Instantons and M5-branes

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Mathematics and Applications of Branes in String and M-theory
Issac Newton Institute, Cambridge

talk based on:

Hee-Cheol Kim (KIAS), S.K. (SNU), Eunkyung Koh (KIAS),
Kimyeong Lee (KIAS), Sungjay Lee (DAMTP),

"On instantons as Kaluza-Klein modes of M5-branes,"
JHEP**1112**, 031, arXiv:1110.2175

related works:
D0-brane threshold bound states: [P. Yi], [Sethi-Stern], …
instanton partition functions: [Nekrasov], [Nekrasov, Okounkov], …
topological string counting of U(1) instantons: [Iqbal, Kozcaz, Shabbir]

5d SYM & 6d (2,0): [Lambert, Papageorgakis, Schmidt-Sommerfeld] [Douglas]
self-dual string anomalies: [Berman, Harvey]
DLCQ description of M5's: [Aharony, Berkooz, Seiberg] [Lambert, Richmond]

Motivation

- Mysterious aspects of M-theory: 10d$\longrightarrow$11d, M2- and M5-branes, no perturbative description, …

- Some issues understood to certain extent: M2's & AdS_4/CFT_3 [Bagger, Lambert] [Gustavsson] [Aharony,Bergman,Jafferis,Maldacena], … , M5's wrapped on Riemann surfaces [Gaiotto] [Alday,Gaiotto,Tachikawa], … , …

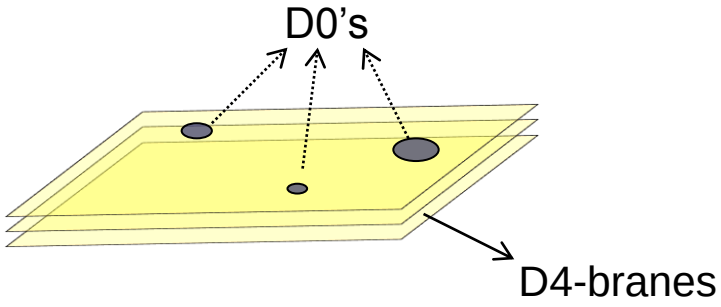
- M5-branes? Still a lot of mysteries: N^3 degrees of freedom on N M5, microscopic description lacking for 6d (2,0) theory…

- Compactify M5 on a circle: D4-branes, **5d maximal SYM** at low energy

Aspects of 5d SYM

- 5d SYM has nonperturbative objects which see the 6th direction.
- Instanton "solitons": D0's on D4's

$$F_{\mu\nu} = \star_4 F_{\mu\nu} \qquad \text{on } R^4$$



- Visible as QFT solitons rather than extra ingredients, unlike D0's in (perturbative) type IIA string/SUGRA.
- 5d theory could see (at least some of) its quantum aspects.

- UV finite? [Lambert,Papageorgakis,Schmidt-Sommerfeld] [Douglas] (2010)
- We study BPS quantities: not too sensitive to UV details.
- Our analysis however will be (almost) self-contained in 5d QFT.

Table of Contents

1. Introduction
2. Index for instantons (& W-bosons) in Coulomb phase
3. Instantons in the symmetric phase, 6d SCFT & $AdS_7 \times S^4$
4. Concluding remarks

Threshold bound states of instantons

- Supersymmetry & BPS bounds (i,j=1,2,3,4, $SO(5)_R$ spinor; I=1,2, … 5 for vector)

$$\{Q^i_M, Q^j_N\} = P_\mu(\Gamma^\mu C)_{MN}\omega^{ij} + i\frac{8\pi^2 k}{g^2_{YM}}C_{MN}\omega^{ij} - i\mathrm{tr}(qv_I)(\Gamma^I C)_{MN}$$

instanton number

electric charge
(in Coulomb phase)

$$M \geq \frac{8\pi^2 k}{g^2_{YM}} + \mathrm{tr}(qv_5)$$

- Threshold bound states (with 0 binding energy) are subtle.

- Continuum: 1-particle bound & multi-particles at rest have same energy.

