D-optimal designs for Two-Variable Binary Logistic Models with Interaction

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Joint Work with

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Problem

Setting

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- Binary response $p = P(success)$

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- Two explanatory variables x_1 and x_2 with interaction

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- Binary response $p = P(success)$
- Two explanatory variables x_1 and x_2 with interaction
- Locally D-optimal designs?

Model

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- $logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$

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- relative potency $\rho = \frac{\beta_2}{\beta_1}$

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- T.E. O’Brien (2004). “Modeling and Design to Detect Interaction of Insecticides, Herbicides and Other Similar Compounds .” Proceedings of the 15th Conference on Applied Statistics in Agriculture, KSU. 303–321.

First Take

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- Multiplicity of designs

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- Taxonomy?

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- Multiplicity of designs
- Taxonomy?
- Algebra?

Preliminaries

No Interaction Model

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- $u = logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

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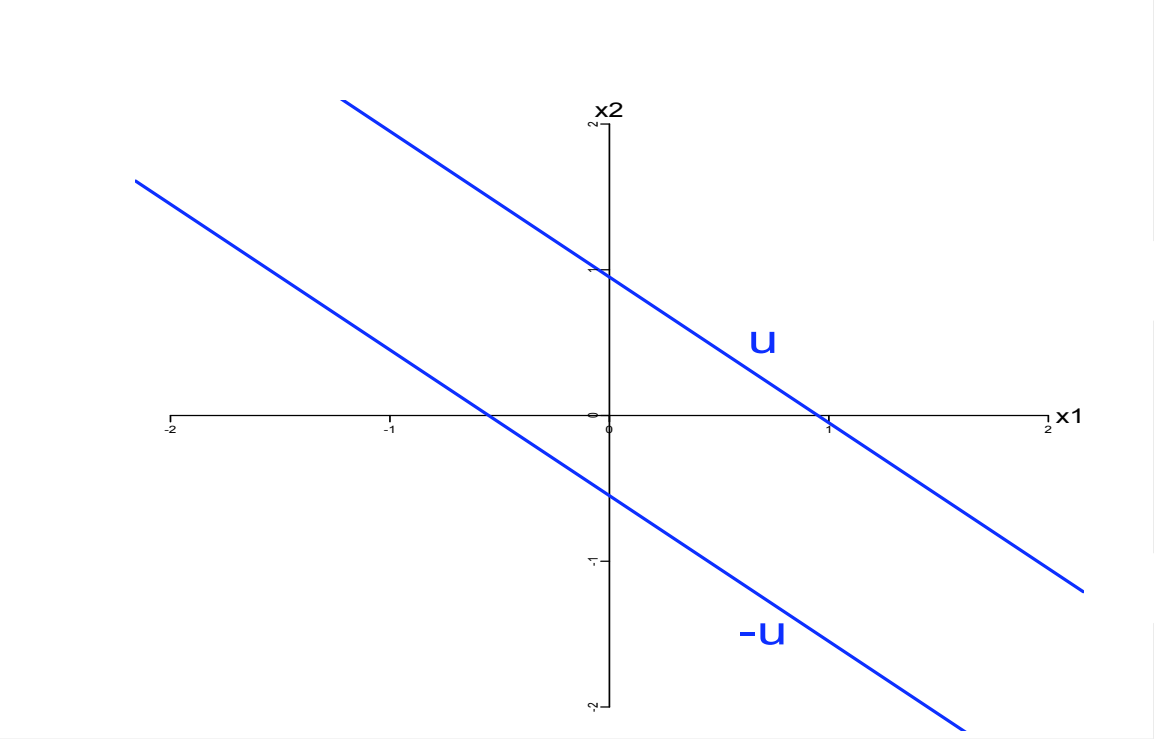
- $u = logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

