TITLE

- FORMULATING and FITTING **COPULA** BASED LATENT VARIABLE MODELS for PAIRED COMPARISONS and RANKING STUDIES
  An application of
- OPTIMAL DESIGN THEORY

MOI

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MAIN CONTENTS

- I: Classes of Models
- II: Use of Multiplicative Algorithm
- for Model Fitting

CONTENTS

1 PAIRED COMPARISONS and RANKING STUDIES

- BINARY or DICHOTOMOUS OUTCOMES
- TREATMENTS or PRODUCTS
- $T_1, T_2, \ldots\ldots\ldots\ldots\ldots, T_J$.
-
- n_{ij} comparisons : T_i to T_j,
-
- T_i is preferred to T_j. O_{ij} times
-
- $O_{ij} + O_{ji} = n_{ij}$

Triplets

An extension of pairwise comparisons
is to invite subjects to place
three treatments T_i , T_j , T_k ,
in order of preference .

Triplets

- EXAMPLE
-
- DATA SET on 4 Treatments
- 40 Responses on each of 4 Triplets

- Pendergrass & Bradley

Triplet-1,2,3

Row	i	j	k	Oijk
1	1	2	3	10
2	1	3	2	8
3	2	1	3	8
4	2	3	1	6
5	3	1	2	4
6	3	2	1	4

Triplet-1,2,4

Row	i	j	k	Oijk
7	1	2	4	12
8	1	4	2	8
9	2	1	4	8
10	2	4	1	6
11	4	1	2	4
12	4	2	1	2

Triplet-1,3,4

Row	i	j	k	Oijk
13	1	3	4	10
14	1	4	3	8
15	3	1	4	8
16	3	4	1	8
17	4	1	3	4
18	4	3	1	2

Triplet-2,3,4

Row	i	j	k	Oijk
19	2	3	4	8
20	2	4	3	6
21	3	2	4	8
22	3	4	2	6
23	4	2	3	6
24	4	3	2	6

2.1 MODELS for PAIRED COMPS.

GENERAL MODEL

$O_{ij} \sim Bi(n_{ij}, \theta_{ij})$

$\theta_{ij} = P(T_i \text{ is preferred to } T_j)$

$\theta_{ij} + \theta_{ji} = 1$

BRADLEY TERRY MODEL

$\theta_{ij} = p_i/(p_i + p_j)$,

$p_i > 0.$

BRADLEY TERRY MODEL LATENT VAR. INTERP.

Let $\lambda_i = \ln(p_i)$

- $\theta_{ij} = \exp(\delta_{ij})/(1 + \exp(\delta_{ij})) = F(\delta_{ij})$
- $= \text{Logistic cdf}$
- $\delta_{ij} = \lambda_i - \lambda_j$

$$\theta_{ij} = F(\lambda_i - \lambda_j) = F\{\, \text{loge}(p_i/p_j)\}$$

Other LATENT VAR. Models

- Bradley-Terry: $F(\delta) = \exp(\delta)/(1 + \exp(\delta))$
- = logistic distribution function.
- THURSTONE MODEL : $F(\delta) = \Phi(\delta)$
- = normal distribution function
- IN GENERAL NEED SYMMETRIC F(.)
- $F(\delta) + F(-\delta) = 1$
- $\theta_{ij} + \theta_{ji} = F(\lambda_i - \lambda_j) + F(\lambda_j - \lambda_i) = 1$

Latent or Utility Scale

- So θ_{ij} uniquely determined by
-
- DIFFERENCE
- in quality characteristics λ_i, λ_j ,
-
- ------------+---------------+-------------------------
- λ_j λ_i

GENERATING MODEL

$X_1, X_2, \ldots\ldots, X_J$ (Utilities)

- $E(X_i) = \lambda_i$, $Var(X_i) = 1 =$ constant
-
- $Z_i = X_i - \lambda_i$ i.i.d
-
- $P(T_i$ is preferred to $T_j) = P(X_i > X_j)$
- $= P(Z_j - Z_i < \lambda_i - \lambda_j)$
- $= F(\lambda_i - \lambda_j)$

