Design of Networked Experiments

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In this work, we investigate how the structure of relationships between subjects affects the design of experiments on these subjects.

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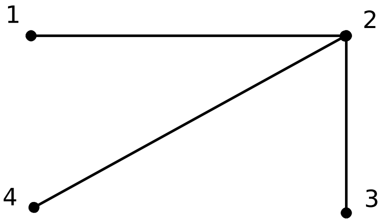

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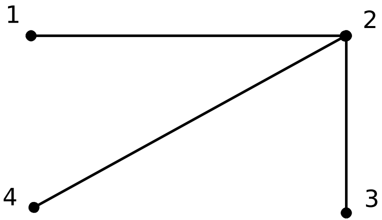

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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If a subject is exposed to marketing, their friend may see this marketing and their response may be altered (positively or negatively). We assume that if a relationship exists between two individuals, the response of the first subject is dependent on the treatment applied to the second.

A less trivial example

The nodes may be subjects in a public health experiment. The links may be some friendship relationship, some geographical relationship, or some familial relationship. The treatments may be different interventions in public health (e.g. "Eat five fruit or vegetables a day") and the response the change in the number of fruits or vegetables eaten after the campaign.

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Optimal design without any network effect

Model

$$Y_i = \mu + \tau_j + \epsilon_i$$

where ϵ_i are i.i.d normally with mean 0, and variance σ^2, and τ_j are the treatment effects for applying treatment j to subject i, $j = 1, \ldots, m$. Let $\tau_m = 0$ for uniqueness.

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For example, we minimise the average variance of all estimates of pairwise differences of treatment effects,

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This is A-optimality for estimating the difference in treatment effects.
For $m = 2$ treatments, it is clear that the balanced design with an equal number of subjects given each treatment is optimal.

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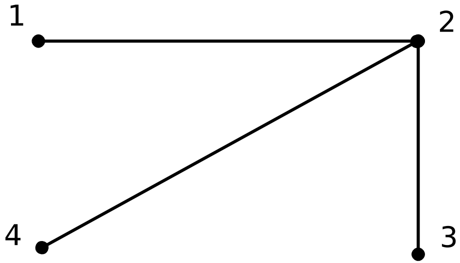

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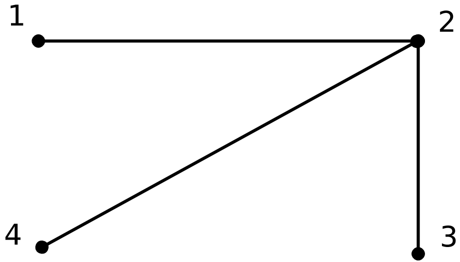

For example, if we apply treatment 1 to subject 2, there will be a network effect of γ_1 on subject 1

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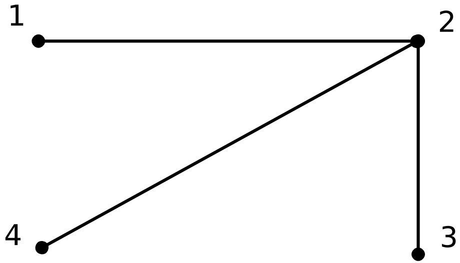

For example, if we apply treatment 1 to subject 2, there will be a network effect of γ_1 on subject 1 as well as the standard subject effect on subject 1 from its own treatment.

Optimal design with network effect

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Linear network effect model

$$Y_i = \mu + \tau_j + \sum_{k=1} A_{ik}\gamma_{l(k)} + \epsilon_i$$

where $l = l(k)$ is the treatment given to subject k, and γ_l is the corresponding network effect, which is the change in the behaviour on a subject due to giving a connected subject a particular treatment.

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It is no longer clear that balanced designs are A-optimal for estimating the difference in treatment effects.