- Lines of constant logit



D-optimal Design (1): $-b \leq x_2 \leq b$

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$(\pm u^*, \pm b)$

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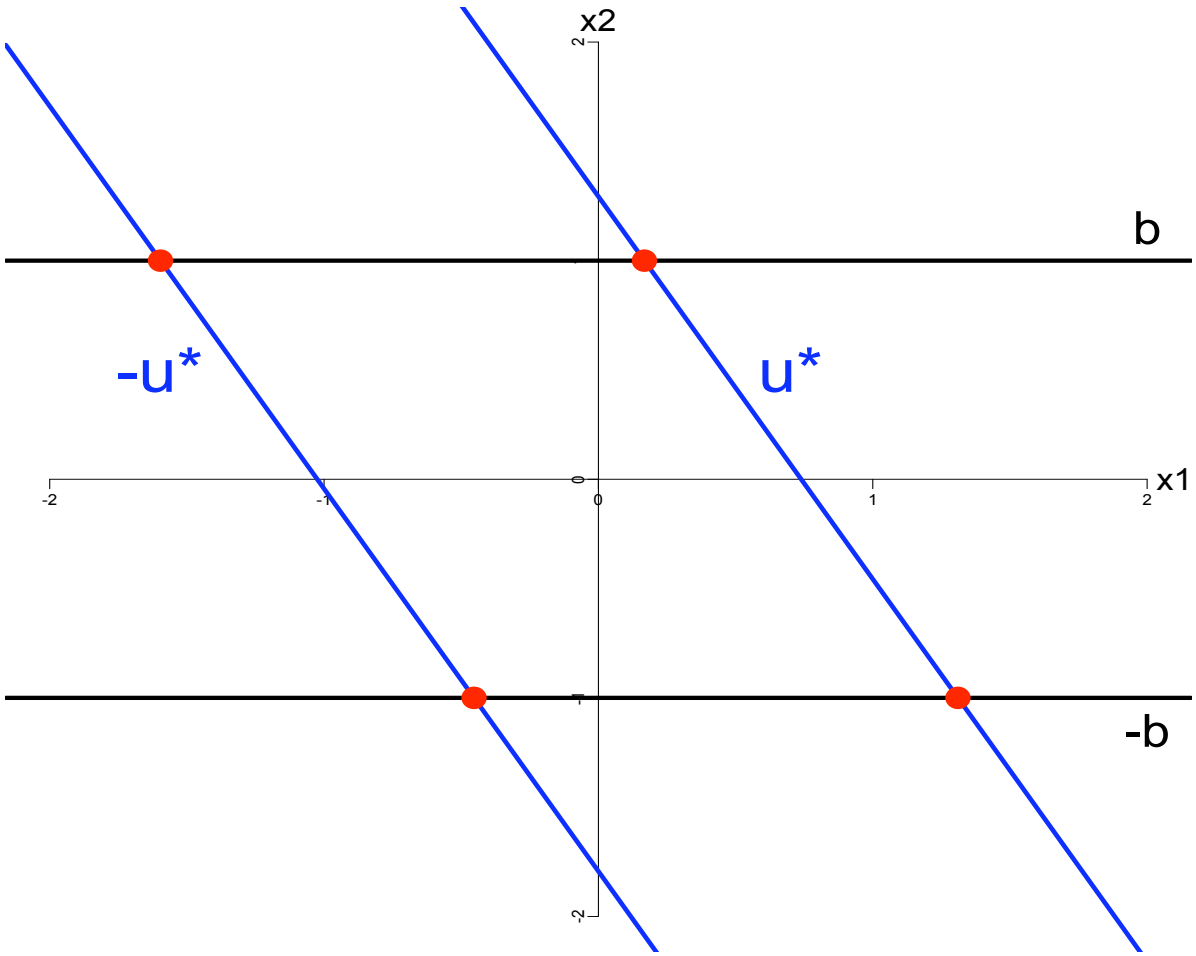

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$$(k+1)u + 2 - ((k+1)u - 2)e^u = 0$$

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- $k = 2, u^* = 1.2229$ and $k = 1, u^* = 1.5434$