ON SYMMETRIZATION

- If $G_1(u)$ = arbitrary univariate cdf
- Symmetric cdfs are:
- $F_1(u) = \{G_1(u) + [1 - G_1(-u)]\}/2$
- **$F_1(u) = G_1(u)/[G_1(u) + G_1(-u)]$**
- If X has cdf $G_1(x)$, $Y = -X$ has cdf $[1- G_1(-y)]$

General Model : Paired Data

- θ_{ij} = P(T_i is preferred to T_j)
- = $G_1(u)/[G_1(u) + G_1(-u)]$ (A)
- = $1/[1 + \{G_1(-u)/G_1(u)\}]$ (B)
- u = $(\lambda_i - \lambda_j)$ = $\log_e(p_i/p_j)$
- $G_1(u)$ = cdf
- For Triplets we develop
- extensions of (A),(B)

2.2 Models forTriplets

Let

θ_{ijk} = P(T_i is preferred to T_j
& T_j is preferred to T_k)

= P($T_i > T_j > T_k$)

2.2.0 Specific Models TRIPLETS

- Various possible extensions of
- Bradley Terry Model include:
- i) Pendergrass and Bradley
-
- $\theta_{ijk} = (p_i)^2 p_j / D$,
- D =
- $(p_i)^2 p_j + (p_j)^2 p_i + (p_i)^2 p_k +$
- $(p_k)^2 p_i + (p_j)^2 p_k + (p_k)^2 p_j$
-
-

2.2.0 Specific Models TRIPLETS

- Has symmetry between ranking and inverse ranking i.e. between by "preference" and by "dislike" .
- 'Treatments' swap roles under $q_i = 1/p_i$.
- But
- $$\theta_{ij} = P(T_i > T_j) \neq p_i/(p_i + p_j)$$

2.2.0 Specific Models TRIPLETS

- ii) Plackett Luce:
-
- $\theta_{ijk} = p_i p_j / \{(p_i + p_j + p_k)(p_j + p_k)\}$
- Ensures:
- $\theta_{ij} = P(T_i > T_j) = p_i/(p_i + p_j)$
- $P(T_i = 1^{st}) = p_i/(p_i + p_j + p_k)$
- No symmetry

2.2.0 Specific Models TRIPLETS

- iii) Extension : Benter Model
-
- θ_{ijk} =
- $(p_i)^\alpha (p_j)^\beta / \{[(p_i)^\alpha + (p_j)^\alpha + (p_k)^\alpha][(p_j)^\beta + (p_k)^\beta]\}$
-
- α, β are positive dampening parameters.
-
- Should possibly satisfy $\alpha + \beta = 1$

General Models for Triplets

- We formulate models of the form:

$$\theta_{ijk} = P(T_i > T_j > T_k) = G_2(u,v)/D$$

- $D = G_2\{u,v\}+G_2\{u+v,-v\}+G_2\{-u,u+v\}+$
- $G_2\{v,-(u+v)\}+G_2\{-(u+v),u\}+G_2\{-v,-u\}$
- $u = (\lambda_i - \lambda_j)$, $v = (\lambda_j - \lambda_k)$, $\lambda_i = \ln(p_i)$

Extensions of Paired Formula (A)

- **θ_{ij} = P(T_i is preferred to T_j)**
- **$= G_1(u)/[G_1(u) + G_1(-u)]$ (A)**

2.2.1 Model-1: Bivariate for Triplets

- $\theta_{ijk} = P(X_i > X_j > X_k)$
- $= P(X_i - X_j > 0, X_j - X_k > 0)$
-
- $= P(Z_j - Z_i < \lambda_i - \lambda_j , Z_k - Z_j < \lambda_j - \lambda_k)$
- $= F_2\{(\lambda_i - \lambda_j),(\lambda_j - \lambda_k)\}$
- $F_2(x,y)$ = Joint cdf.
-

