New Classes of Second-Order Equivalent-Estimation Split-Plot Designs

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Outline

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What is a split-plot experiment?

- one or more experimental factors are not independently reset for several runs
- runs partitioned in whole plots
- a factor that is held constant is called a whole-plot factor
- other factors are called subplot factors
- two separate randomizations
    1. randomization of whole plots gives rise to whole-plot error
    2. randomization of runs in whole plots gives rise to subplot error

Statistical model

- Response surface model
  - quantitative factors
  - linear, quadratic, two-factor interaction effects
- $\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$
- $\mathsf{E}(\boldsymbol{\gamma}) = 0, \mathsf{E}(\boldsymbol{\epsilon}) = 0, \mathsf{var}(\boldsymbol{\gamma}) = \sigma_\gamma^2 \mathbf{I}_b, \mathsf{var}(\boldsymbol{\epsilon}) = \sigma_\epsilon^2 \mathbf{I}_n, \mathsf{cov}(\boldsymbol{\gamma}, \boldsymbol{\epsilon}) = 0$
- $\mathbf{V} = \sigma_\epsilon^2 \mathbf{I}_N + \sigma_\gamma^2 \mathbf{Z}\mathbf{Z}'$

Equivalent-estimation designs

- in general the split-plot design requires the use of generalized least squares to estimate the model
- not implemented in some software packages
- development of various methods for constructing split-plot designs for which the ordinary and generalized least squares estimators produce the same point estimates
- OLS-GLS equivalence condition:
  - $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = (\mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\mathbf{V}^{-\mathbf{1}}\mathbf{Y}$
  - $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\mathbf{Z}'\mathbf{X} = \mathbf{Z}\mathbf{Z}'\mathbf{X}$

Equivalent-estimation designs

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- Goos, P. and Vandebroek, M. (2003). D-optimal split-plot designs with given numbers and sizes of whole plots, *Technometrics* **45**: 235-245.

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- Vining, G. G., Kowalski, S. M. and Montgomery, D. C. (2005). Response surface designs within a split-plot structure, *Journal of Quality Technology* **37**: 115-129.

Equivalent-estimation designs

- Parker, P. A., Kowalski, S. M. and Vining, G. G. (2006). Classes of split-plot response surface designs for equivalent estimation, *Quality and Reliability Engineering International* **22**: 291-305.
- Parker, P. A., Kowalski, S. M. and Vining, G. G. (2007). Construction of balanced equivalent estimation second-order split-plot designs, *Technometrics* **49**: 56-65.
- Parker, P. A., Kowalski, S. M. and Vining, G. G. (2007). Unbalanced and minimal point equivalent estimation second-order split-plot designs, *Journal of Quality Technology* **39**: 376-388.

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Outline

Subset designs

- introduced by Gilmour, (2006), for completely randomized designs
- each subset S_r, $r = 0, \ldots, M$ contains the runs that have r different factors acting at their high or low level and the $M - r$ remaining factors acting at their middle level
- a design which gives:
  1. for the one-factor projections, a good compromise between no replication and double replication of points at 0 relative to those at ± 1
  2. for the two-factor projections, as high as possible replication of points at $(\pm 1, \pm 1)$
  3. as little inflation in the variances due to nonorthogonality as possible
- special cases of subset designs are face-centered central composite designs and Box-Behnken designs

Four subsets for the 3^3 factorial design

S_3			S_2			S_1			S_0		
-1	-1	-1	-1	-1	0	-1	0	0	0	0	0
-1	-1	1	-1	1	0	1	0	0			
-1	1	-1	1	-1	0	0	-1	0			
-1	1	1	1	1	0	0	1	0			
1	-1	-1	-1	0	-1	0	0	-1			
1	-1	1	-1	0	1	0	0	1			
1	1	-1	1	0	-1						
1	1	1	1	0	1						
			0	-1	-1						
			0	-1	1						
			0	1	-1						
			0	1	1						

Equivalent-estimation designs based on subset designs

1. Sort the runs within every subset S_r in increasing order of the hard-to-change factors' levels.
2. Split these groups into smaller ones.

