs is continuous as a function of design points, an "optimal" regular design may not exist.
- It turns out that for the design problems with a product covariance function, it is simpler to first find a method of constructing optimal designs with fixed initial and final design points, and then find the optimal position of the initial and the final design points.

Optimal regular sampling designs

Under the design $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$, the n-dimensional random vector $\boldsymbol{Y}(\tau) = (Y(t_1), ..., Y(t_n))'$ of observations satisfies the linear regression model with correlated errors

$$\boldsymbol{Y}(\tau) = \beta \boldsymbol{f}(\tau) + \epsilon(\tau),$$

where $\boldsymbol{f}(\tau) = (f(t_1), ..., f(t_n))'$, and $\epsilon(\tau) = (\epsilon(t_1), ..., \epsilon(t_n))'$ has zero mean value and a regular covariance matrix $\Sigma(\tau)$.

Under the design $\tau \in \mathfrak{T}_n$, the weighted least squares estimator of β is

$$\begin{aligned} \hat{\beta}(\tau) &= M^{-1}(\tau)\boldsymbol{f}'(\tau)\Sigma^{-1}(\tau)\boldsymbol{Y}(\tau), \text{ where} \\ M(\tau) &= \boldsymbol{f}'(\tau)\Sigma^{-1}(\tau)\boldsymbol{f}(\tau) = \mathrm{Var}^{-1}(\hat{\beta}(\tau)) \end{aligned}$$

is the “information” about β (the 1×1 information matrix).

A design $\tau^* \in \mathfrak{T}_n$ is an optimal n-point regular design for estimating β if

$$M(\tau^*) \geq M(\tau) \text{ for all } \tau \in \mathfrak{T}_n.$$

A simple formula for $M(\tau)$

It turns out that the assumption of the product covariance function significantly simplifies calculation of the bilinear forms defined by $\Sigma^{-1}(\tau)$ and, consequently, calculation of the information about β.

Lemma

Assume the model (f, u, v) *on the experimental domain* $[T_*, T^*] \subset (A, B)$*. Let* $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$*. Then,*

$$M(\tau) = \frac{f^2(t_1)}{u(t_1)v(t_1)} + \sum_{i=2}^{n} \frac{\left(\frac{f(t_i)}{v(t_i)} - \frac{f(t_{i-1})}{v(t_{i-1})}\right)^2}{\frac{u(t_i)}{v(t_i)} - \frac{u(t_{i-1})}{v(t_{i-1})}}.$$

- The position of t_i contributes to the information about β only via its relation with the neighboring design points t_{i-1} (if $i \geq 2$) and t_{i+1} (if $i \leq n-1$).
- If the covariance function has the product form, the information is, in a sense, "additive", resembling the situation in the uncorrelated models.

The Wiener equivalence theorem

Theorem

Let (f, u, v) *be a model and let* $r = \frac{u}{v} : (A, B) \to (0, \infty)$ *satisfy the assumptions from the introduction.*
Let $\tau^* = (t_1, ..., t_n)'$ *be a regular design on* $[T_*, T^*]$ *and let* $\tilde{\tau}^* = (\tilde{t}_1, ..., \tilde{t}_n)'$ *be a regular design on* $[\tilde{T}_*, \tilde{T}^*]$, *where*

$$\tilde{t}_i = r(t_i) \text{ for } i = 1, ..., n, \tilde{T}_* = r(T_*), \tilde{T}^* = r(T^*).$$

Then the following two statements are equivalent:

1. *The design* τ^* *is an optimal n-point design for* (f, u, v) *on* $[T_*, T^*]$.
2. *The design* $\tilde{\tau}^*$ *is optimal n-point design for* $(\tilde{f}, u_W, v_W)$ *on* $[\tilde{T}_*, \tilde{T}^*]$, *where*

$$\tilde{f}(\cdot) = \frac{f(r^{-1}(\cdot))}{v(r^{-1}(\cdot))}, \; u_W(s) = s, v_W(t) \equiv 1 \text{ for all } s, t \geq 0.$$

The Wiener equivalence theorem

Consequences:

- Any design problem with the product covariance function is equivalent to a design problem with the Wiener covariance function.
- Hence, we can focus on the design problems with the Wiener covariance function, which can be taken as “canonical” for all product covariance functions.