2.2.1 Model-1: Bivariate for Triplets

- Requirements:
- 1) $\sum\theta_{ijk} = 1$
- 2) Negative correlation   (? of -1/2)
-
- Desirable:
- 3) $\theta_{ij} = P(T_i \text{ is preferred to } T_j) = F_1(\lambda_i - \lambda_j)$
- 4) Symmetry between ranking and inverse ranking

2.2.1 Model-1: Bivariate for Triplets

- Implied Properties of $F_2(x,y)$:
-
- 4) is ensured given symmetry in x,y
- 3) points to common symmetric marginals
- of $F_1(z)$
- 2) negative correlation
-

2.2.1 Model-1: Bivariate for Triplets

- 1) requires:
-
- $D^* = F_2\{u,v\}+F_2\{u+v,-v\}+F_2\{-u,u+v\}+$
- $F_2\{v,-(u+v)\}+F_2\{-(u+v),u\}+F_2\{-v,-u\}$
- $= 1$
- $u = (\lambda_i - \lambda_j),\ v = (\lambda_j - \lambda_k)$

2.2.1 Model-1: Bivariate for Triplets

- True for $F_2(x,y)$ = Bivariate Normal
- i.e. $X_i \sim N(\lambda_i,1)$ iid
- Any two paired differences are
- Bivariate Normal,
- Variances = 2,Correlation = ±½

- Gold Standard

ON SYMMETRIZATION

- BIVARIATE:
-
- $G_2(z_1,z_2)$ = arbitrary joint cdf
- Common Marginal cdf = $G_1(z)$
-
- SYMMETRIC cdf in z_1 , z_2 :
- $F_2(z_1,z_2) = [G_2(z_1,z_2) + G_2(z_2,z_1)]/2$

ON SYMMETRIZATION

- SYMMETRIC cdf in Origin:
-
- $F_2(z_1,z_2) =$
- $\{G_2(z_1,z_2) + [1 - G_1(-z_1) - G_1(z_2) + G_2(-z_1,-z_2)]\}/2$
-
- $[1 - G_1(-z_1) - G_1(z_2) + G_2(-z_1,-z_2)] =$
- joint cdf of $(-Z_1,-Z_2)$

COPULA Choice of $G_2(u,v)$

$G_2(u,v) \quad = \; H\{F_1(u), F_1(v)\}$

- Marginal cdf's are $F_1(u), F_1(v)$

- $H(r,s)$ = Copula
- $H(r,s)$ = joint cdf on unit square $0<r,s<1$
- $H(r,1) = r$, $H(1,s) = s$
- {Assume symmetry: $H(r,s) = H(s,r)$}

COPULA Choice of $G_2(u,v)$

- e.g. Plackett Copula
- $H(r,s) = \{(1+(\psi-1)(r+s)) -$
- $\sqrt{[}\{(1+(\psi-1)(r+s))^2 - 4\psi(\psi-1)rs]\}/2(\psi-1)$
-
- $\psi < 1$ implies negative association

  $\psi > 1$ implies positive association

Model-1 Bivariate for Triplets

- Possible Choice for
- $\theta_{ijk} = F_2\{(\lambda_i - \lambda_j),(\lambda_j - \lambda_k)\}$
- Let $G_2(u,v) = H\{F_1(u), F_1(v)\}$
- $F_2(u,v) = G_2(u,v)/D$
- $D = G_2\{u,v\}+G_2\{u+v,-v\}+G_2\{-u,u+v\}+$
- $G_2\{v,-(u+v)\}+G_2\{-(u+v),u\}+G_2\{-v,-u\}$

Model-1 Bivariate for Triplets

- 'Marginals' are:
-
- $H_1(u) = F_1(u)/[F_1(u) + F_1(-u)]$,
- $H_1(v) = F_1(v)/[F_1(v) + F_1(-v)]$
-

2.2.2 Model-2: Best-Worst

- Placket Luce Model is
- $\theta_{ijk} = p_i p_j / \{(p_i + p_j + p_k)(p_j + p_k)\}$
- $\quad = [p_i/(p_i + p_j + p_k)][p_j/(p_j + p_k)]$
- $\quad = P(C)P(E|C)$
- C = ‘T_i = 1^{st} Preference’
- E = ‘T_j prefered to T_k‘
- (In fact E,C (conditionally) independent)

