# Partial profile paired comparison designs for avoiding information overload

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# Outline

Motivation

Preliminaries: Model, notation

Optimal approximate designs

Exact designs

# Types of profiles



**Full profiles** ...

... use **all** attributes in every pair

| | |
|---|---|
| A2 | A1 |
| B1 | B2 |
| C2 | C1 |

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**Partial profiles ...**

... differ only in **subset** of attributes with a given **profile strength**

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|---|---|
| **A3** | **A1** |
| B1 | B1 |
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|---|---|
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| | |
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# Setting for PCs of partial profiles



- $K$ attributes: $k^{\text{th}}$ attribute has levels $1, 2, \ldots, v_k$
- Profiles: $K$-tuples of levels denoted by **s**, **t**
- Pairs: $(\mathbf{s}, \mathbf{t})$

# Setting for PCs of partial profiles

- $K$ attributes: $k^{\text{th}}$ attribute has levels $1, 2, \ldots, v_k$
- Profiles: $K$-tuples of levels denoted by **s**, **t**
- Pairs: $(\mathbf{s}, \mathbf{t})$
- Profile strength: $S$
- Design space for **partial profiles**

$$\mathcal{X} = \{(\mathbf{s}, \mathbf{t}) : \mathbf{s} \text{ and } \mathbf{t} \text{ differ on exactly } S \text{ attributes}\}$$

# Linear main effects model

Mean overall utility of single profile $\mathbf{s} = (s_1, \ldots, s_K)$

$$u(\mathbf{s}) = \mu + \sum_{k=1}^{K} \mathbf{f}_k(s_k)^\top \beta_k \qquad \mathbf{f}_k(s_k) = \begin{cases} \mathbf{e}_{v_k-1, s_k} & s_k \neq v_k \\ -\mathbf{1}_{v_k-1} & s_k = v_k \end{cases}$$

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Model equation for paired comparisons

$$\boxed{Y(\mathbf{s}, \mathbf{t}) = u(\mathbf{s}) - u(\mathbf{t}) + \varepsilon = \sum_{k=1}^{K} [\mathbf{f}_k(s_k) - \mathbf{f}_k(t_k)]^\top \beta_k + \varepsilon} \qquad (1)$$

**Exact design** $\xi_N$ **of size** $N$

- sequence $(\mathbf{s}_1, \mathbf{t}_1), \ldots, (\mathbf{s}_N, \mathbf{t}_N)$ of pairs in $\mathcal{X}$
- can also be given by design matrix $\mathbf{X}$ in model (1)
- has normalized information matrix $\mathbf{M}(\xi_N) = \frac{1}{N}\mathbf{X}^\top \mathbf{X}$

**Approximate design** $\xi$

- discrete probability measure on $\mathcal{X}$
- with information matrix $\mathbf{M}(\xi)$

# The problem

**Problem considered here**

Find $D$-optimal designs for PCs of partial profiles.

**Approach taken**

(Invariance) $\Rightarrow$ Optimal approximate designs $\Rightarrow$ Exact designs

Some notation

# An essentially complete class of designs

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- $M = C(K, S)$
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There always exists a $D$-optimal approximate design of the form

$$\boxed{\xi = \sum_{m=1}^{M} \sum_{(\mathbf{s},\mathbf{t}) \in \mathcal{G}_m} \frac{w_m}{|\mathcal{G}_m|} \xi_{(\mathbf{s},\mathbf{t})}} \qquad (2)$$

# Information matrix

# Information matrix

For designs $\xi$ as in (2)

$$\mathbf{M}(\xi) = \begin{pmatrix} c_1(\xi)\mathbf{M}_{v_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & c_K(\xi)\mathbf{M}_{v_K} \end{pmatrix}$$

where

- $\mathbf{M}_a = \frac{2}{a-1}(\mathbf{I}_{a-1} + \mathbf{1}_{a-1}\mathbf{1}_{a-1}^\top)$ for every positive integer $a > 1$
- $c_k(\xi) = \sum_{1 \le m \le M : k \in \mathcal{F}_m} w_m$ for every $k$.

# Two groups of factors

From now on consider only two groups of factors.

The first $1 \leq K_1 < K$ factors have $u_1$ levels; each of the remaining $K_2 = K - K_1$ factors has $u_2$ levels where $u_1 < u_2$.