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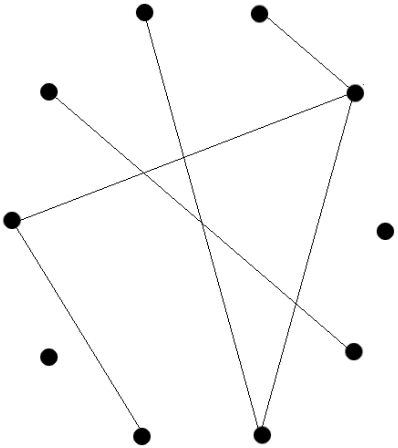

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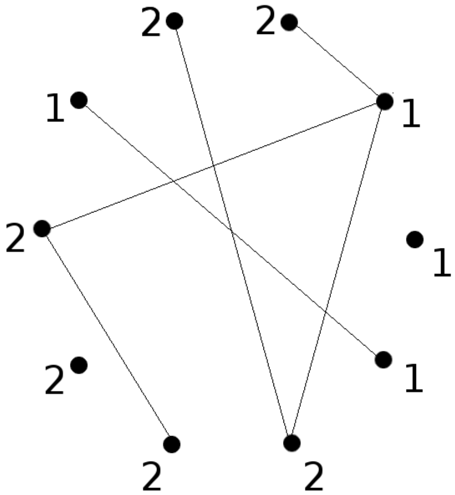

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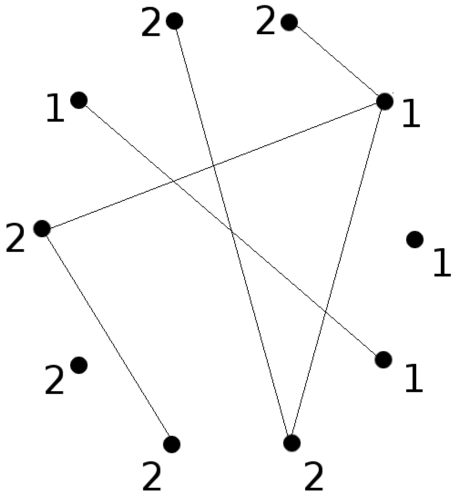

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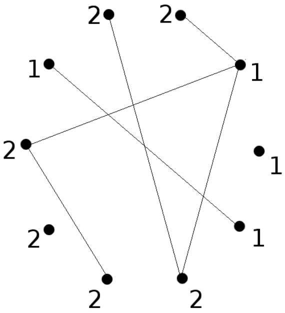

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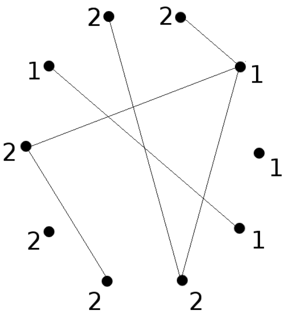
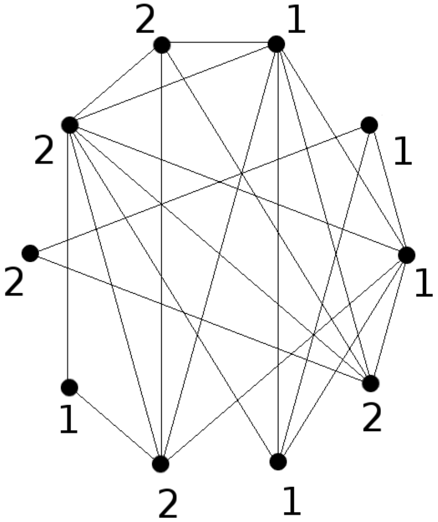


By performing exhaustive search over all possible designs, we find that the optimal designs for the $m = 2$ two treatment case is :

- $\{1, 1, 2, 2, 2, 1, 2, 1, 1, 1\}$ for optimality in estimating the difference in the subject effects (i.e we give treatment 1 to subjects 1,2,6,8,9, and 10 and treatment 2 to the other subjects).
- $\{2, 2, 1, 1, 1, 1, 2, 2, 2, 1\}$ for optimality in estimating the difference in the network effects γ_i.