D-optimal Design (2a): $x_1 \geq 0, x_2 \geq 0$

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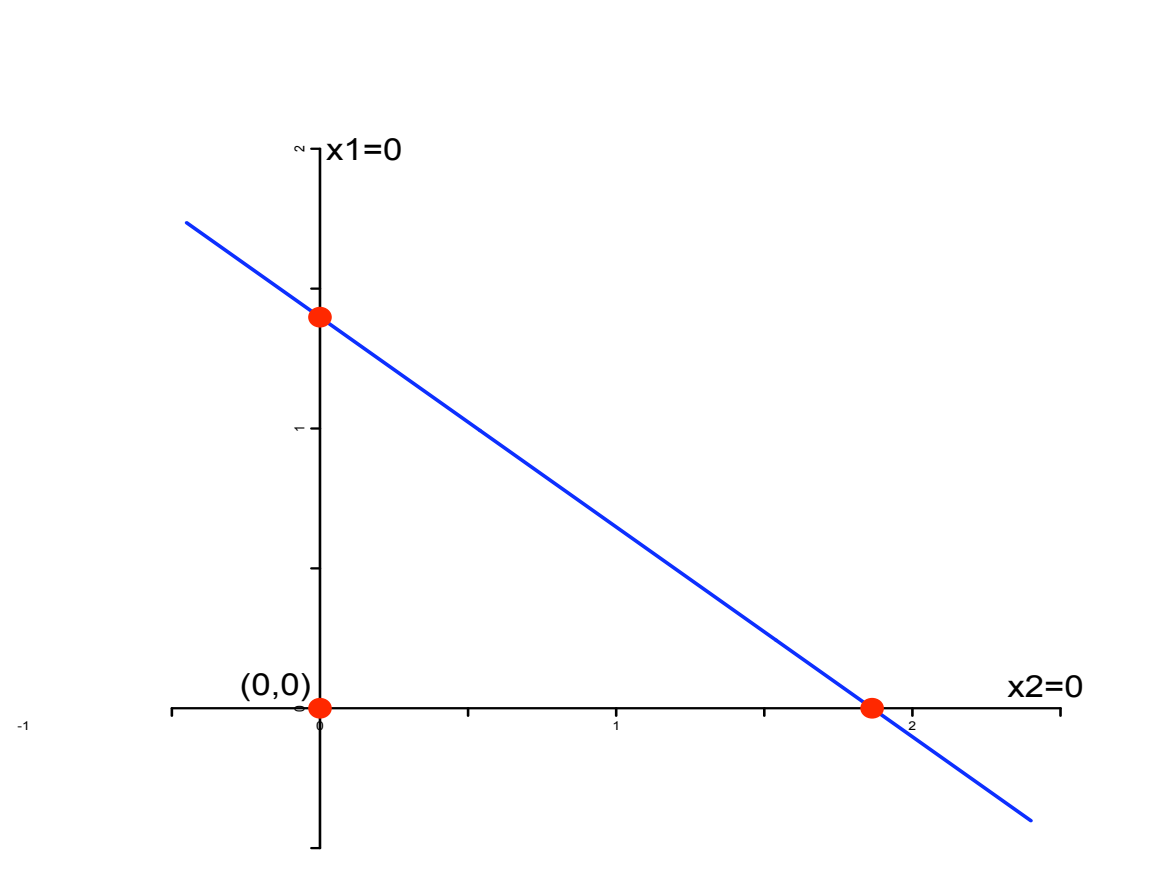