Total number of parameters is $p = K_1 q_1 + K_2 q_2$, where $q_i = u_i - 1$.

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Total number of parameters is $p = K_1 q_1 + K_2 q_2$, where $q_i = u_i - 1$.

A set $\mathcal{G}_m$ is of type $(n_1, n_2)$ with $n_1 + n_2 = S$, if

for every pair $(\mathbf{s}, \mathbf{t}) \in \mathcal{G}_m$ the profiles $\mathbf{s}$ and $\mathbf{t}$ differ in

$n_1$ factors with $u_1$ levels and

$n_2$ factors with $u_2$ levels.

# Optimal approximate designs

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**Theorem**

The designs $\xi$ in (2) with weights $w_m$ specified as follows are $D$-optimal:

(a) If $K_1, K_2 \geq S$, then $w_m = C(K_1-1, S-1)^{-1}\frac{q_1 S}{p}$ for all $\mathcal{G}_m$ of type $(S, 0)$ and $w_m = C(K_2-1, S-1)^{-1}\frac{q_2 S}{p}$ for all $\mathcal{G}_m$ of type $(0, S)$.

(b) If $K_2 \geq S > K_1$, then $w_m = C(K_2, S-K_1)^{-1}\frac{q_1 S}{p}$ for all $\mathcal{G}_m$ of type $(K_1, S-K_1)$ and $w_m = C(K_2, S)^{-1}(1-\frac{q_1 S}{p})$ for all $\mathcal{G}_m$ of type $(0, S)$.

(c) If $K_1 \geq S > K_2$ and $q_2 S < p$, then $w_m = C(K_1, S-K_2)^{-1}\frac{q_2 S}{p}$ for all $\mathcal{G}_m$ of type $(S-K_2, K_2)$ and $w_m = C(K_1, S)^{-1}(1-\frac{q_2 S}{p})$ for all $\mathcal{G}_m$ of type $(S, 0)$.

(d) If $S > K_1, K_2$ and $q_2 S < p$, then $w_m = C(K_2-1, S-K_1)^{-1}(1-\frac{q_2 S}{p})$ for all $\mathcal{G}_m$ of type $(K_1, S-K_1)$ and $w_m = C(K_1-1, S-K_2)^{-1}(1-\frac{q_1 S}{p})$ for all $\mathcal{G}_m$ of type $(S-K_2, K_2)$.

(e) If $S > K_2$ and $q_2 S \geq p$, then $w_m = C(K_1, S-K_2)^{-1}$ for all $\mathcal{G}_m$ of type $(S-K_2, K_2)$.

For the sets $\mathcal{G}_m$ of any other type not explicitly specified in the statements the weights $w_m$ are equal to zero.

# Information matrix revisited

For the designs in the theorem the constants in

$$\mathbf{M}(\xi) = \begin{pmatrix} c_1(\xi)\mathbf{M}_{v_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & c_K(\xi)\mathbf{M}_{v_K} \end{pmatrix}$$

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are equal to

- cases (a)-(d):

$$c_k(\xi) = \begin{cases} q_1 S/p & \text{for } k = 1, \ldots, K_1 \\ q_2 S/p & \text{for } k = K_1 + 1, \ldots, K \end{cases}$$

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- case (e):

$$c_k(\xi) = \begin{cases} 1 - (K - S)/K_1 & \text{for } k = 1, \dots, K_1 \\ 1 & \text{otherwise.} \end{cases}$$

# Summary so far

There exist optimal approximate designs whose support consists of pairs of at most **two** different types.

Information matrices of these are **known**.

Goal of second part of talk:

Find **exact** designs with the same information matrices.