If we have the model (f, u_W, v_W), and if $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$ then the bilinear forms defined by $\Sigma^{-1}(\tau)$ and the information about β have the following simple forms:

$$\boldsymbol{x}'\Sigma^{-1}(\tau)\boldsymbol{y} = \frac{x_1 y_1}{t_1} + \sum_{i=2}^{n} \frac{(x_i - x_{i-1})(y_i - y_{i-1})}{t_i - t_{i-1}}.$$

$$M(\tau) = \frac{f^2(t_1)}{t_1} + \sum_{i=2}^{n} \frac{(f(t_i) - f(t_{i-1}))^2}{t_i - t_{i-1}}.$$

The model with the Wiener covariance function

Assume a design $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$. What happens if we add a design point t_+ between the points t_k and t_{k+1}?

Lemma

Let $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$, $\tau_+ = (t_1, ..., t_k, t_+, t_{k+1}, ..., t_n)' \in \mathfrak{T}_{n+1}$, *where* $t_+ \in (t_k, t_{k+1})$. *Then*

$$M(\tau_+) - M(\tau) = \frac{\left[\det \begin{pmatrix} t_+ - t_k & f(t_+) - f(t_k) \\ t_+ - t_{k+1} & f(t_+) - f(t_{k+1}) \end{pmatrix}\right]^2}{(t_+ - t_k)(t_{k+1} - t_+)(t_{k+1} - t_k)}.$$

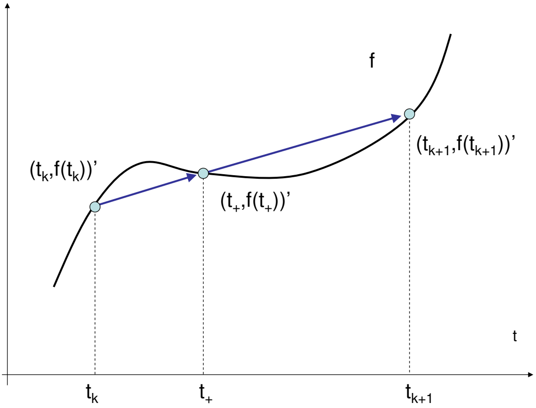
Consequences:

- $M(\tau_+) \geq M(\tau)$, i.e., adding a point can not decrease the information about β.
- $\lim_{t_+ \to_+ t_k} M(\tau_+) = M(\tau)$ and $\lim_{t_+ \to_- t_{k+1}} M(\tau_+) = M(\tau)$, i.e., the increase in information diminishes as t_+ approaches t_k or t_{k+1}.

The model with the Wiener covariance function

When

$$M(\tau_+) - M(\tau) = \frac{\left[\det\begin{pmatrix} t_+ - t_k & f(t_+) - f(t_k) \\ t_+ - t_{k+1} & f(t_+) - f(t_{k+1}) \end{pmatrix}\right]^2}{(t_+ - t_k)(t_{k+1} - t_+)(t_{k+1} - t_k)} = 0 \ ?$$



The model with the Wiener covariance function

- If the points $(t_k, f(t_k))'$, $(t_+, f(t_+))'$ and $(t_{k+1}, f(t_{k+1}))'$ do not lie on a common line, adding observation in t_+ strictly increases the information about β compared to the information from observations $Y(t_1), ..., Y(t_n)$. (Holds for the model (f, u_W, v_W).)

Theorem

Assume the model (f, u_W, v_W) on the experimental domain $[T_*, T^*] \subset (0, \infty)$. Then there exists an optimal n-point regular design τ_n^*. If f is not linear on any interval $(a, b) \subset [T_*, T^*]$, then $M(\tau_n^*)$ is an increasing function of n.

Using the Wiener equivalence theorem we obtain

Theorem

Assume the model (f, u, v) on the experimental domain $[T_*, T^*] \subset (A, B)$. Then there exists an optimal n-point regular design τ^*. Let $r = u/v$. If $f(r^{-1}(\cdot))/v(r^{-1}(\cdot))$ is not linear on any interval $(a, b) \subset [T_*, T^*]$, then $M(\tau_n^*)$ is an increasing function of n.

The parabolic class of regression functions

Theorem

Let

$$f(t) = a_0 + a_1 t + a_2 t^2,$$

where $a_0, a_1, a_2 \in \mathbb{R}$ *are given constants, at least one of them nonzero. Then the optimal n-point regular design for the model* (f, u_W, v_W) *on* $[T_*, T^*] \subset (0, \infty)$ *is* $\tau_n^* = (t_1^*, ..., t_n^*)'$*, where*

$$t_i^* = \frac{n-i}{n-1} T_* + \frac{i-1}{n-1} T^*; \; i = 1, ..., n.$$

That is, the optimal design points form an arithmetic progression.