D-optimal Design (2b): $x_1 \geq 0, x_2 \geq 0$

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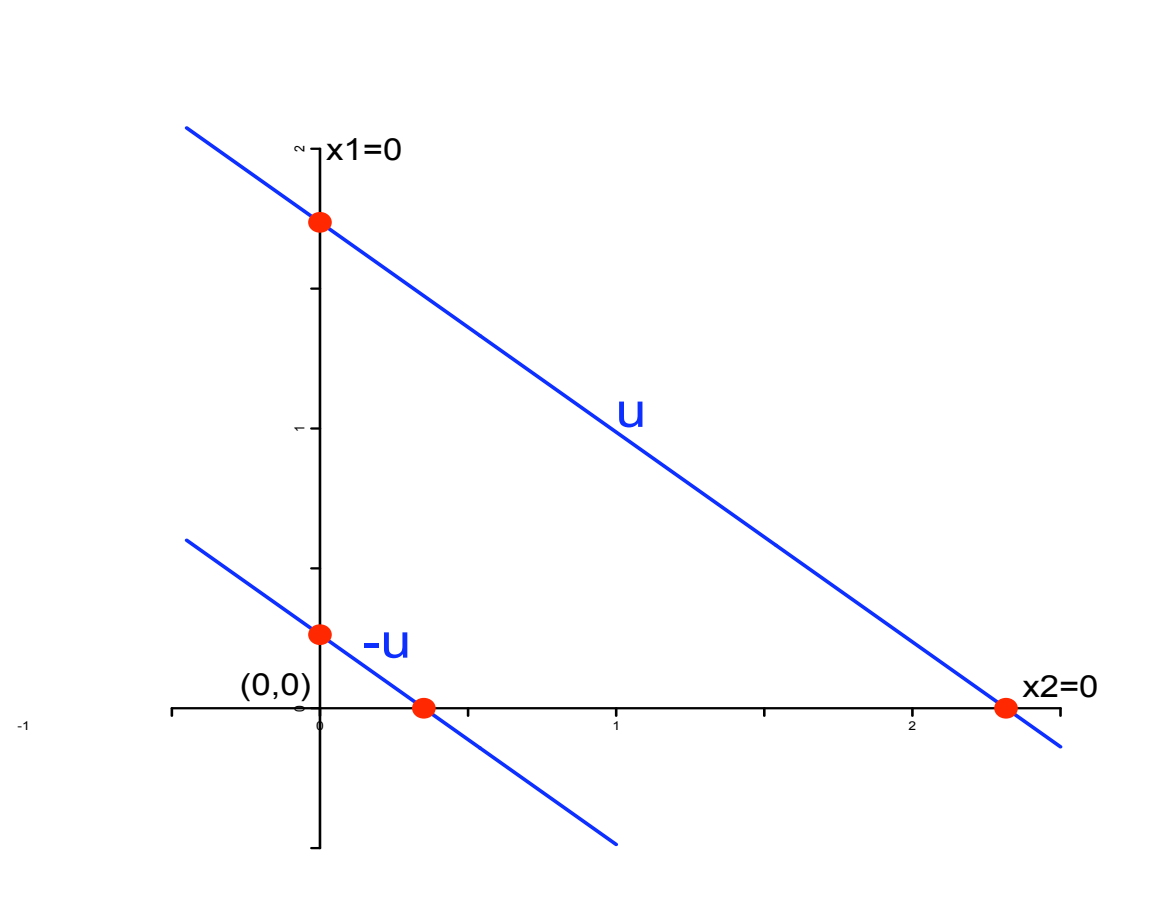

Results

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where $\beta_0^* = \beta_0 - \frac{\beta_1 \beta_2}{\beta_{12}}$

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$$(x_1, x_2) \longleftrightarrow (z_1, z_2) \longleftrightarrow (u, x_2) \longleftrightarrow (u, z_2)$$

where $u = \beta_0 + b_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 = \beta_0^* + \beta_{12} z_1 z_2$

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- Logit lines

$$x_1 = 0, x_2 = 0 \quad \text{lies on} \quad u = \beta_0$$
$$z_1 = 0, z_2 = 0 \quad \text{lies on} \quad u = \beta_0^*$$