Example 2



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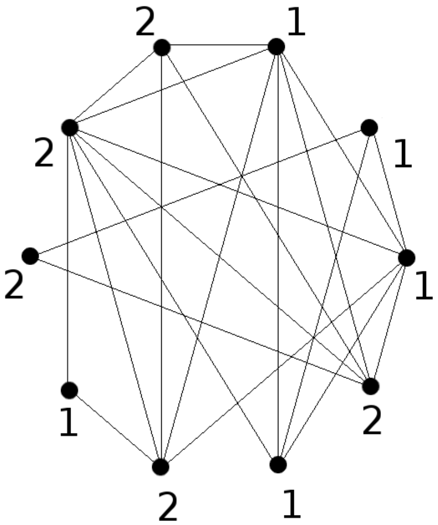

A-optimal criterion: 0.406

Neglecting the network effect

Let us imagine our linear network effect model

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- This is not the case for our model.

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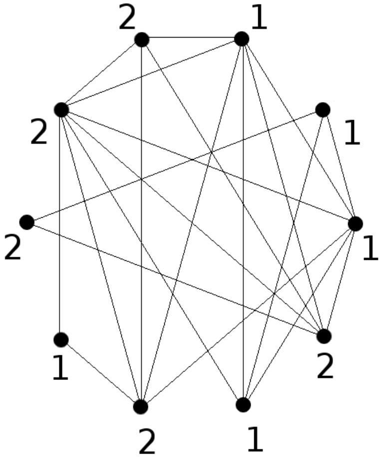

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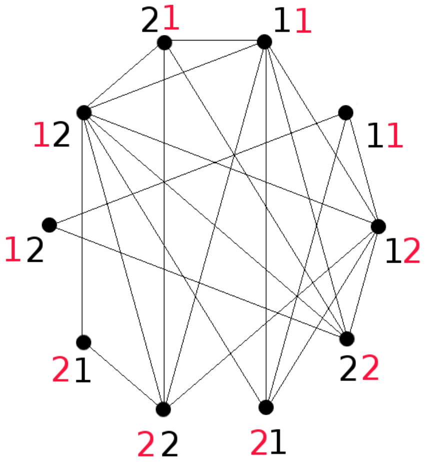

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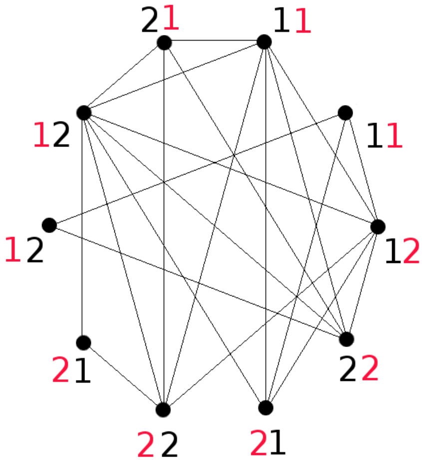

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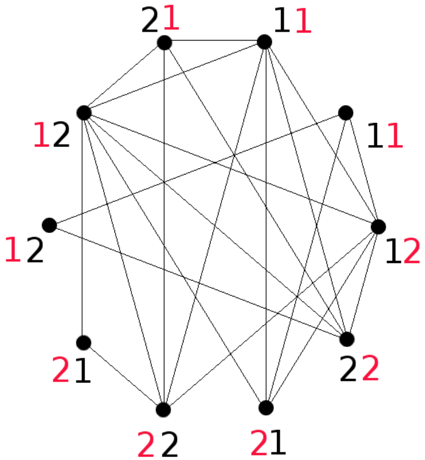