2.2.2 Model-2: Best-Worst

- $\theta_{ijk} = P(X_i > \{X_j, X_k\})P(X_j > X_k)$
- $= P(X_i - X_j > 0, X_i - X_k > 0)P(X_j - X_k > 0)$

2.2.2 Model-2: Best-Worst

- In fact
- $\theta_{ijk} = F_2\{(\lambda_i - \lambda_j),(\lambda_i - \lambda_k)\}F_1\{(\lambda_j - \lambda_k)\}$
- $=F_2(u,u+v)\ F_1(v)$
- $=M_2(u,v)$,
- with F_1,F_2 being univariate & bivariate Logistic cdf's
- $F_2(x,y)$ symmetric in x,y

2.2.2 Model-2: Best-Worst

- Possible Choice for $M_2(u,v)$
- Let $G_2(u,v) = H\{F_1(u),F_1(u+v)\}F_1(v)$
- $M_2(u,v) = G_2(u,v)/D$

2.2.3 Model-3: Worst-Best

- $\theta_{ijk} = P(X_k<\{X_j, X_i\})P(X_j<X_i)$
- $= P(X_i-X_k>0, X_j-X_k>0)P(X_i-X_j>0)$
- $=F_2(u+v,v)\ F_1(u)$
- $=R_2(u,v)$

2.2.3 Model-3: Worst-Best

- Possible Choice for $R_2(u,v)$
- Let $G_2(u,v) = H\{F_1(u+v),F_1(v)\}F_1(u)$
- $R_2(u,v) = G_2(u,v)/D$

2.2.4 Models-4: Possible Benter Extensions

- 4a Best-Worst
- $G_2(u,v) \;=\; H\{[F_1(u)]^\alpha,[F_1(u+v)]^\alpha\}[F_1(v)]^\beta$
- 4b Worst-Best
- $G_2(u,v) \;=\; H\{[F_1(u+v)]^\beta,[F_1(v)]^\beta\}[F_1(u)]^\alpha$

Extensions of Paired Formula (B)

- θ_{ij} = P(T_i is preferred to T_j)
- $= 1/[1 + \{G_1(-u)/G_1(u)\}]$ (B)

- **Note: Bradley Terry Model:**
- $\theta_{ij} = 1/(1+y)$,
- $y = p_j / p_i = \{G_1(-u)/G_1(u)\}$
- $G_1(u)$ = Logistic

2.2.5 Model-5: Pendergrass Bradley Generalised

- $\theta_{ijk} = F_2(u,v) = G_2(u,v)/D$

- $G_2(u,v) = 1/[1+y+z+y^2z+yz^2+y^2z^2]$,

- $y = F_1(-u)/ F_1(u)$, $z = F_1(-v)/ F_1(v)$
- $F_1(.)$ = univarite cdf
- (logistic for Pend. Brad.)

2.2.6 Models-6:
Plackett Luce Generalised

- $\theta_{ijk} = F_2(u,v) = G_2(u,v)/D$
- (a) Best-Worst
- $G_2(u,v) = 1/[(1+y+yz)(1+z)]$,
- (b) Worst-Best
- $G_2(u,v) = 1/[(1+z+yz)(1+y)]$,

- $y = F_1(-u)/ F_1(u)$, $z = F_1(-v)/ F_1(v)$
- $F_1(.)$ = univarite cdf
- (logistic for Plack. Luce)

2.2.7 Models-7: Benter Generalised

- $\theta_{ijk} = F_2(u,v) = G_2(u,v)/D$
- (a) Best-Worst
- $G_2(u,v) = 1/[(1+y^{\alpha}+y^{\alpha}z^{\alpha})(1+ z^{\beta})]$,
- (b) Best-Worst
- $G_2(u,v) = 1/[(1+z^{\beta} +y^{\beta}z^{\beta})(1+ y^{\alpha})]$,