Lines of Constant Logit: $u > \beta_0^*$

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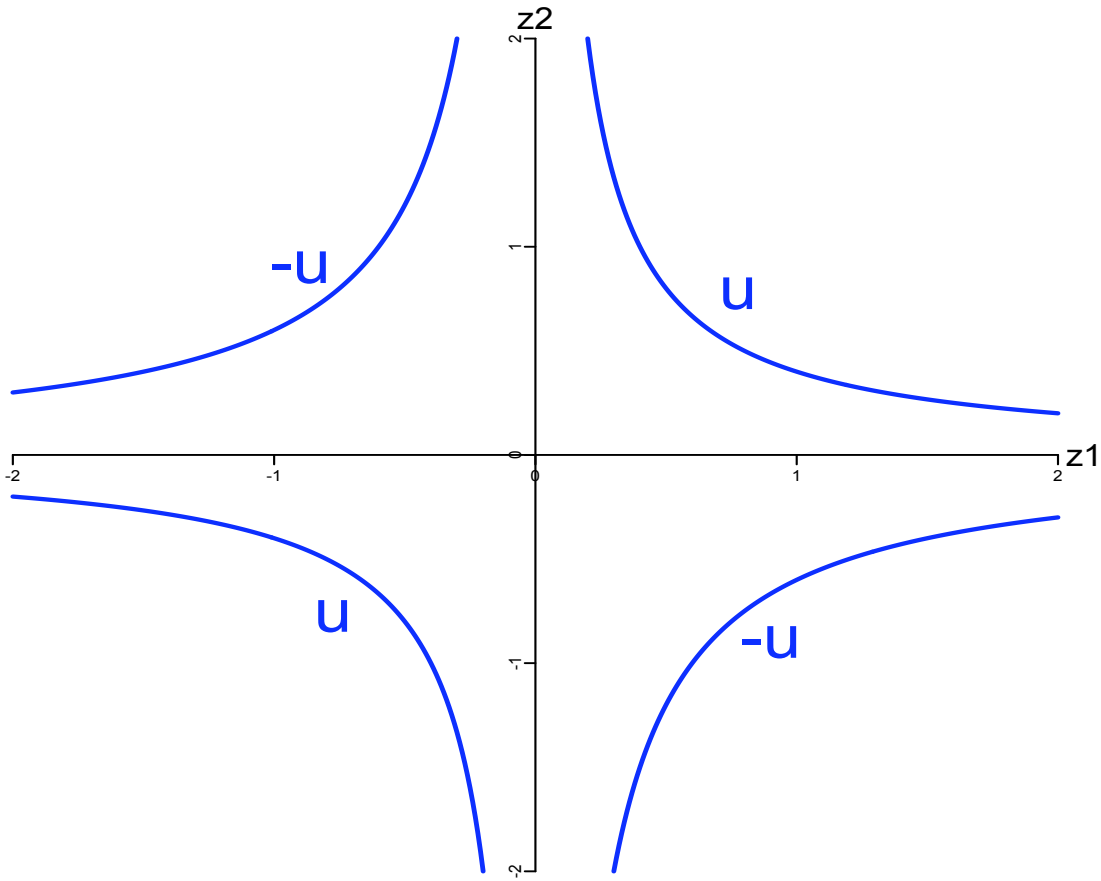

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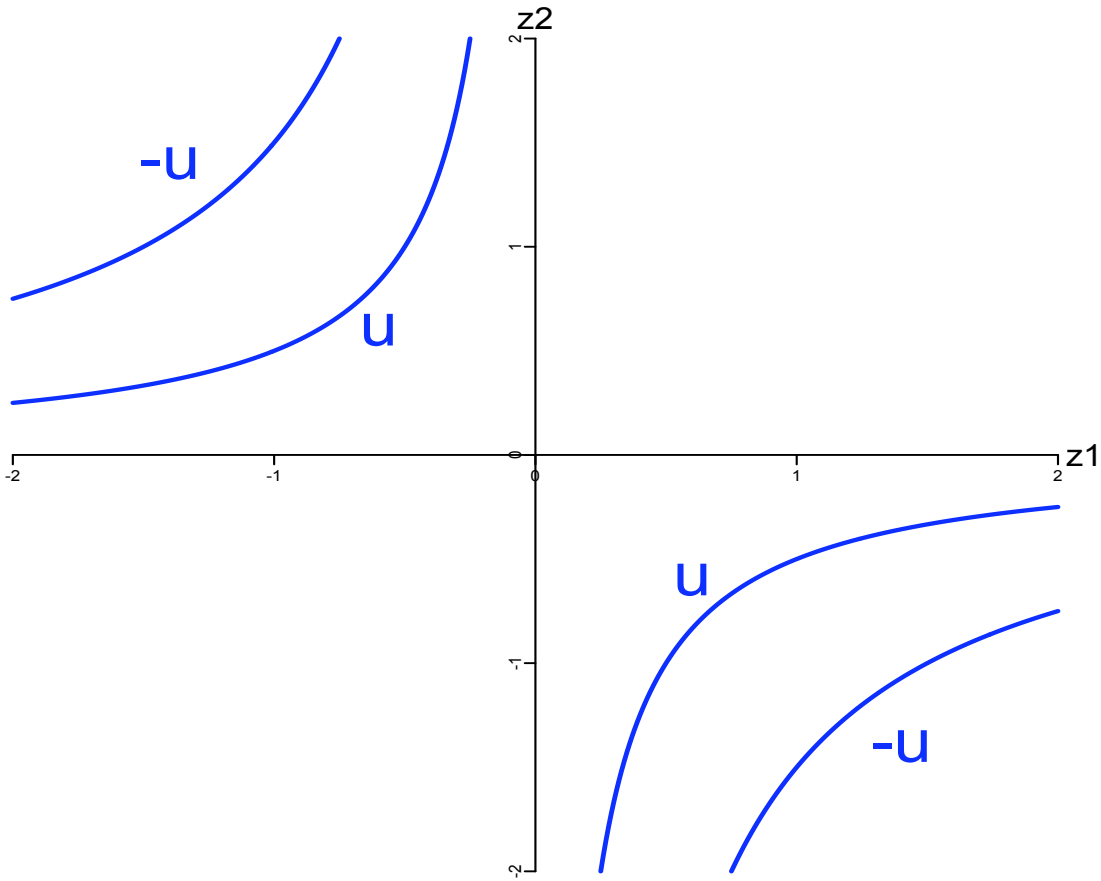