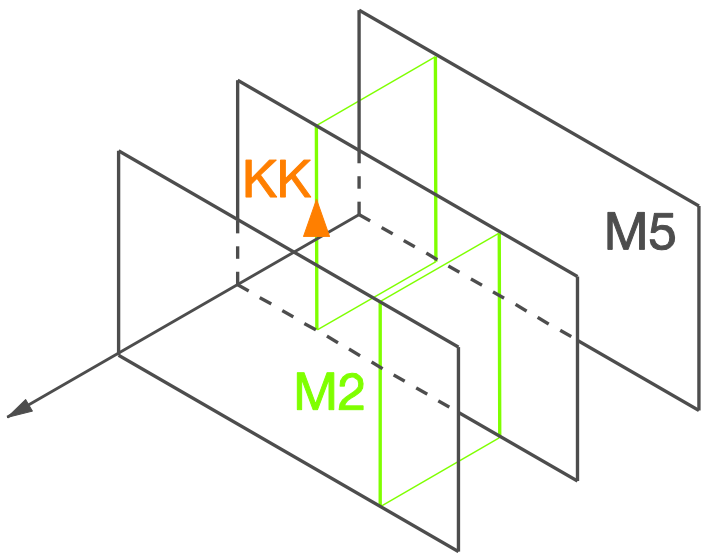
  Small relative kinetic energy makes a continuum. translation 0-modes

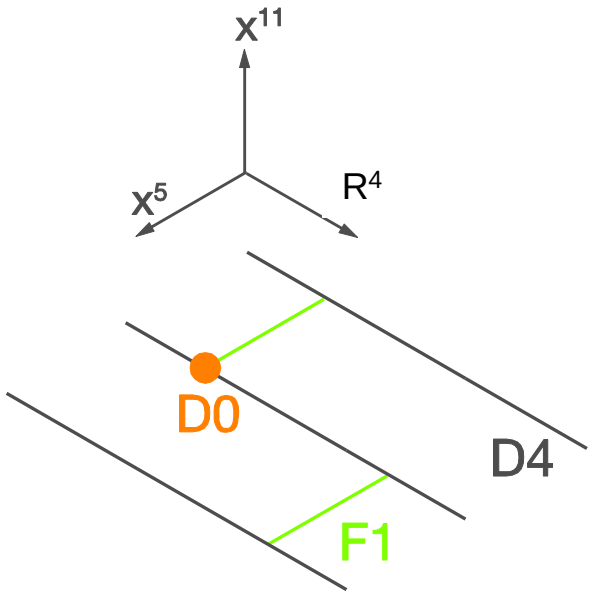
- Instantons also have internal noncompact 0-modes: instanton sizes

- Calculating Witten index is subtle for threshold bounds. [P.Yi] [Sethi,Stern]

The Coulomb phase index

- Coulomb phase: VEV to a scalar, $U(N) \to U(1)^N$. Instanton/W-boson bounds (dyonic instantons [Lambert,Tong]) lift to self-dual strings on M5 wrapping S^1 with momenta





- Size 0-modes lifted in the Coulomb phase (instantons tend to shrink)
- Translation 0-modes lifted by SO(4) chemical potential, or Omega background. (works for D0-D4, but not completely for D0 [Moore,Nekrasov,Shatashvili])

An index (continued)

- ¼-BPS: we only need 2 real SUSY to define Witten index

$\boxed{16 \text{ SUSY } Q^i_\alpha,\ \overline{Q}^i_{\dot\alpha} \text{ on D4}} \rightarrow \boxed{8 \text{ SUSY } \overline{Q}^i_{\dot\alpha} \text{ on D0-D4}} \rightarrow \boxed{4 \text{ SUSY } \overline{Q}^a_{\dot\alpha} \text{ on D0-D4-F1 } (a=1,2)}$

- Index:

U(1)^N electric charges

$$I(\mu^i,\gamma_1,\gamma_2,\gamma_R) = \mathrm{Tr}\left[(-1)^F e^{-\beta Q^2} e^{-\mu^i \Pi_i} e^{-i\gamma_1(2J_{1L})-i\gamma_2(2J_{2L})-i\gamma_R(2J_R)}\right]$$

SU(2)$_L$ of spatial SO(4)

SU(2)$_L$ of internal SO(4) in SO(5)

sum of two SU(2)$_R$ charges in two SO(4)

preserves two SUSY $\bar{Q}^+_+,\ \bar{Q}^-_-$

- Spatial chemical potentials: “masses” to translation 0-modes
- Counts all (single/multi-particle) BPS states: careful study of γ_1, γ_R dependence will let us to separate out single particle states
- U(1)^N chemical potentials: lifts internal 0-modes & size moduli.