- $y = F_1(-u)/ F_1(u)$, $z = F_1(-v)/ F_1(v)$
- $F_1(.)$ = univarite cdf
- (logistic for Benter)

2.2.8 Further points
Models 1-7: ‘Marginal’ cdf’s

- $F_2(u,v)$ has ‘marginal cdf’s’ as follows:
- Models 1,2,3,5,6a,6b:
- $F_1(w)/ [F_1(w)+F_1(-w)]$, $w = u,v$
- Models 4a,4b,7a,7b:
- $\{F_1(u)\}^\alpha/[\{F_1(u)\}^\alpha+\{F_1(-u)\}^\alpha]$
- $\{F_1(v)\}^\beta/[\{F_1(v)\}^\beta+\{F_1(-v)\}^\beta]$

Models Summarised

- $\theta_{ijk} = F_2(u,v) = G_2(u,v)/D$

- $D = G_2\{u,v\}+G_2\{u+v,-v\}+G_2\{-u,u+v\}+$
- $G_2\{v,-(u+v)\}+G_2\{-(u+v),u\}+G_2\{-v,-u\}$

Models Summarised

$G_2(u,v)$	Model
$H\{F_1(u),F_1(v)\}$	1
$H\{F_1(u),F_1(u+v)\}F_1(v)$	2
$H\{F_1(u+v),F_1(v)\}F_1(u)$	3
$H\{[F_1(u)]^\alpha,[F_1(u+v)]^\alpha\}[F_1(v)]^\beta$	4a
$H\{[F_1(u+v)]^\beta,[F_1(v)]^\beta\}[F_1(u)]^\alpha$	4b

Models Summarised

$G_2(u,v)$	Model
$1/[1+y+z+y^2z+yz^2+y^2z^2]$	5
$1/[(1+y+yz)(1+z)]$	6a
$1/[(1+z+yz)(1+y)]$	6b
$1/[(1+y^\alpha+y^\alpha z^\alpha)(1+ z^\beta)]$	7a
$1/[(1+z^\beta +y^\beta z^\beta)(1+ y^\alpha)]$	7b

$y = F_1(-u)/ F_1(u)$, $z = F_1(-v)/ F_1(v)$

Symmetrising or Not!

- Symmetric models are:
- Averages of ‘$G_2(u,v)/D$’-Ratios of Models
- 2 & 3 or 6a & 6b or 7a & 7b
- OR
- Ratios of corresponding Averages

- Assymmetric Models are corresponding
- Convex Combinations

Flexible Classes of Models

- Copulae H(.,.),
- parameters ψ, α, β,
- convex weights
- create a flexible class of models

Induced Models

- Given a model for a Triplet Ranking we
- Can derive models for:
- First Preference (Best)
- Least Preference (Worst)
- (Middle Preference!)

3.1 Model Fitting

- All models could be transformed to generalised linear models via e.g.
- $p_i = \exp(\underline{\beta}^T \underline{x}_i)$
- (with scaling constraints needed on $\underline{\beta}$)
- So class of design problems created.
- We focus on direct estimtion of the p_i's.

3.1 Model Fitting

- In all models:
- $u = (\lambda_i - \lambda_j)$, $v = (\lambda_j - \lambda_k)$
- $u = \log e(p_i/p_j)$, $v = \log e(p_j/p_k)$
- $u = \log e(cp_i/cp_j)$, $v = \log e(cp_j/cp_k)$
- u,v invariant to proportional changes in p_i's

3.1 Model Fitting

- u,v and hence θ_{ijk} and (log-)likelihood are homogeneous functions of degree zero in the p_i's.
- A constraint is needed
- One possibility: $\sum_i p_i = 1$ (or $\prod_i p_i = 1$)
- Given $p_i > 0$, the p_i's are like weights.

3.2 PROBLEM (P)

Maximise $\phi(p)$ subject to

$p_i \geq 0,$

$\sum p_i = 1.$

p is a probability distribution.