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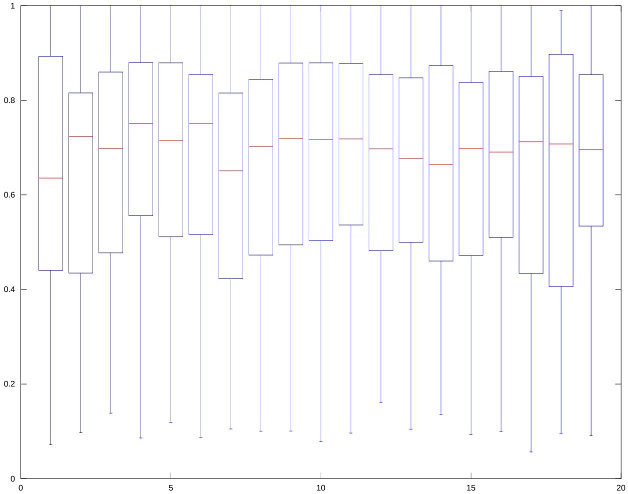
A-criterion for design 1,1,1,1,1,2,2,2,2,2 : 0.430 (Efficiency : 94.3%)

Boxplot of efficiencies of balanced designs for 20 random networks



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Writing R for the (reduced) model without network effects, and C for that with network effects, the bias in the subject and network effects β caused by neglecting a network effect in a model can be shown to be

$$\mathrm{E}\,(\hat{\beta}_R - \hat{\beta}_C) = \left[\begin{pmatrix} (F_R^T F_R)^{-1} F_R^T \\ 0_{m \times n} \end{pmatrix} - (F_C^T F_C)^{-1} F_C^T \right] F_C \beta_C,$$

where F is the extended design matrix for our experiment.

Bias

e.g. The bias in the two examples seen so far, for $\beta^T = (\tau_1 \quad \tau_2 \quad \gamma_1 \quad \gamma_2)$ are

$$\begin{pmatrix} 0 & 0.8 & 1.2 & 0.2 \\ 0 & -1.4 & -1 & 0.6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \beta$$

and

$$\begin{pmatrix} 0 & 0.6 & 2.4 & 2.2 \\ 0 & -1.2 & -0.4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \beta$$

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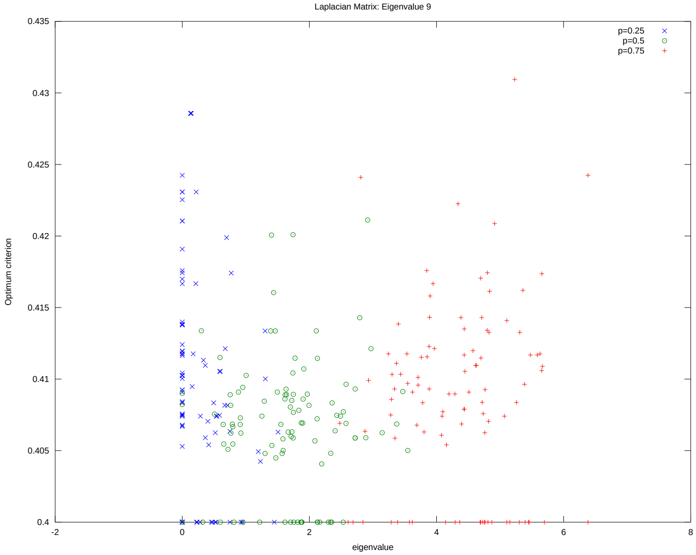
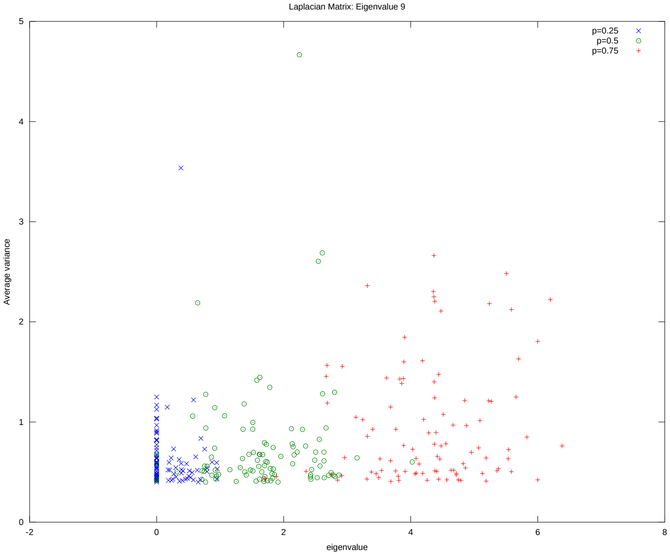
These spectra are a way of summarising some important macroscopic properties of the graph; for example, the second smallest eigenvalue of the Laplacian matrix is the algebraic connectivity of our network. This gives a measure of how well connected the overall graph is.