# Tools

- Weighing matrix of weight $w$ and order $n$: Square matrix $\mathbf{W}$ of order $n$ with elements $0, \pm 1$ such that

$$\mathbf{W}\mathbf{W}^\top = w\mathbf{I}$$

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- Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$

Construction c1 using points from sets $\mathcal{G}_m$ of types $(S, 0)$ and $(S - K_2, K_2)$

# A construction for optimal exact designs



Construction c1 using points from sets $\mathcal{G}_m$ of types $(S, 0)$ and $(S - K_2, K_2)$

$$\mathbf{X} = \left(\begin{array}{c|c} \mathbf{W}_1 \otimes \mathbf{1}_{a_1} \otimes \mathbf{X}_1 & \mathbf{0} \\ \hline \mathbf{H}_2^{\perp} \otimes \mathbf{W}_{1,2,z_2} \otimes \mathbf{1}_{a_2} \otimes \mathbf{X}_1 & \mathbf{H}_2 \otimes \mathbf{L}_2 \otimes \mathbf{X}_2 \end{array}\right)$$

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Construction c1 using points from sets $\mathcal{G}_m$ of types $(S, 0)$ and $(S - K_2, K_2)$

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Information matrix

$$\frac{1}{N}\mathbf{X}^\top \mathbf{X} = \frac{1}{N} \begin{pmatrix} \alpha_1(\mathbf{I}_{K_1} \otimes \mathbf{M}_{u_1}) & \mathbf{0} \\ \mathbf{0} & \alpha_2(\mathbf{I}_{K_2} \otimes \mathbf{M}_{u_2}) \end{pmatrix}$$

# Summary of constructions



| ID | Design matrix $\mathbf{X}$ | Number of pairs $N$ |
|---|---|---|
| a1 | $\begin{pmatrix} \mathbf{1}_{N_1}\otimes\mathbf{W}_1\otimes\mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_2}\otimes\mathbf{W}_2\otimes\mathbf{X}_2 \end{pmatrix}$ | $K_1N_1C(u_1,2)+K_2N_2C(u_2,2)$ |
| a2 | $\begin{pmatrix} \mathbf{1}_{N_1}\otimes\mathbf{W}_1\otimes\mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_2}\otimes\mathbf{P}_2 \end{pmatrix}$ | $K_1N_1C(u_1,2)+N_2\#(\mathbf{P}_2)$ |
| a3 | ((a2)) | $K_2N_2C(u_2,2)+N_1\#(\mathbf{P}_1)$ |
| a4 | $\begin{pmatrix} \mathbf{1}_{N_1}\otimes\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_2}\otimes\mathbf{P}_2 \end{pmatrix}$ | $N_1\#(\mathbf{P}_1)+N_2\#(\mathbf{P}_2)$ |
| b1 | $\begin{pmatrix} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \mathbf{0} & \mathbf{W}_2\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_2 \end{pmatrix}$ | $(a_1m_1/z_1+a_2)K_2C(u_2,2)$ |
| b2 | $\begin{pmatrix} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \mathbf{0} & \mathbf{1}_{a_2}\otimes\mathbf{P}_2 \end{pmatrix}$ | $a_1m_1K_2C(u_2,2)/z_1+a_2\#(\mathbf{P}_2)$ |
| b3 | $\begin{pmatrix} \mathbf{1}_{N_1}\otimes\mathbf{F}_1\otimes\mathbf{1}_{M_1} & \mathbf{1}_{N_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{P}_{2,1} \\ \mathbf{0} & \mathbf{W}_2\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_2 \end{pmatrix}$ | $a_1N_1\#(\mathbf{P}_{2,1})+a_2K_2C(u_2,2)$ |
| b4 | $\begin{pmatrix} \mathbf{1}_{N_1}\otimes\mathbf{F}_1\otimes\mathbf{1}_{M_1} & \mathbf{1}_{N_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{P}_{2,1} \\ \mathbf{0} & \mathbf{1}_{a_2}\otimes\mathbf{P}_2 \end{pmatrix}$ | $a_1N_1\#(\mathbf{P}_{2,1})+a_2\#(\mathbf{P}_2)$ |
| c1 | ((b1)) | $(a_2m_2/z_2+a_1)K_1C(u_1,2)$ |
| c2 | ((b2)) | $a_2m_2K_1C(u_1,2)/z_2+a_1\#(\mathbf{P}_1)$ |
| c3 | ((b3)) | $a_2N_2\#(\mathbf{P}_{1,2})+a_1K_1C(u_1,2)$ |
| c4 | ((b4)) | $a_2N_2\#(\mathbf{P}_{1,2})+a_1\#(\mathbf{P}_1)$ |
| d1 | $\begin{pmatrix} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \mathbf{H}_2^{\perp}\otimes\mathbf{W}_{1,2,z_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_1 & \mathbf{H}_2\otimes\mathbf{L}_2\otimes\mathbf{X}_2 \end{pmatrix}$ | $m_1n_1C(u_1,2)+m_2n_2C(u_2,2)$ |
| d2 | $\begin{pmatrix} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \mathbf{1}_{N_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{P}_{1,2} & \mathbf{1}_{N_2}\otimes\mathbf{F}_2\otimes\mathbf{1}_{M_2} \end{pmatrix}$ | $m_1n_1C(u_1,2)+a_2N_2\#(\mathbf{P}_{1,2})$ |
| d3 | ((d2)) | $m_2n_2C(u_2,2)+a_1N_1\#(\mathbf{P}_{2,1})$ |
| d4 | $\begin{pmatrix} \mathbf{1}_{N_1}\otimes\mathbf{F}_1\otimes\mathbf{1}_{M_1} & \mathbf{1}_{N_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{P}_{2,1} \\ \mathbf{1}_{N_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{P}_{1,2} & \mathbf{1}_{N_2}\otimes\mathbf{F}_2\otimes\mathbf{1}_{M_2} \end{pmatrix}$ | $a_1N_1\#(\mathbf{P}_{2,1})+a_2N_2\#(\mathbf{P}_{1,2})$ |
| e1 | $\begin{pmatrix} \mathbf{H}_2^{\perp}\otimes\mathbf{W}_{1,2,z_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_1 & \mathbf{H}_2\otimes\mathbf{L}_2\otimes\mathbf{X}_2 \end{pmatrix}$ | $m_2n_2C(u_2,2)$ |
| e2 | $\begin{pmatrix} \mathbf{1}_{a_2}\otimes\mathbf{P}_{1,2} & \mathbf{F}_2\otimes\mathbf{1}_{M_2} \end{pmatrix}$ | $a_2\#(\mathbf{P}_{1,2})$ |