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- The axial runs for the subplot factors should be kept together when subset S_1 is used.

Equivalent-estimation designs based on subset designs

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2. Split these groups into smaller ones.

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- The axial runs for the subplot factors should be kept together when subset S_1 is used.
- The axial runs for the whole-plot factors need to be replicated.

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- The axial runs for the whole-plot factors need to be replicated.
- All designs involving the center run must have the center run replicated.

1 whole-plot factor w_1 and 2 subplot factors s_1 and s_2

j	S_3			S_2			S_1			S_0		
	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2
1	−1	−1	−1	−1	−1	0	−1	0	0	0	0	0
2	−1	−1	1	−1	1	0	0	−1	0			
3	−1	1	−1	−1	0	−1	0	1	0			
4	−1	1	1	−1	0	−1	0	0	−1			
5	1	−1	−1	0	−1	−1	0	0	1			
6	1	−1	1	0	−1	1	1	0	0			
7	1	1	−1	0	1	−1						
8	1	1	1	0	1	1						
9				1	−1	0						
10				1	1	0						
11				1	1	0						
12				1	0	1						

Option 1

j	S_3			S_2			S_1			S_0		
	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2
1	−1	−1	−1	−1	−1	0	−1	0	0	0	0	0
2	−1	−1	1	−1	1	0	0	−1	0			
3	−1	1	−1	−1	0	−1	0	1	0			
4	−1	1	1	−1	0	−1	0	0	−1			
5	1	−1	−1	0	−1	−1	0	0	1			
6	1	−1	1	0	−1	1	1	0	0			
7	1	1	−1	0	1	−1						
8	1	1	1	0	1	1						
9				1	−1	0						
10				1	1	0						
11				1	1	0						
12				1	0	1						

Option 2

j	S_3			S_2			S_1			S_0		
	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2
1	-1	-1	-1	-1	-1	0	-1	0	0	0	0	0
2	-1	-1	1	-1	1	0	0	-1	0			
3	-1	1	-1	-1	0	-1	0	1	0			
4	-1	1	1	-1	0	-1	0	0	-1			
5	1	-1	-1	0	-1	-1	0	0	1			
6	1	-1	1	0	-1	1	1	0	0			
7	1	1	-1	0	1	-1						
8	1	1	1	0	1	1						
9				1	-1	0						
10				1	1	0						
11				1	1	0						
12				1	0	1						

Option 3

j	S_3			S_2			S_1			S_0		
	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2	w_1	s_1	s_2
1	-1	-1	-1	-1	-1	0	-1	0	0	0	0	0
2	-1	-1	1	-1	1	0	0	-1	0			
3	-1	1	-1	-1	0	-1	0	1	0			
4	-1	1	1	-1	0	-1	0	0	-1			
5	1	-1	-1	0	-1	-1	0	0	1			
6	1	-1	1	0	-1	1	1	0	0			
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9				1	-1	0						
10				1	1	0						
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Supplementary difference set designs

- Introduced by Koukouvinos et al., submitted, for completely randomized designs
- The original designs involve two parts:
    1. a factorial part, in which each run has one factor acting at its middle level and $M-1$ factors are set at either -1 or $+1$, and
    2. $2M$ different axial points at a distance α from the center, where the distance α is determined so as to achieve rotatability in completely randomized experiments.