Path integral & index

- Path integral rep. of D0-D4 quantum mechanics (same result from QFT)
- Many ways to calculate: e.g. index invariant under following deformation:

$$\text{non-commutativity (FI-term on D0-D4 mechanics, a UV regulator): } \zeta \to \infty$$

$$\text{regulator } \beta \to 0$$

- Dominated by Gaussian path integrals over a set of saddle points

- The result is actually a well-known one: Nekrasov's instanton partition function for 5d N=2* theory on $R^4 \times S^1$ $\quad q = e^{2\pi i \tau}$: instanton # fucacity

$$a_i = \frac{\mu_i}{2}\,, \quad \epsilon_1 = i\frac{\gamma_1 - \gamma_R}{2}\,, \quad \epsilon_2 = i\frac{\gamma_1 + \gamma_R}{2}\,, \quad m = \frac{i\gamma_2}{2}$$

scalar VEV Omega background hypermultiplet mass

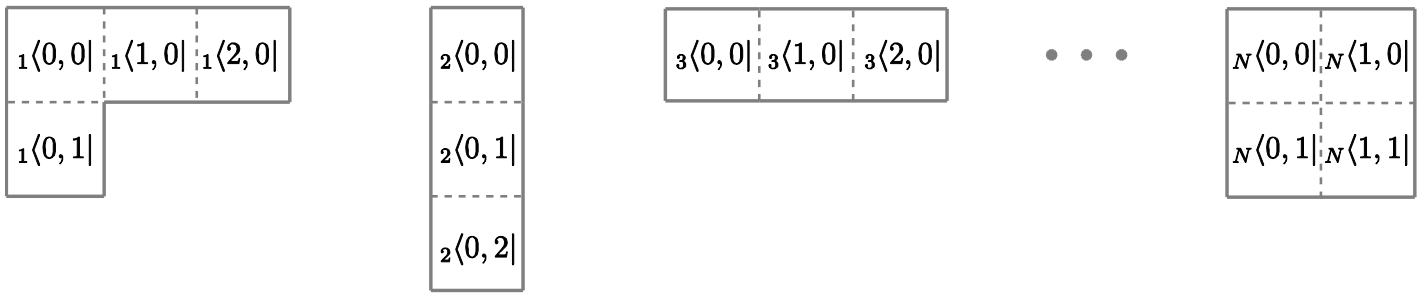
(all parameters made dimensionless by multiplying β)

Path integral & index *(continued)*

- Saddle points: solution to (generalized) ADHM, N-colored Young diagrams

$$[\phi, B_1] = \frac{i(\gamma_R - \gamma_1)}{\beta} B_1 \ , \quad [\phi, B_2] = \frac{i(\gamma_R + \gamma_1)}{\beta} B_2$$

$$[B_1, B_2] + \bar{x}^{\dot{-}} x_{\dot{+}} = 0 \ , \quad [B_1^\dagger, B_1] + [B_2^\dagger, B_2] + \bar{x}^{\dot{+}} x_{\dot{+}} - \bar{x}^{\dot{-}} x_{\dot{-}} = \zeta$$



- After evaluating Gaussian path integrals, one obtains

$$I_{\{Y_1, Y_2, \cdots, Y_N\}} = \prod_{i,j=1}^{N} \prod_{s \in Y_i} \frac{\sinh \frac{E_{ij} - i(\gamma_2 + \gamma_R)}{2} \sinh \frac{E_{ij} + i(\gamma_2 - \gamma_R)}{2}}{\sinh \frac{E_{ij}}{2} \sinh \frac{E_{ij} - 2i\gamma_R}{2}}$$

$$E_{ij} = \mu_i - \mu_j + i(\gamma_1 - \gamma_R) h_i(s) + i(\gamma_1 + \gamma_R)(v_j(s) + 1)$$