| ID | Conditions |
|---|---|
| a1 | $N_1 = N_2 \frac{u_2}{u_1}$ |
| a2 | $N_1 = N_2 \frac{u_2}{u_1} \frac{H(S)}{\gcd(K_2,S)}$ |
| a3 | $N_1 = N_2 \frac{u_2}{u_1} \frac{\gcd(K_1,S)}{H(S)}$ |
| a4 | $N_1 = N_2 \frac{u_2}{u_1} \frac{\gcd(K_1,S)}{\gcd(K_2,S)}$ |
| b1 | $n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}, a_2 = a_1 \frac{m_1}{z_1} \left(\frac{p}{q_1 S} - 1\right)$ |
| b2 | $n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}, a_2 = a_1 \frac{m_1}{z_1} \left(\frac{p}{q_1 S} - 1\right) \frac{\gcd(K_2,S)}{H(S)}$ |
| b3 | $M_1 = \#(\mathbf{P}_{2,1}),\ a_1 = \#(\mathbf{F}_1)$ <br> $a_2 = N_1 u_1 \left(\frac{p}{S} - q_1\right) \frac{H(K_1)H(S-K_1)}{\gcd(K_2,S-K_1)} \left(1 - \frac{u_2 \bmod 2}{2}\right)$ |
| b4 | $M_1 = \#(\mathbf{P}_{2,1}), a_1 = \#(\mathbf{F}_1)$ <br> $a_2 = N_1 u_1 \left(\frac{p}{S} - q_1\right) \frac{\gcd(K_2,S)}{\gcd(K_2,S-K_1)} \frac{H(K_1)H(S-K_1)}{H(S)} \left(1 - \frac{u_2 \bmod 2}{2}\right)$ |
| c1 | $n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2},\ a_1 = a_2 \frac{m_2}{z_2} \left(\frac{p}{q_2 S} - 1\right)$ |
| c2 | $n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2}, a_1 = a_2 \frac{m_2}{z_2} \left(\frac{p}{q_2 S} - 1\right) \frac{\gcd(K_1,S)}{H(S)}$ |
| c3 | $M_2 = \#(\mathbf{P}_{1,2}), a_2 = \#(\mathbf{F}_2)$ <br> $a_1 = N_2 u_2 \left(\frac{p}{S} - q_2\right) \frac{H(K_2)H(S-K_2)}{\gcd(K_1,S-K_2)} \left(1 - \frac{u_1 \bmod 2}{2}\right)$ |
| c4 | $M_2 = \#(\mathbf{P}_{1,2}), a_2 = \#(\mathbf{F}_2)$ <br> $a_1 = N_2 u_2 \left(\frac{p}{S} - q_2\right) \frac{\gcd(K_1,S)}{\gcd(K_1,S-K_2)} \frac{H(K_2)H(S-K_2)}{H(S)} \left(1 - \frac{u_1 \bmod 2}{2}\right)$ |
| d1 | $n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}, n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2}$ <br> $m_2 = m_1 \frac{a_1}{a_2} \frac{z_2}{z_1} \frac{u_2 q_2}{u_1 q_1} \frac{q_1 S - p}{q_2 S - p}$ |
| d2 | $M_2 = \#(\mathbf{P}_{1,2}), a_2 = \#(\mathbf{F}_2), n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}$ <br> $m_1 = N_2 u_1 q_1 \frac{z_1}{a_1} \frac{p - q_2 S}{p - q_1 S} \frac{H(K_2)H(S-K_2)}{\gcd(K_1,S-K_2)} \left(1 - \frac{u_1 \bmod 2}{2}\right)$ |
| d3 | $M_1 = \#(\mathbf{P}_{2,1}), a_1 = \#(\mathbf{F}_1), n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2}$ |