As a special case $a_0 = a_1 = 0$ we obtain a theorem in:
Harman R, Štulajter F (2011): *Optimal sampling designs for the Brownian motion with a quadratic drift*, Journal of Statistical Planning and Inference, Volume 141, Issue 8, pp. 2750–2758

The square-root class of regression functions

Theorem

Let

$$f(t) = b_0 + b_1 t + b_2 \sqrt{t},$$

where $b_0, b_1, b_2 \in \mathbb{R}$ *are given constants, at least one of them nonzero. Then the optimal n-point regular design for the model* (f, u_W, v_W) *on* $[T_*, T^*] \subset (0, \infty)$ *is* $\tau^* = (t_1^*, ..., t_n^*)'$*, where*

$$t_i^* = (T_*)^{\frac{n-i}{n-1}} (T^*)^{\frac{i-1}{n-1}};\ i = 1, ..., n.$$

That is, the optimal design points form a geometric progression.

Using the Wiener equivalence theorem we can get optimal designs for other models with product covariance functions. For instance, since $r_{OU}(t) = \sigma^2 \exp(2\lambda t)$ transforms an arithmetic progression of design points on $(-\infty, \infty)$ onto a geometric progression of design points on $(0, \infty)$ we obtain:

The square-root class of regression functions

Corollary

Let

$$f(t) = b_0 exp(-\lambda t) + b_1 \exp(\lambda t) + b_2,$$

where $\lambda > 0$*,* $b_0, b_1, b_2 \in \mathbb{R}$ *are given constants, at least one of them nonzero. Then the optimal n-point regular design for the model* (f, u_{OU}, v_{OU}) *on* $[T_*, T^*] \subset (-\infty, \infty)$ *is* $\tau^* = (t_1^*, ..., t_n^*)'$*, where*

$$t_i^* = \frac{n-i}{n-1} T_* + \frac{i-1}{n-1} T^*; \; i = 1, ..., n.$$

That is, the optimal design points form an arithmetic progression.

A special case $b_0 = b_1 = 0$ of the theorem appeared in, e.g. Kiseľák J, Stehlík M (2008) *Equidistant and D-optimal designs for parameters of Ornstein - Uhlenbeck process*, Statistics and Probability Letters 78, pp. 1388-1396.

The hyperbolic class of regression functions

Theorem

Let

$$f(t) = c_0 + c_1 t + \frac{c_2}{t},$$

where c_0, c_1, c_2 *are given constants, at least one of them nonzero. Then the optimal n-point regular design for the model* (f, u_W, v_W) *on* $[T_*, T^*] \subset (0, \infty)$ *is* $\tau^* = (t_1^*, ..., t_n^*)'$*, where*

$$t_i^* = \frac{n-1}{\frac{n-i}{T_*} + \frac{i-1}{T^*}}; \; i = 1, ..., n.$$

That is, the optimal design points form a harmonic progression.

Since $r_{BB}(t) = \sigma^2 \frac{t}{1-t}$ transforms a harmonic progression of design points on $(0, 1)$ onto a harmonic progression of design points on $(0, \infty)$ we obtain:

The hyperbolic class of regression functions

Corollary

Let

$$f(t) = d_0 + d_1 t + \frac{d_2}{t},$$

where d_0, d_1, d_2 *are given constants, at least one of them nonzero. Then the optimal design for the model* (f, u_{BB}, v_{BB}) *on* $[T_*, T^*] \subset (0, 1)$ *is* $\tau^* = (t_1^*, ..., t_n^*)'$*, where*

$$t_i^* = \frac{n-1}{\frac{n-i}{T_*} + \frac{i-1}{T^*}};\ i = 1, ..., n.$$

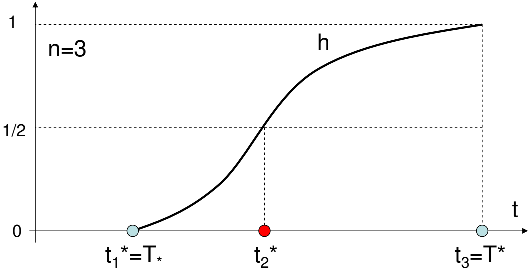
That is, the optimal design points form a harmonic progression.