3.3 OPTIMALITY CONDITIONS

- Let $d_j = \partial\phi/\partial p_j$
- (= $F_j = d_j - \sum p_i d_i$
- = Dircetional Derivative, given $\sum p_i d_i = 0$)
- Above Likelihoods : $p_j^* > 0$
- If $\phi(.)$ differentiable at p^*, necessary condition for $\phi(p^*)$ to be local maximum :
- $d_j^* = 0$

4.1 MULTIPLICATIVE (FIXED POINT) ALGORITHMS

- Let $f(d,\delta)$ be a function satisfying (for $\delta \geq 0$):
-
- $f(d,\delta) > 0$,
- $\partial f(d,\delta)/\partial d > 0$ (for $\delta \geq 0$),
- $f(d,0) =$ constant
-
- (e.g. $f(d,\delta) = \Phi(\delta d)$ or $f(d,\delta) = d^{\delta}$ (if $d > 0$))

4.1 MULTIPLICATIVE (FIXED POINT) ALGORITHMS

- Algorithm:
-
- $p_j^{(r+1)} = p_j^{(r)} f(d_j^{(r)},\delta) / \sum p_i^{(r)} f(d_i^{(r)},\delta)$
- ((OR) $p_j^{(r+1)} = p_j^{(r)} f(F_j^{(r)},\delta) / \sum p_i^{(r)} f(F_i^{(r)},\delta)$)
- Ensures $p_j^{(r)} > 0$, $\sum p_i^{(r)} = 1$
-
- (OR $p_j^{(r)} f(d_j^{(r)},\delta) / \{\prod p_i^{(r)} f(d_i^{(r)},\delta)\}^{(1/J)}$
- Ensures $p_j^{(r)} > 0$, $\prod p_i^{(r)} = 1$)

PROPERTIES:

-
- (i) $p^{(r)}$ is always feasible.
-
- (ii) $F_\phi\{p^{(r)},p^{(r+1)}\} \geq 0$, with equality when the d_j's corresponding to nonzero p_j's have a common value d ($= \sum p_i d_i$), in which case $p^{(r)} = p^{(r+1)}$.
-

PROPERTIES:

- (iii) If $\delta = 0$ there is no change in $p^{(r)}$,
- given $f(d,\delta)$ = constant
- (iv) So the algorithm should be monotonic for small positive δ.
-
- (v) An iterate $p^{(r)}$ is a fixed point of the iteration if derivatives $d_j^{(r)}$ corresponding to nonzero $p_j^{(r)}$ are equal; i.e. if corresponding vertex directional derivatives $F_j^{(r)}$ are zero.

PROOF of (ii)

$F_{\phi}\{p^{(r)},p^{(r+1)}\} = \mathrm{Cov}\{D,f(D)\}/E\{f(D)\}$

- D = Random Variable : $P(D = d_j^{(r)}) = p_j^{(r)}$
-
- $\mathrm{Cov}\{D,f(D)\} > 0$
- if $\partial f(D)/\partial D > 0$;
- and $E\{f(D)\} > 0$
- if $f(D) > 0$

4.2 CHOICE of f(.)

- We choose
- f(.) = Φ(.) = Standard Normal cdf.
- So iterations are:
- $p_j^{(r+1)} = p_j^{(r)}\Phi(\delta d_j^{(r)})/\sum p_i^{(r)}\Phi(\delta d_i^{(r)})$
- Suitable δ:
- $\delta = \alpha(1/N)$ for $\phi(p)$,
- $N = \sum\sum O_{ij}$, $0< \alpha<1$
- (i.e. $\delta \sim \alpha$ for $\{\phi(p)/N\}$)

Initial Iterate

- $p_j^{(0)} = 1/4$, $j = 1,2,3,4$

4.3 Models Fitted and Results

- Fit to Full Triplet Ranking:
- Bivariate & Best-Worst
- Plackett Copula:
- Marginals: Logistic & Normal
- Ψ = .1,.3,.5,.7,.9,2,4,6,8,10

4.3 Models Fitted and Results

- Fit to Full Triplet Ranking:
- Worst-Best
- Plackett Copula:
- Marginals: Logistic & Normal
- $\Psi = .5, 2$