Algebraic connectivity for optimal design



Algebraic connectivity for an arbitrary balanced design



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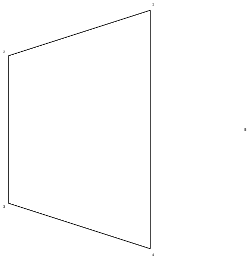

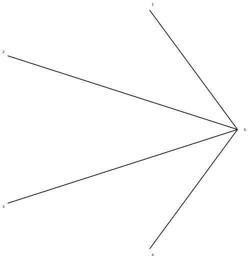

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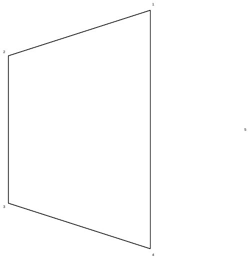
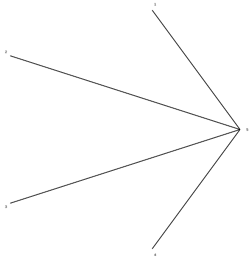




These graphs are clearly very different, although have the same spectrum (-2,0,0,0,2). However, the left hand graph has optimal design for estimating the treatment effects $\{1, 2, 2, 2, 1\}$ with value 2, whereas for the right-hand graph all factors are not estimable; there is no optimal design, or rather all designs are equally ineffective.

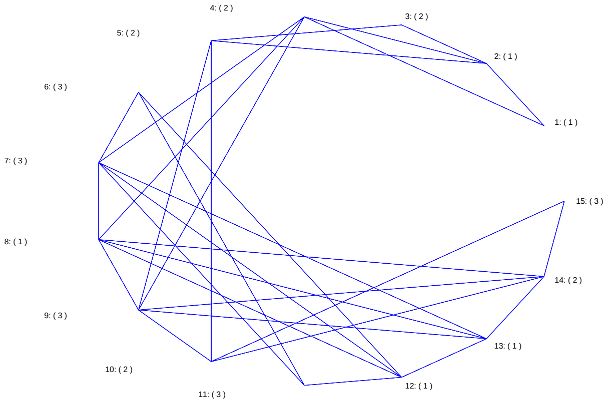
Strange shaped fields

Most field trial experiments focus on rectangular fields. Let us assume we have an irregularly shaped field divided into plots as follows:

A field layout

		1	2	3
		4		5
6	7	8	9	10
11	12	13	14	15

Example 3: Non-rectangular field trials



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- The A-optimal design function value is 0.4055.
- Minimum average variance possible with no restriction on comparisons $0.4\sigma^2$.
- It is interesting to note how close we can get to the variance for the unrestricted case with this unusual shape of field (0.4/0.4055=98.6% efficiency).

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Usually in a crossover trial, including a planned drop out in the design be tricky. Let us assume we are performing a crossover trial

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We can represent our network as below, with links now unidirectional

Crossover Trial

```
a:  1-> 2-> 3->4
b:  5-> 6   x    7
c:  8-> 9->10->  11
d: 12->13->14->15
```

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This has optimality criterion 0.4128 (efficiency of 96.9% of the unrestricted design with 15 treatments and no drop out).

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- We would very much like to hear of any experimental work in this area.