# Parameters of $D$-optimal exact designs

| $K$ | $K_1$ | $K_2$ | $u_1$ | $u_2$ | $S$ | ID | $N_1$ | $N_2$ | $M_1$ | $M_2$ | $a_1$ | $a_2$ | $m_1$ | $r_1$ | $n_1$ | $s_1$ | $z_1$ | $m_2$ | $r_2$ | $n_2$ | $s_2$ | $z_2$ | Pairs |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 3 | 2 | 3 | 3 | b4 | 1 | – | 18 | – | 1 | 2 | – | – | – | – | – | – | – | – | – | – | 42 |
| 4 | 2 | 2 | 2 | 3 | 2 | a1 | 3 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 18 |
| 4 | 2 | 2 | 2 | 3 | 3 | e1 | – | – | – | – | – | 3 | – | – | – | – | – | 2 | 1 | 2 | 2 | 1 | 12 |
| 4 | 2 | 2 | 2 | 4 | 2 | a1 | 2 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 16 |
| 4 | 2 | 2 | 2 | 4 | 3 | e1 | – | – | – | – | – | 3 | – | – | – | – | – | 4 | 2 | 1 | 1 | 1 | 24 |
| 4 | 2 | 2 | 2 | 5 | 2 | a1 | 5 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 50 |
| 4 | 2 | 2 | 2 | 5 | 3 | e1 | – | – | – | – | – | 5 | – | – | – | – | – | 4 | 2 | 1 | 1 | 1 | 40 |
| 4 | 2 | 2 | 3 | 4 | 2 | a1 | 4 | 3 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 60 |
| 4 | 2 | 2 | 3 | 5 | 2 | a1 | 5 | 3 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 90 |
| 4 | 3 | 1 | 2 | 3 | 2 | c2 | – | – | – | – | 1 | 1 | – | – | – | – | – | 8 | 1 | 1 | 1 | 1 | 30 |
| 4 | 3 | 1 | 2 | 3 | 3 | e2 | – | – | – | 12 | – | 3 | – | – | – | – | – | – | – | – | – | – | 36 |
| 4 | 3 | 1 | 2 | 4 | 2 | e1 | – | – | – | – | – | 2 | – | – | – | – | – | 2 | 1 | 1 | 1 | 1 | 12 |
| 4 | 3 | 1 | 2 | 4 | 3 | e2 | – | – | – | 12 | – | 6 | – | – | – | – | – | – | – | – | – | – | 72 |
| 4 | 3 | 1 | 2 | 5 | 2 | e1 | – | – | – | – | – | 10 | – | – | – | – | – | 2 | 1 | 3 | 1 | 1 | 60 |
| 4 | 3 | 1 | 3 | 4 | 2 | c2 | – | – | – | – | 1 | 2 | – | – | – | – | – | 2 | 1 | 3 | 1 | 1 | 54 |
| 5 | 1 | 4 | 2 | 3 | 3 | b3 | 1 | – | 12 | – | 1 | 2 | – | – | – | – | – | – | – | – | – | – | 36 |
| 5 | 2 | 3 | 2 | 3 | 2 | a2 | 3 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 24 |
| 5 | 2 | 3 | 2 | 3 | 3 | b2 | – | – | – | – | 1 | 5 | 4 | 2 | 9 | 1 | 1 | – | – | – | – | – | 96 |
| 5 | 2 | 3 | 2 | 4 | 2 | a2 | 4 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 44 |
| 5 | 2 | 3 | 2 | 5 | 2 | a2 | 5 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 70 |
| 5 | 3 | 2 | 2 | 3 | 2 | a3 | 3 | 4 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 42 |
| 5 | 3 | 2 | 2 | 3 | 3 | c2 | – | – | – | – | 1 | 1 | – | – | – | – | – | 8 | 2 | 1 | 1 | 1 | 28 |
| 5 | 3 | 2 | 2 | 4 | 2 | a3 | 1 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 18 |