Nested optimal designs

Assume that there exists a fixed bijection $h : [T_*, T^*] \to [0, 1]$ such that for any $n \geq 2$ the optimal n-point design has the following form:

$$\tau^* = \left(h^{-1}(0), h^{-1}\left(\frac{1}{n-1}\right), h^{-1}\left(\frac{2}{n-1}\right), \dots, h^{-1}(1) \right)',$$

$h^{-1}(0) = T_*$, $h^{-1}(1) = T^*$. Let us call such optimal designs “nested”.

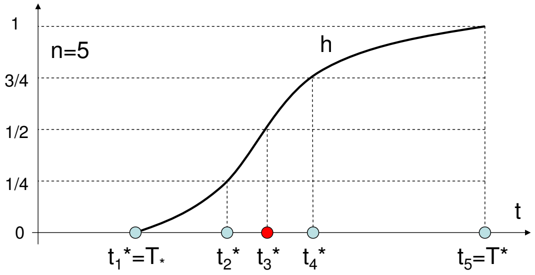


Nested optimal designs

Assume that there exists a fixed bijection $h : [T_*, T^*] \to [0, 1]$ such that for any $n \geq 2$ the optimal n-point design has the following form:

$$\tau^* = \left(h^{-1}(0), h^{-1}\left(\frac{1}{n-1}\right), h^{-1}\left(\frac{2}{n-1}\right), ..., h^{-1}(1)\right)',$$

$h^{-1}(0) = T_*$, $h^{-1}(1) = T^*$. Let us call such optimal designs "nested".

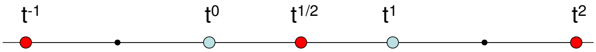


Conjecture about nested optimal designs

We can search for the candidates for regression functions *f* that lead to the nested optimal designs in a systematic way by numerically calculating the 3-point and the 5-point optimal designs and comparing the “middle” points. If the middle points do not coincide, then *f* does not lead to a nested class of optimal designs.

Conjecture

For the model (f, u_W, v_W) *on* $[T_*, T^*] \subset (0, \infty)$ *there exist only three classes of regression functions f that lead to the nested optimal regular designs, namely the parabolic class, the square-root class and the hyperbolic class. The corresponding optimal design points form the arithmetic, the geometric and the harmonic progressions.*



Ultimate efficiency

For a model (f, u_W, v_W) on $[T_*, T^*]$ let τ_n^* be the n-point optimal regular design. Let the "ultimate" efficiency of a regular design τ on $[T_*, T^*]$ (with any number of design points) be defined by

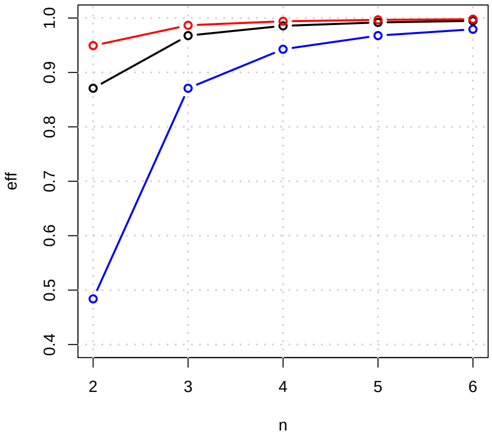
$$\mathrm{eff}_f(\tau) = \frac{M(\tau)}{\lim_{n\to\infty} M(\tau_n^*)}.$$

- The value $\lim_{n\to\infty} M(\tau_n^*)$ is a maximum possible information about β that can be gained from an observation of the process $(Y(t); t \in [T_*, T^*])$.
- It is typical for the models with correlated observations that the maximum possible information is finite. That is, even knowing the full trajectory of the process, we can not estimate β with zero variance.
- It turns out that for all three classes of regression functions, the ultimate efficiency $\mathrm{eff}_f(\tau_n^*)$ of the n-point optimal regular design tends to be very high, even for a small number n of observations.

