The Specification of Robust Binary Block Designs Against the Loss of Whole Blocks

Janet Godolphin
j.godolphin@surrey.ac.uk

Department of Mathematics
University of Surrey

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Description of Problem

Planned Design $D = \mathsf{BD}(v, b, n, N)$

D is a binary block design in which v treatments are allocated to n experimental units arranged in b blocks with the $v \times b$ treatment-block incidence matrix N.

b_* = number of blocks missing from D due to unforseen accident, damage or other random event in the field.

Eventual Design $D_{\#,b_*} = \mathsf{BD}(v, b - b_*, n_\#, N_\#)$

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The problem is to find conditions on the parameters of D which ensure that $D_{\#,b_*}$ is a connected design.

Background

Ghosh (1982) introduced the concept of maximal robustness:

D is described as being maximally robust if $D_{\#,r_0-1}$ is a connected design for every set of $r_0 - 1$ blocks, where r_0 is the minimum treatment replication.

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Related Work

Baksalary and Tabis (1987)

Sathe and Satam (1992)

Godolphin and Warren (2011)

Resolvable Designs

Theorem Let D be a resolvable block design with r replicates, such that the number of blocks in the ith replicate is s_i where $s_i \geq 2$ for $i = 1, 2, \ldots, r$.

If each block in the ith replicate has a treatment in common with more than $\frac{1}{2}s_j$ blocks in the jth replicate for each $i, j = 1, 2, \ldots, r; i \neq j$, then D is maximally robust.

Proof Outline $D_{\#,r-1}$ contains at least one complete replicate from D and at least one further replicate which has lost no more than one block. Label these replicates 1 and 2 in D.

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S_1 and S_2 each contain at least one block from replicate 1, blocks 1, 2 say. Block 1 has a common treatment with at least one block in the remnant of replicate 2. But this block has a common treatment with more than $\frac{1}{2}s_1$ blocks in replicate 1, therefore S_1 contains more than $\frac{1}{2}s_1$ blocks from replicate 1.

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From a similar argument, S_2 contains more than $\frac{1}{2}s_1$ blocks from replicate 1. Consequently, disjoint sets S_1, S_2 do not exist, which implies that $D_{\#,r-1}$ is connected.

Affine-resolvable Designs

Affine-resolvable Designs satisfy the condition of Theorem 1.

Example 1 Four replicates of twelve treatments are arranged in eight blocks of size 6:

1	3	2	1	1	2	2	1
2	7	3	4	3	5	4	3
4	8	5	8	4	9	5	6
5	9	6	9	6	10	7	10
6	11	7	10	7	11	8	11
10	12	11	12	8	12	9	12

Bailey, Monod and Morgan (1995)

Rectangular Lattice Designs

Rectangular Lattice Designs satisfy the condition of Theorem 1.

Example 2 Three replicates of twelve treatments are arranged in twelve blocks of size 3:

0	3	6	9	3	0	1	2	4	2	0	1
1	4	7	10	6	7	4	5	8	6	5	3
2	5	8	11	9	10	11	8	10	11	9	7

John and Williams (1995)

Concurrence Conditions

The concurrence matrix of D is NN'.

Definition L_D denotes the set of the $\frac{1}{2}v(v-1)$ concurrences for all pairs of treatments, arranged in increasing order.

Definition Let m be an integer such that $1 \leq m \leq \frac{1}{2}v$. Λ_m is the sum of the first $m(v-m)$ elements of L_D.

Concurrence Conditions

For $D_{\#,b_*}$ to be disconnected, $N_\# N'_\#$ is block diagonal.

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Assume $k \leq v/2$ - otherwise D is necessarily maximally robust.

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Each block contributes at most $\frac{1}{8}\{2k^2 - 1 + (-1)^k\}$ to the sum of terms in N_{12}.

Any set of b_* blocks contributes at most $\frac{b_*}{8}\{2k^2 - 1 + (-1)^k\}$ to the sum of terms in N_{12}.

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A sufficient condition for D to be robust against the loss of b_* blocks is

$$\frac{8\Lambda_k}{2k^2 - 1 + (-1)^k} > b_*$$

Rectangles of Concurrences

Example 3 PBIB(3) design: 4 replications of 9 treatments are arranged in 12 blocks of size 3, with $n_1 = 2, n_2 = 4, n_3 = 2$ and $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$.

1	1	2	1	1	2	3	5	6	2	3	3
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$$\begin{array}{cccccccccccc} 1 & 1 & 2 & 1 & 1 & 2 & 3 & 5 & 6 & 2 & 3 & 3 \\ 4 & 4 & 5 & 2 & 3 & 4 & 4 & 7 & 7 & 6 & 5 & 6 \\ 7 & 7 & 8 & 5 & 6 & 8 & 9 & 8 & 9 & 9 & 9 & 8 \end{array}$$

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Λ_3, the sum of the 3×6 smallest concurrences is 9.
Design is robust against the loss of b_* blocks if:

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Group Divisible Designs

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Theorem Let D be a PBIB[2] design with parameter set $\big(v, b, r, k; (\lambda_1, \lambda_2)\big)$ and let b_* satisfy $1 \le b_* \le r-1$. Define set $U = \{k, k+1, \ldots, \mathrm{Int}(\frac{1}{2}v)\}$. If

$$\min_{u \in U} \left\{ \frac{8[ur(k-1) - u(u-1)\lambda_2 + 2\Pi_u(\lambda_2 - \lambda_1)]}{2k^2 - 1 + (-1)^k} \right\} > b_* \,,$$

then D is robust against the loss of b_* blocks.

$\Pi_u =$ maximum (minimum) number of pairs of first associates in a set of u treatments when $\lambda_1 > \lambda_2$ ($\lambda_1 < \lambda_2$).

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$\Rightarrow$ Semi-regular GDDs are maximally robust.

Singular GDDs are maximally robust.

Some Regular GDDs with $\lambda_1 > \lambda_2$ are not maximally robust.

For a GDD with m groups of n_0 treatments and $\lambda_1 > \lambda_2$:

$$\Pi_u = \frac{n_0(n_0-1)}{2}\,\gamma + \frac{\delta(\delta-1)}{2} = \frac{u(n_0-1)-\delta(n_0-\delta)}{2},$$

where $u = \gamma n_0 + \delta$.

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Condition for robustness against the loss of any b_* blocks is:

$$\min_{u \in U}\left\{\frac{8\left[\lambda_2 u(v-u) + (\lambda_1-\lambda_2)\delta(n_0-\delta)\right]}{2k^2-1+(-1)^k}\right\} > b_*.$$

References

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