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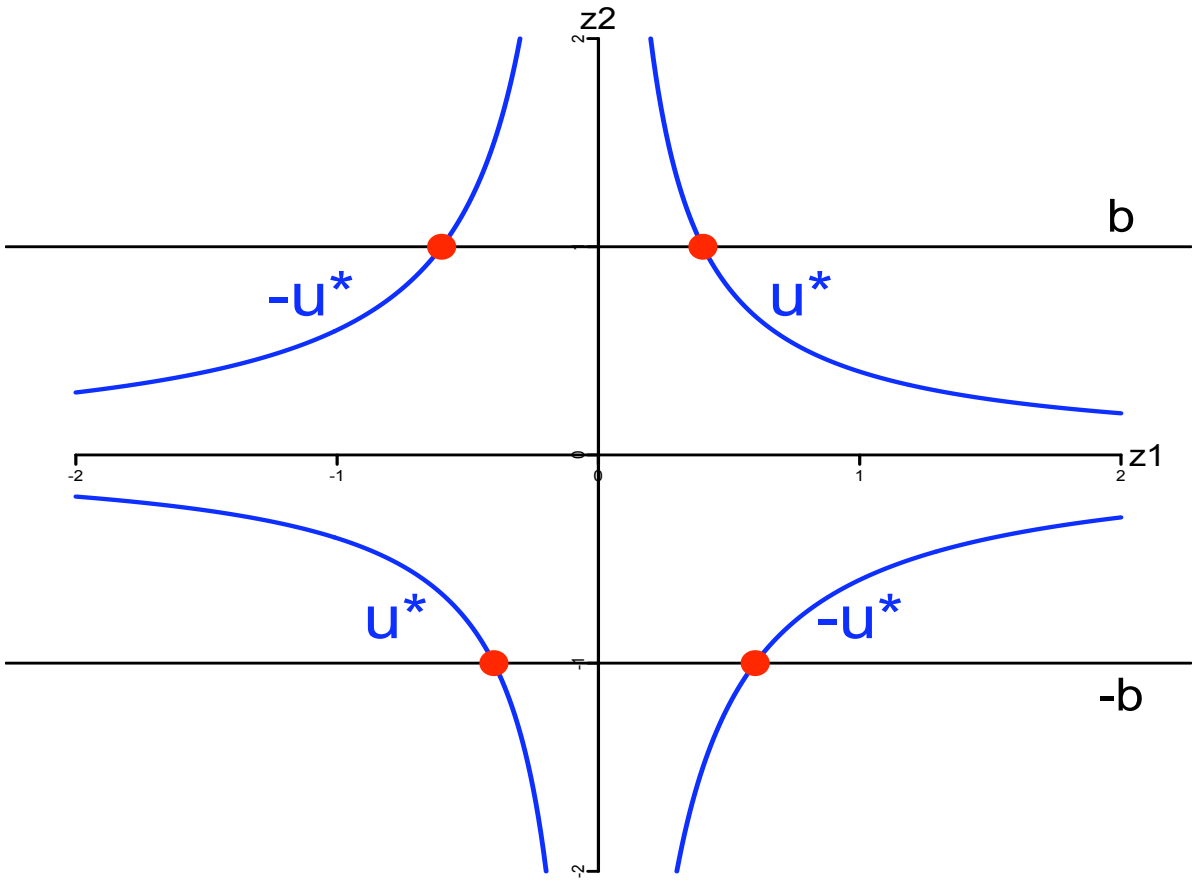