Supplementary difference set design with 4 factors and 40 runs

$\mathbf{0}_8$	$\pm\mathbf{1}_8$	$\pm\mathbf{1}_8$	$\pm\mathbf{1}_8$
$\pm\mathbf{1}_8$	$\mathbf{0}_8$	$\pm\mathbf{1}_8$	$\pm\mathbf{1}_8$
$\pm\mathbf{1}_8$	$\pm\mathbf{1}_8$	$\mathbf{0}_8$	$\pm\mathbf{1}_8$
$\pm\mathbf{1}_8$	$\pm\mathbf{1}_8$	$\pm\mathbf{1}_8$	$\mathbf{0}_8$
$\pm\boldsymbol{\alpha}$	$\mathbf{0}_2$	$\mathbf{0}_2$	$\mathbf{0}_2$
$\mathbf{0}_2$	$\pm\boldsymbol{\alpha}$	$\mathbf{0}_2$	$\mathbf{0}_2$
$\mathbf{0}_2$	$\mathbf{0}_2$	$\pm\boldsymbol{\alpha}$	$\mathbf{0}_2$
$\mathbf{0}_2$	$\mathbf{0}_2$	$\mathbf{0}_2$	$\pm\boldsymbol{\alpha}$

Equivalent-estimation split-plot designs based on supplementary difference set designs

1. We rearrange the runs of the factorial portion and of the axial portion in increasing order of the whole-plot factor levels.
2. The subplot designs must be orthogonal to the whole plots.

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Remarks:

- The axial runs for the subplot factors form a single whole plot.

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- The axial runs for the subplot factors form a single whole plot.
- If necessary, the axial points are repeated in order to obtain a balanced design.

Equivalent-estimation split-plot designs based on supplementary difference set designs

1. We rearrange the runs of the factorial portion and of the axial portion in increasing order of the whole-plot factor levels.
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Remarks:

- The axial runs for the subplot factors form a single whole plot.
- If necessary, the axial points are repeated in order to obtain a balanced design.
- We did not use the axial points corresponding to whole-plot factors.

2 whole-plot factors w_1 and w_2 and 2 subplot factors s_1 and s_2

Whole plot	w_1	w_2	s_1	s_2	Whole plot	w_1	w_2	s_1	s_2
1	0	-1	-1	-1	6	1	-1	0	1
1	0	-1	-1	1	6	1	-1	0	-1
1	0	-1	1	1	6	1	-1	-1	0
1	0	-1	1	-1	6	1	-1	1	0
2	0	1	-1	1	7	-1	1	0	1
2	0	1	-1	-1	7	-1	1	0	-1
2	0	1	1	-1	7	-1	1	-1	0
2	0	1	1	1	7	-1	1	1	0
3	-1	0	-1	-1	8	1	1	0	-1
3	-1	0	-1	1	8	1	1	0	1
3	-1	0	1	1	8	1	1	-1	0
3	-1	0	1	-1	8	1	1	1	0
4	1	0	-1	1	9	0	0	$-\alpha$	0
4	1	0	-1	-1	9	0	0	α	0
4	1	0	1	-1	9	0	0	0	$-\alpha$
4	1	0	1	1	9	0	0	0	α
5	-1	-1	0	-1					
5	-1	-1	0	1					
5	-1	-1	-1	0					
5	-1	-1	1	0					

Outline

D-efficiencies and I-efficiencies

D-efficiencies:

$$\left\{ \frac{|\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|}{|\mathbf{X}'_{\text{opt}}\mathbf{V}^{-1}\mathbf{X}_{\text{opt}}|} \right\}^{1/p}$$

D-efficiencies and I-efficiencies

D-efficiencies:

$$\left\{ \frac{|\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|}{|\mathbf{X}'_{\text{opt}}\mathbf{V}^{-1}\mathbf{X}_{\text{opt}}|} \right\}^{1/p}$$

I-efficiencies:

$$\frac{\int_{\chi} \mathbf{f}'(\mathbf{x})(\mathbf{X}'_{\text{opt}}\mathbf{V}^{-1}\mathbf{X}_{\text{opt}})^{-1}\mathbf{f}(\mathbf{x})d\mathbf{x}}{\int_{\chi} \mathbf{f}'(\mathbf{x})(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{f}(\mathbf{x})d\mathbf{x}}$$