4.3 Models Fitted and Results

- Fit to First Preference:
- Bivariate & Best-Worst & Worst-Best
- Plackett Copula:
- Marginals: Logistic & Normal
- $\Psi = .5, 2$

4.3 Models Fitted and Results

- Convergence in all but 1 case to
- `d`

```
0.0000000
0.0000000
0.0000000
0.0000000
```

- with $\alpha = 1/2$

4.3 Models Fitted and Results

- Exception:
- Best-Worst Model
- Normal
- $\Psi=0.1$
- Oscillation realised.
- Convergence for $\alpha=1/20=(1/2)/10$

Further Results

- Models: Original
- Pendergrass Bradley & Plackett Luce
- Fit To Full Triplet Ranking;
- Fit First Preference (Best)
- Fit Least Preference (Worst)

Further Results

- Convergence in all but 1 case to
- d

  0.0000000

  0.0000000

  0.0000000

  0.0000000
- with $\alpha = 1$

Further Results

- Exception:
- Pendergrass Bradley Model
- Full Triplet Ranking
- Divergence
- Convergence for $\alpha=1/2$

Performance of Algorithm

- Fast initial convergence
- Number of iterations required to converge
- ranged from < 100 to < 500

5 Other Problems
5.1 Complete Orderings

- Gormley and Murphy (2005/6) fit extensions of both the Plackett Luce and Benter models to Irish voting data with at least 9 candidates
-
-
- They also fit mixtures of these models.
- So several sets of parameter vectors $(p_1, p_2, \ldots\ldots, p_J)$
- plus mixing weights.
-

Complete Orderings

- Further scope for work on latent variable models
- based on Multivariate Distributions
-
- $\theta_{ijklmn} = F(\lambda_i - \lambda_j, \lambda_j - \lambda_k, \lambda_k - \lambda_l, \lambda_l - \lambda_m, \lambda_m - \lambda_n)$
- F(.,.,.,.,.) = Joint CDF
- $\lambda_i - \lambda_j = \log e(p_i/p_j)$

5.2 TREATMENTS with a FACTORIAL STRUCTURE

- COFFEE EXAMPLE
-
- Three Factors in Coffee production :
-
- α : Brew Strength ; K = 2 Levels
-
- β : Roast Colour : L = 2 Levels
-
- γ : Coffee Brand : M = 2 Levels
-

5.2 FACTORIAL STRUCTURE

- Bradley Terry Models:
- $p_i \quad \rightarrow \quad p_{klm} = \alpha_k \beta_l \gamma_m$
-
- $\ln(p_{klm}) = \ln(\alpha_k) + \ln(\beta_l) + \ln(\gamma_m)$
- where $\alpha_k, \beta_l, \gamma_m > 0$.
-
- Likelihood is again a homogeneous function of degree zero in each of the three sets of main effect parameters.

5.2 FACTORIAL STRUCTURE

- Suitable constraints:
-
- $$\sum\alpha_k = \sum\beta_l = \sum\gamma_m = 1$$
-
- $f_\alpha(.), f_\beta(.), f_\gamma(.)$ positive increasing
-
-
-

5.2 FACTORIAL STRUCTURE

- Optimality Conditions are:

- $F_k^{(\alpha)} = d_k^{(\alpha)} = 0$
- $F_l^{(\beta)} = d_l^{(\beta)} = 0$
- $F_m^{(\gamma)} = d_m^{(\gamma)} = 0$
- $d_k^{(\alpha)} = \partial\phi/\partial\alpha_k$ at $\alpha = \alpha^{(r)}$ etc.
- $F_k^{(\alpha)}$ = Corresponding Directional Derivative

5.2 FACTORIAL STRUCTURE

- Suitable set of iterations are:
-
- $\alpha_k^{(r+1)} = \alpha_k^{(r)} f_\alpha(d_k^{(\alpha)}) / \sum \alpha_t^{(r)} f_\alpha(d_t^{(\alpha)})$
- $\beta_l^{(r+1)} = \beta_l^{(r)} f_\beta(d_l^{(\beta)}) / \sum \beta_t^{(r)} f_\beta(d_t^{(\beta)})$
- $\gamma_m^{(r+1)} = \gamma_m^{(r)} f_\gamma(d_m^{(\gamma)}) / \sum \gamma_t^{(r)} f_\gamma(d_t^{(\gamma)})$
-
- $f_\alpha(.)$, $f_\beta(.)$, $f_\gamma(.)$ positive increasing