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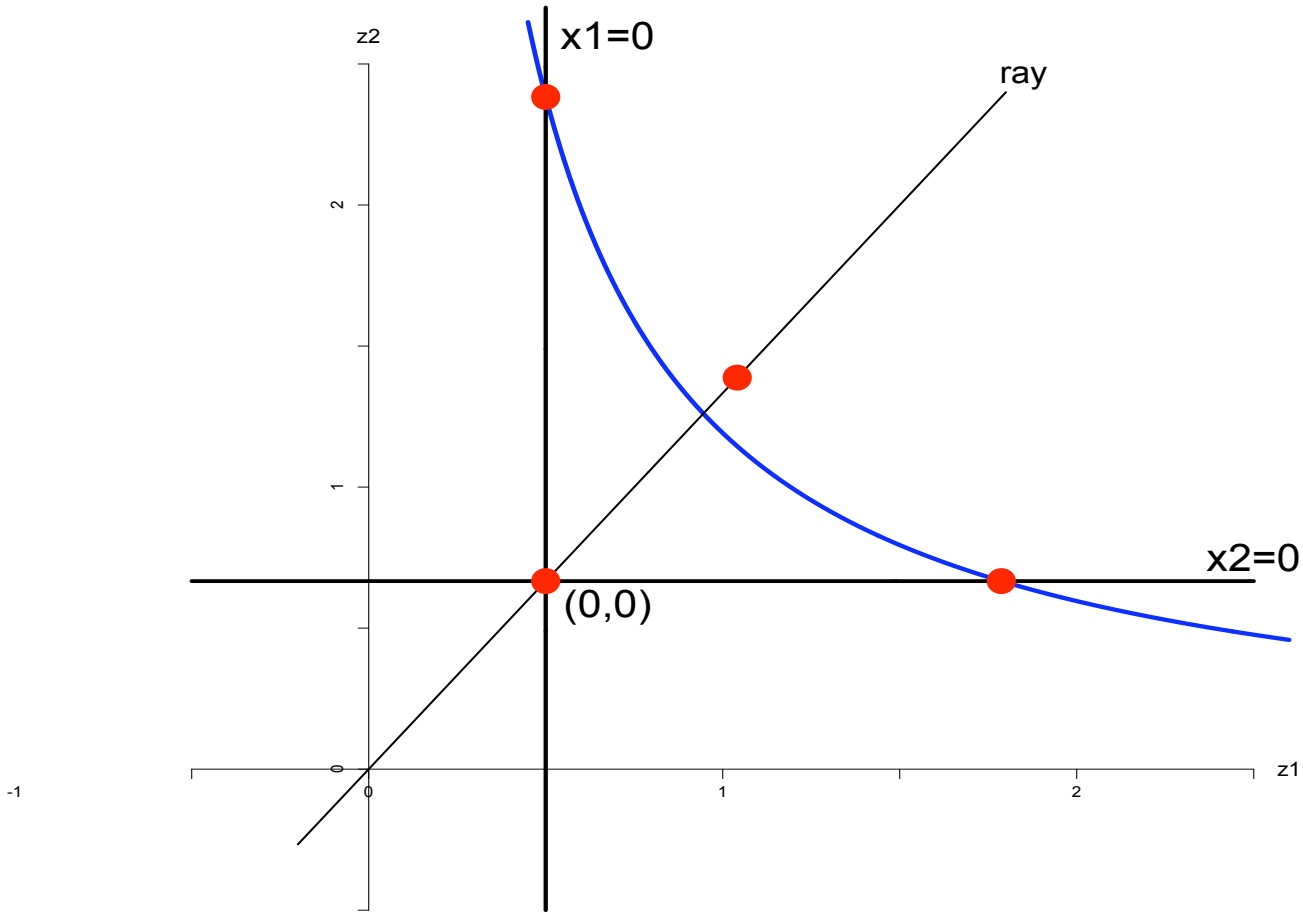