| 5 | 3 | 2 | 2 | 4 | 3 | e1 | – | – | – | – | – | 2 | – | – | – | – | – | 4 | 2 | 1 | 1 | 1 | 24 |
| 5 | 3 | 2 | 3 | 4 | 2 | a3 | 2 | 3 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 72 |
| 5 | 3 | 2 | 3 | 4 | 3 | c2 | – | – | – | – | 2 | 2 | – | – | – | – | – | 4 | 2 | 3 | 1 | 1 | 96 |
| 5 | 4 | 1 | 2 | 3 | 2 | c1 | – | – | – | – | 3 | 3 | – | – | – | – | – | 2 | 1 | 4 | 1 | 1 | 36 |
| 5 | 4 | 1 | 2 | 3 | 3 | e1 | – | – | – | – | – | 3 | – | – | – | – | – | 4 | 1 | 2 | 1 | 2 | 24 |
| 5 | 4 | 1 | 2 | 4 | 2 | c1 | – | – | – | – | 1 | 3 | – | – | – | – | – | 2 | 1 | 2 | 1 | 1 | 28 |
| 5 | 4 | 1 | 2 | 4 | 3 | e1 | – | – | – | – | – | 3 | – | – | – | – | – | 4 | 1 | 1 | 1 | 2 | 24 |
| 5 | 4 | 1 | 2 | 5 | 2 | e1 | – | – | – | – | – | 5 | – | – | – | – | – | 2 | 1 | 2 | 1 | 1 | 40 |
| 5 | 4 | 1 | 2 | 5 | 3 | e1 | – | – | – | – | – | 5 | – | – | – | – | – | 4 | 1 | 1 | 1 | 2 | 40 |
| 6 | 2 | 4 | 2 | 3 | 2 | a1 | 3 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 30 |
| 6 | 2 | 4 | 2 | 4 | 2 | a1 | 2 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 28 |
| 6 | 2 | 4 | 2 | 5 | 2 | a1 | 5 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 90 |
| 6 | 2 | 4 | 3 | 4 | 2 | a1 | 4 | 3 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 96 |
| 6 | 3 | 3 | 2 | 3 | 2 | a4 | 3 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 54 |
| 6 | 3 | 3 | 2 | 3 | 3 | a4 | 3 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 36 |
| 6 | 3 | 3 | 2 | 4 | 2 | a4 | 2 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 48 |
| 6 | 3 | 3 | 2 | 4 | 3 | a4 | 2 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 32 |
| 6 | 3 | 3 | 2 | 5 | 3 | a4 | 5 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 100 |
| 6 | 4 | 2 | 2 | 3 | 2 | a1 | 3 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 24 |
| 6 | 4 | 2 | 2 | 3 | 3 | c1 | – | – | – | – | 2 | 3 | – | – | – | – | – | 2 | 1 | 4 | 2 | 1 | 32 |
| 6 | 4 | 2 | 2 | 4 | 2 | a1 | 2 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 20 |
| 6 | 4 | 2 | 2 | 4 | 3 | c1 | – | – | – | – | 2 | 3 | – | – | – | – | – | 6 | 1 | 2 | 2 | 1 | 80 |
| 6 | 4 | 2 | 2 | 5 | 2 | a1 | 5 | 2 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 60 |
| 6 | 4 | 2 | 2 | 5 | 3 | e1 | – | – | – | – | – | 5 | – | – | – | – | – | 2 | 1 | 2 | 2 | 1 | 40 |
| 6 | 4 | 2 | 3 | 4 | 2 | a1 | 4 | 3 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 84 |
| 6 | 5 | 1 | 2 | 4 | 2 | c2 | – | – | – | – | 2 | 6 | – | – | – | – | – | 2 | 1 | 5 | 1 | 1 | 80 |
| 6 | 5 | 1 | 2 | 5 | 2 | c2 | – | – | – | – | 1 | 2 | – | – | – | – | – | 8 | 1 | 1 | 1 | 1 | 90 |