Ultimate efficiency

Example: Ultimate efficiency of the n-point optimal regular designs in models (f, u_W, v_W) on $[T_*, T^*] = [1, 5]$ for selected functions f:

$$f_p(t) = t^2 - 1, \; f_s(t) = \sqrt{t} - 1, \; f_h(t) = \frac{1}{t} - 1$$



The model with a multidimensional parameter

Assume a random process $(Y(t) : t \in (A, B))$, $-\infty \leq A < B \leq +\infty$ defined by

$$Y(t) = \beta' \boldsymbol{f}(t) + \epsilon(t).$$

- $\beta \in \mathbb{R}$ is an unknown parameter, which is to be estimated from observations of the process in a finite number of times t chosen in an optimal way within a given experimental domain.
- $\boldsymbol{f} = (f_1, ..., f_m)' : (A, B) \to \mathbb{R}^m$ is a vector of known continuously differentiable regression functions.
- $(\epsilon(t) : t \in (A, B))$ is a Gaussian random process of errors with the zero mean value function and the covariance function $R(s, t)$, $A < s, t < B$.

We will again assume that $R(s, t)$ has the product form:

$$R(s, t) = u(s)v(t) \text{ for all } s, t \text{ such that } A < s \leq t < B.$$

This model will be denoted by $(\boldsymbol{f}, u, v)$.

The regression model

Under the design $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$, the n-dimensional random vector $\boldsymbol{Y}(\tau) = (Y(t_1), ..., Y(t_n))'$ of observations satisfies the linear regression model with correlated errors

$$\boldsymbol{Y}(\tau) = \boldsymbol{F}(\tau)\beta + \epsilon(\tau),$$

where $\boldsymbol{F}(\tau) = (\boldsymbol{f}_1(\tau), ..., \boldsymbol{f}_m(\tau))'$ is the matrix of plan, $\boldsymbol{f}_i(\tau) = (f_i(t_1), ..., f_i(t_n))'$ for $i = 1, ..., m$, $\epsilon(\tau) = (\epsilon(t_1), ..., \epsilon(t_n))'$ has zero mean value and a regular covariance matrix $\Sigma(\tau)$.

If $\tau \in \mathfrak{T}_n$ such that $\boldsymbol{F}(\tau)$ has full rank, the weighted least squares estimator of β is

$$\begin{aligned} \hat{\beta}(\tau) &= \boldsymbol{M}^{-1}(\tau)\boldsymbol{F}'(\tau)\Sigma^{-1}(\tau)\boldsymbol{Y}(\tau), \text{ where} \\ \boldsymbol{M}(\tau) &= \boldsymbol{F}'(\tau)\Sigma^{-1}(\tau)\boldsymbol{F}(\tau) \end{aligned}$$

is the $m \times m$ information matrix.
A design $\tau^* \in \mathfrak{T}_n$ is an n-point Φ-optimal design for estimating β if

$$\Phi(\boldsymbol{M}(\tau^*)) \geq \Phi(\boldsymbol{M}(\tau)) \text{ for all } \tau \in \mathfrak{T}_n,$$

where Φ is an optimality criterion (D-, A-, E-...).

A simple formula for $\boldsymbol{M}(\tau)$

Similarly as in the case of a one-dimensional parameter, the assumption of a product covariance function leads to a relatively simple form of the information matrix.

Lemma

Assume the model $((f_1, ..., f_m)', u, v)$ *on the experimental domain* $[T_*, T^*] \subset (A, B)$. *Let* $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$. *Then*

$$
\begin{aligned}
\boldsymbol{M}(\tau) &= \boldsymbol{M}_1(t_1) + \sum_{i=2}^{n} \boldsymbol{M}(t_{i-1}, t_i), \text{ where} \\
(\boldsymbol{M}_1(t_1))_{j,k} &= \frac{f_j(t_1) f_k(t_1)}{u(t_1) v(t_1)}, \\
(\boldsymbol{M}(t_{i-1}, t_i))_{j,k} &= \frac{\left(\frac{f_j(t_i)}{v(t_i)} - \frac{f_j(t_{i-1})}{v(t_{i-1})}\right)\left(\frac{f_k(t_i)}{v(t_i)} - \frac{f_k(t_{i-1})}{v(t_{i-1})}\right)}{\frac{u(t_i)}{v(t_i)} - \frac{u(t_{i-1})}{v(t_{i-1})}}, \text{ for } 1 \leq j, k \leq m.
\end{aligned}
$$

The Wiener equivalence theorem

Theorem

Let $(\boldsymbol{f}, u, v)$ *be a model and let* $r = \frac{u}{v} : (A, B) \to (0, \infty)$ *satisfy the assumptions from the introduction.*
Let $\tau^* = (t_1, ..., t_n)'$ *be a regular design on* $[T_*, T^*]$ *and let* $\tilde{\tau}^* = (\tilde{t}_1, ..., \tilde{t}_n)'$ *be a regular design on* $[\tilde{T}_*, \tilde{T}^*]$*, where*

$$\tilde{t}_i = r(t_i) \text{ for } i = 1, ..., n, \tilde{T}_* = r(T_*), \tilde{T}^* = r(T^*).$$