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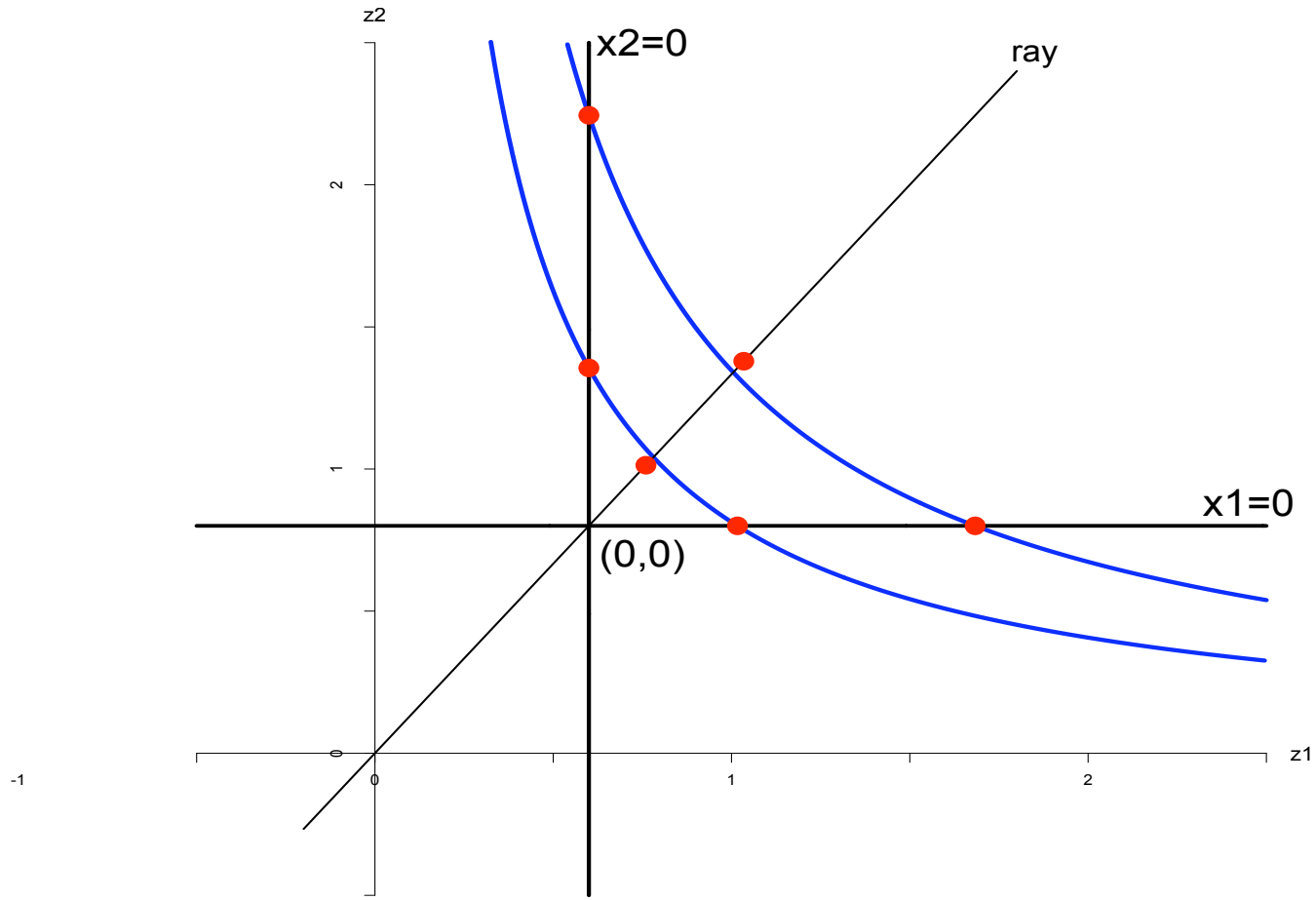

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- Proof?

Conclusions

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  - $\beta_{12} < 0$

Binary response in two variables

- Algebra and patterns $\longrightarrow$ proofs
- Taxonomy for $x_1 \geq 0, x_2 \geq 0$
  - $\beta_1 > 0$ and $\beta_2 < 0$
  - $\beta_{12} < 0$
- Further constraints on x_1 and x_2