# Web tables

Tables of designs with up to 100 pairs available at

`http://www.maths.qmul.ac.uk/~hg/PP2G/`

# Concluding remarks

Designs are also (locally) optimal for the **multinomial logit model** under assumption that parameters are equal to 0.

General method used here can also be used to generate efficient designs with **smaller** numbers of pairs.

Extension to **three** groups of factors is possible, but more complicated.

# References


Großmann, H., Holling, H., Graßhoff, U. & Schwabe, R. (2006). Optimal designs for asymmetric linear paired comparisons with a profile strength constraint. *Metrika* 64, 109–119.

Großmann, H., Graßhoff, U. & Schwabe, R. (2009). Approximate and exact optimal designs for paired comparisons of partial profiles when there are two groups of factors. *J. Statist. Plann. Inference* 139, 1171–1179.

# Example

Design for $K_1 = 4$, $K_2 = 2$, $u_1 = 2$, $u_2 = 3$ and $S = 3$.

# Example



Design for $K_1 = 4$, $K_2 = 2$, $u_1 = 2$, $u_2 = 3$ and $S = 3$.

Construct optimal exact design with $N = 32$ pairs using method c1.

# Example



Design for $K_1 = 4$, $K_2 = 2$, $u_1 = 2$, $u_2 = 3$ and $S = 3$.

Construct optimal exact design with $N = 32$ pairs using method c1.

Building blocks

- $\mathbf{X}_1 = \begin{pmatrix} 2 \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$
- $\mathbf{W}_1 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$ and $\mathbf{W}_{1,2,z_2} = \mathbf{I}_4$
- $\mathbf{1}_{a_1} = \mathbf{1}_2$ and $\mathbf{1}_{a_2} = \mathbf{1}_3$
- $\mathbf{H}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{H}_2^{\perp} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{L}_2 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$

# Example

Design matrix

$$\mathbf{X} = \left(\begin{array}{cccc|cccc}
0 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\
2 & 0 & 0 & 0 & -2 & -1 & -2 & -1 \\
2 & 0 & 0 & 0 & -1 & -2 & -1 & -2 \\
0 & 2 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & 2 & 0 & 0 & -2 & -1 & 2 & 1 \\
0 & 2 & 0 & 0 & -1 & -2 & 1 & 2 \\
0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & 2 & 0 & 2 & 1 & 2 & 1 \\
0 & 0 & 2 & 0 & 1 & 2 & 1 & 2 \\
0 & 0 & 0 & 2 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 2 & 2 & 1 & -2 & -1 \\
0 & 0 & 0 & 2 & 1 & 2 & -1 & -2 \\
-2 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\
-2 & 0 & 0 & 0 & -2 & -1 & -2 & -1 \\
-2 & 0 & 0 & 0 & -1 & -2 & -1 & -2 \\
0 & -2 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & -2 & 0 & 0 & -2 & -1 & 2 & 1 \\
0 & -2 & 0 & 0 & -1 & -2 & 1 & 2 \\
0 & 0 & -2 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & -2 & 0 & 2 & 1 & 2 & 1 \\
0 & 0 & -2 & 0 & 1 & 2 & 1 & 2 \\
0 & 0 & 0 & -2 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & -2 & 2 & 1 & -2 & -1 \\
0 & 0 & 0 & -2 & 1 & 2 & -1 & -2
\end{array}\right)$$