Then the following two statements are equivalent:

1. *The design* τ^* *is a* Φ*-optimal n-point design for* $(\boldsymbol{f}, u, v)$ *on* $[T_*, T^*]$*.*
2. *The design* $\tilde{\tau}^*$ *is a* Φ*-optimal n-point design for* $(\tilde{\boldsymbol{f}}, u_W, v_W)$ *on* $[\tilde{T}_*, \tilde{T}^*]$*, where*

$$\tilde{\boldsymbol{f}}(\cdot) = \frac{\boldsymbol{f}(r^{-1}(\cdot))}{v(r^{-1}(\cdot))},\ u_W(s) = s, v_W(t) \equiv 1 \text{ for all } s, t \geq 0.$$

Example: Quadratic regression with a Brownian drift

Assume the quadratic regression model with a Brownian drift, i.e., the model $(\boldsymbol{f}, u_W, v_W)$, where $\boldsymbol{f}(t) = (1, t, t^2)'$.

Let $\tau = (t_1, ..., t_n)'$ be a regular design on $[T_*, T^*]$. Then

$$\boldsymbol{M}(\tau) = \frac{1}{\sigma^2} \begin{pmatrix} t_1^{-1} & 1 & t_1 \\ 1 & t_n & t_n^2 \\ t_1 & t_n^2 & t_n^3 + \alpha_n(\tau) \end{pmatrix}, \; \alpha_n(\tau) = \sum_{i=1}^{n-1} t_{i+1} t_i (t_{i+1} - t_i).$$

Lemma

For any design $\tau = (t_1, ..., t_n)' \in \mathfrak{T}_n$ we have

$$\alpha_n(\tau) \leq \frac{1}{3} \left(t_n^3 - t_1^3 - \frac{(t_n - t_1)^3}{(n-1)^2} \right),$$

and the equality is attained if and only if the design points of τ form an arithmetic progression.

Example: Quadratic regression with a Brownian drift

Theorem

Let $\tau_n^* = (t_1^*, ..., t_n^*)' \in \mathfrak{T}_n$ *such that* $t_1^*, ..., t_n^*$ *form an arithmetic progression. Then the information matrix* $\boldsymbol{M}(\tau_n^*)$ *Loewner-dominates the information matrix of any regular n-point design* τ*. That is, if* Φ *is any Loewner isotonic criterion, then* τ_n^* *is the n-point* Φ*-optimal regular design for* $(\boldsymbol{f}, u_W, v_W)$ *on* $[T_*, T^*] \subset (0, \infty)$*.*

Theorem

Let $\boldsymbol{M}_\infty = \lim_{n \to \infty} \boldsymbol{M}(\tau_n^*)$*. Then* $\boldsymbol{M}(\tau_n^*) \succeq \gamma_n \boldsymbol{M}_\infty$*, where* $\gamma_n = \frac{n^2 - 2n}{n^2 - 2n + 1}$*.*

Therefore, the efficiency of τ_n^* cannot be lower than γ_n relative to *any* other design (even with a larger number of design points) and with respect to any homogeneous and Loewner isotonic optimality criterion. The value γ_n is thus a lower bound on the "ultimate efficiency" of the design τ_n^*. Note that $\gamma_3 = 0.75$, $\gamma_4 \approx 0.89$, $\gamma_5 \approx 0.94$, and $\gamma_6 = 0.96$.

Conclusions

- Product covariance functions form an infinite class covering the covariance function of the Wiener process, the Ornstein - Uhlenbeck process and the Brownian bridge as special cases.
- The optimal design problems with the product covariance function can be transformed one into another, and the Wiener covariance function can be taken as “canonical”.
- For the Wiener covariance function, the parabolic, the square-root and the hyperbolic classes of regression functions lead to optimal designs forming the arithmetic, the geometric and the harmonic progressions respectively. It seems that these three classes of regression functions are the only ones that lead to “nested” optimal design progressions.
- The information provided by the optimal n-point designs approaches the maximum achievable value very rapidly, i.e., there is no need to perform more that several observations to estimate β.

Thank you for attention!