Pure instanton bound states

- M-theory demands k U(1) instantons form unique threshold bound state (1 supermultiplet): KK modes for a free 6d tensor multiplet
- Direct study: normalizable harmonic forms on instanton moduli space, very difficult for higher k, only studied up to k=2 [K.Lee, D.Tong, S.Yi]
- Index (rewrite using topological vertex technique) [Iqbal, Kozcaz, Shabbir]

$$I = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} I_{\rm sp}(q^n, n\gamma)\right] \quad , \quad I_{\rm sp} = \boxed{\left(\frac{\sin\frac{\gamma_1+\gamma_2}{2}\sin\frac{\gamma_1-\gamma_2}{2}}{\sin\frac{\gamma_1+\gamma_R}{2}\sin\frac{\gamma_1-\gamma_R}{2}}\right)} \frac{q}{1-q} \equiv I_{com}(\gamma_1, \gamma_2, \gamma_R)\frac{q}{1-q}$$

index from the c.o.m. 0-modes & fermion 0-modes

- One c.o.m. supermultiplet at $O(q^k)$: 5d QFT derivation of uniqueness

Charged instanton bound states: SU(2)

- Chemical potential $e^{-(\mu_1-\mu_2)}$ measures # of W-bosons or self-dual strings.
- Single particle index for one self-dual string:

$$z_{\rm sp} = 1+8q+40q^2+160q^3+552q^4+1712q^5+4896q^6+13120q^7+33320q^8+80872q^9+188784q^{10}+\cdots$$

set $\gamma_2 = \pi$: kills $(-1)^F$ sign set $\gamma_R = 0$, $\gamma_1 \to 0$ and factor out I_{com}

- This can be written in a closed form as: $z_{\rm sp} = I_{com} \prod_{n=1}^{\infty} \frac{(1+q^n)^4}{(1-q^n)^4}$

or… $$\left(\frac{\sin\frac{\gamma_2+\gamma_R}{2}\sin\frac{\gamma_2-\gamma_R}{2}}{\sin\frac{\gamma_1+\gamma_R}{2}\sin\frac{\gamma_1-\gamma_R}{2}}\right)\prod_{n=1}^{\infty}\frac{(1-q^n e^{i(\gamma_2+\gamma_R)})(1-q^n e^{i(\gamma_2-\gamma_R)})(1-q^n e^{i(-\gamma_2+\gamma_R)})(1-q^n e^{i(-\gamma_2-\gamma_R)})}{(1-q^n e^{i(\gamma_1+\gamma_R)})(1-q^n e^{i(\gamma_1-\gamma_R)})(1-q^n e^{i(-\gamma_1+\gamma_R)})(1-q^n e^{i(-\gamma_1-\gamma_R)})}$$

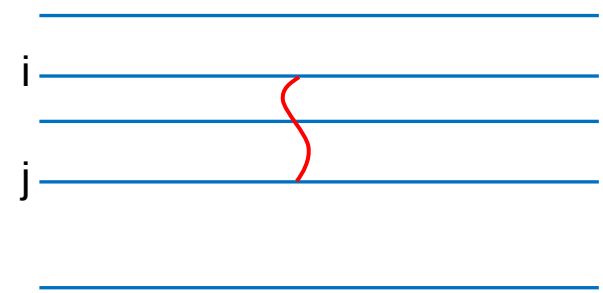
- 1+1d fluctuations on 0-modes of an SU(2) self-dual string: instantons again form a KK tower of M5, all visible from 5d QFT framework.

- Our index also predicts nontrivial bounds of multiple self-dual strings

Charged bound states: SU(N)

- Physics to expect is less clear:

Single W-bosons ending on i & j'th D4's:
SU(j – i + 1) matters.