5.2 FACTORIAL STRUCTURE

- Optimality Conditions are:
-
- $F_k^{(\alpha)} = d_k^{(\alpha)} = 0$
-
- $F_l^{(\beta)} = d_l^{(\beta)} = 0$
-
- $F_m^{(\gamma)} = d_m^{(\gamma)} = 0$
-
- $d_k^{(\alpha)} = \partial\phi/\partial\alpha_k$ at $\alpha = \alpha^{(r)}$ etc.

5.2 FACTORIAL STRUCTURE

- Taking $\delta = 1/N = 1/728 = q(1/N)$, $q=1$
- $f_\theta(d^{(\theta)},\delta) = \Phi(\delta d^{(\theta)})$ and
- $\theta_j^{(0)} = ½$, $j = 1,2$, for $\theta = \alpha, \beta, \gamma$,
- numbers of iterations needed to achieve
- $\max|d_j^{(\theta)}| = \max|F_j^{(\theta)}| \leq 10^{-n}$, $n = 0,1,\ldots\ldots,7$
- are 7, 12,15,19, 23,27, 31, 36.

- Iterations were monotonic.
- Extends to Interactions

5.3 Multivariate Data

- Comparisons in respect of several attributes.
-
- Models depend on several sets of ‘weights’
- or Bradley Terry parameters, one for each attribute
- and other parameters.
-
- Case of 2 attributes and no ties:
-

5.3+ Bivariate Data

- Need joint cdf $F_2\{x,y\}$ satisfying
-
- $$\sum\sum F_2\{\pm u,\pm v\} = 1$$
-
- $$u = \lambda_i - \lambda_j = \text{loge}(p_i/p_j)$$
- $$v = \tau_i - \tau_j = \text{loge}(q_i/q_j)$$
-

5.3+ Bivariate Data

- e.g.
- $F_2(u,v) = G_2(u,v)/D$
- $D = \sum\sum G_2\{\pm u,\pm v\}$
-
- OR
-
- $F_2(u,v) = \{ G_2(u,v) + G_1(u) - G_2(u,-v)$
- $+ G_1(v) - G_2(-u,v)$
- $+ [1- G_1(-u) - G_1(-v) +G_2(-u,-v)]\}/4$
-

5.4 Several Distribution Problems

- NOTE
- Other problems involving several distributions are:
- determining optimal conditional designs
- Transition matrices for Markov Chains
- e.g. find monthly transitions rates between HIV states, given 6 monthly rates.

5.5 Other Problems (P)

a----x_1---- x_2-------------------- x_n----b

- Problem: choose x_i‘s optimally in [a,b]
- x_i‘s = design points, cutpoints,
- knots in spline regressions,
- roots of a function.

5.5 Other Problems (P)

- Let $a=x_0$, $b=x_{n+1}$,
- $w_i = (x_i-x_{(i-1)})/(b-a)$ (Linear)
- $w_i = \geq 0$,
- $\sum w_i = 1$
- Choose w_i's optimally

5.6 GLM's

- Generalised Linear Models
- One class:
- $p_i = \exp\{ \underline{x}_i^T \log_e(\underline{\beta})\} = p_i(\underline{\beta})$, $\underline{\beta} > \underline{0}$
- Suppose $\underline{1}^T \underline{x}_i = 1$ for all $i = 1, 2, \ldots\ldots, J$
- e.g. $\underline{x}_i$ = mixture (product specific)

5.6 GLM's

- p_i is homogeneous of degree 1 in $\underline{\beta}$
- p_r / p_s is homogeneous of degree 0 in $\underline{\beta}$
- For MLEs need constraint: e.g. $\underline{1}^T\underline{\beta} = 1$

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