where χ represents the (cuboidal or spherical) design region

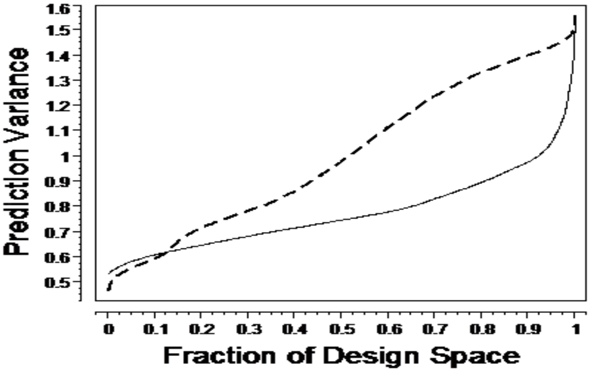
Efficiencies of the equivalent-estimation split-plot designs obtained from subset designs.

N	b	M_w	M_s	Region	D-efficiency	I-efficiency
16	4	1	2	spherical	0.8262	1.2454
20	5	1	2	cuboidal	0.8482	0.3819
20	5	1	2	cuboidal	0.7336	1.1210
24	6	1	2	cuboidal	0.8485	1.2905
48	6	1	3	cuboidal	0.6895	0.3231
48	6	1	3	spherical	0.5430	1.3858
36	9	2	2	spherical	0.9488	0.8385
36	9	2	2	cuboidal	0.7437	1.2708
72	9	2	2	cuboidal	0.8992	1.5836
68	17	2	2	cuboidal	0.8220	1.2729
72	18	2	2	cuboidal	0.7971	1.4359
80	10	2	3	cuboidal	0.6878	0.8114
80	10	2	3	spherical	0.5296	1.2842
68	17	3	2	spherical	0.7683	1.0958
88	22	3	2	cuboidal	0.8457	0.6825
84	21	4	2	cuboidal	0.8910	0.6067
84	21	4	2	cuboidal	0.7886	1.0653
104	26	4	2	cuboidal	0.8770	0.3945
104	26	4	2	cuboidal	0.7827	1.2314

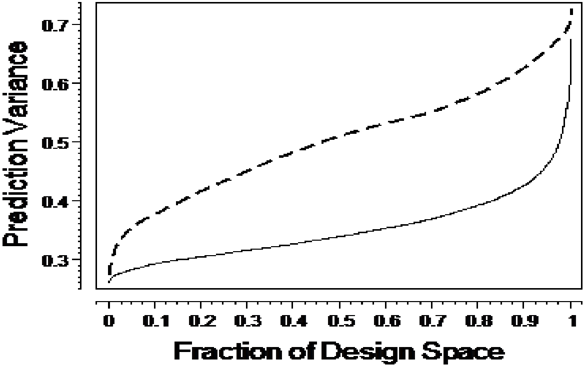
Efficiencies of the equivalent-estimation split-plot designs obtained from supplementary difference set designs.

N	b	M_w	M_s	D-efficiency	I-efficiency
16	4	1	2	0.7832	1.0083
64	4	1	4	0.7212	1.1570
36	9	2	2	0.9222	0.9800
40	10	2	2	0.9078	1.0565
38	9	2	3	0.9405	0.9343
44	10	2	3	0.7181	1.1034
60	13	2	3	0.8034	1.0450
54	13	2	3	0.8005	0.9848
144	9	2	4	0.8430	1.0193
84	21	3	2	0.9975	0.9533
88	22	3	2	1.0015	0.9964
166	21	3	3	0.9491	0.9312
172	22	3	3	0.9472	0.9643
336	21	3	4	0.8944	0.9737
352	22	3	4	0.8829	0.9991
198	49	4	3	0.8904	0.9119
204	50	4	3	1.0583	0.8662

24 runs: Six whole plots of four runs



72 runs: 18 whole plots of four runs



Outline

Discussion

- How to modify two families of second-order response surface designs in order to obtain equivalent-estimation split-plot designs.
- We offer to practitioners without access to advanced software an additional set of equivalent-estimation split-plot designs to choose from.
- Most of the designs are highly D- and/or I-efficient.

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