- Partition function for the "longest string" in SU(N)

$$I_{\rm sp}^{SU(3)} = I_{com} \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^4 (1 + 16q + 96q^2 + 448q^3 + 1728q^4 + 5856q^5 + \cdots)$$

$$I_{\rm sp}^{SU(4)} = I_{com} \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^4 (1 + 32q + 448q^2 + 3968q^3 + 27008q^4 + \cdots)$$

$$I_{\rm sp}^{SU(5)} = I_{com} \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^4 (1 + 48q + 1056q^2 + 14656q^3 + 149568q^4 + \cdots)$$

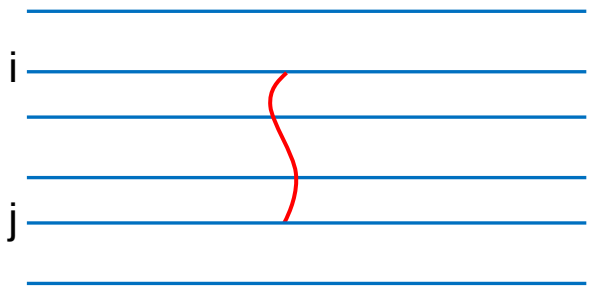
- For SU(N), general closed form for the `relative part' of the index is

$$\left[\oint \frac{dz}{2\pi i z} \prod_{n=1}^{\infty} \left(\frac{(1+q^{\frac{2n-1}{2}}z)(1+q^{\frac{2n-1}{2}}z^{-1})}{(1-q^{\frac{2n-1}{2}}z)(1-q^{\frac{2n-1}{2}}z^{-1})} \right)^2 \right]^{N-2}$$

$$z^B_{\rm letter}(q) = z^F_{\rm letter}(q) = \frac{2q^{1/2}}{1-q}\left(z + \frac{1}{z}\right)$$

Charged SU(N) bound states *(continued)*

- Possible interpretation: degrees of freedom living on the worldvolume of self-dual strings ~ (j - i - 1) of them for ij string.



- Not all "physical": $U(1)^{j-i-1}$ singlet conditions
- For large momentum (~ 6d (2,0) limit, or self-dual string on a large circle), the last constraint effectively becomes irrelevant (saddle point at z=1)
- 1+1d theories for N^2 self-dual strings have more degrees than just giving momenta to the N^2 0-modes: So how many 1+1d "degrees" do we find?

Degrees of freedom on self-dual strings?

- M5-branes in symmetric phase should exhibit N^3 degrees of freedom.
- Coulomb phase sometimes shows remnants of the symmetric phase degrees as massive states: e.g. N^2 massive BPS W-bosons of SYM
- Self-dual strings on S^1: N^2 with 0 momentum. Large momentum regime?
- Previous 2d d.o.f. are effectively "unconstrained" in UV
- Count all of them, giving ½ contribution to fermions

$$\#_B^{int} = \#_F^{int} = {}_4C_3 = \frac{2}{3}N(N-1)(N-2)$$

$$\#_B^{ext} = \#_F^{ext} = {}_4C_2 = 2N(N-1)$$

$$\#_B + 1/2\ \#_F = N(N^2-1)$$

The instanton index in the symmetric phase

- U(N) unbroken if VEV=0. Still, one can turn on $U(1)^N$ chemical potentials. Regularize divergent volume of internal moduli. (similar to SO(4) Ω-background)
- Technically, same path integral & same expression: so what is this…?

- Non-relativistic superconformal sigma model in IR: DLCQ M5's [Aharony, Berkooz,Kachru,Seiberg,Silverstein] [Aharony,Berkooz,Seiberg] [Lambert,Richmond]
- 6d superconformal generators commuting with $P_- = M_{6-} + M_{7-}$

$$x^\pm = x^0 \pm x^5$$

SO(6,2) : $M_{AB},\ A,B = 0,1,2,\cdots,6,7$

$$P_\mu = M_{6\mu} + M_{7\mu}\ ,\quad K_\mu = -M_{6\mu} + M_{7\mu}\ ,\quad \Delta = M_{67}$$

$$\longrightarrow\ H \sim P_+\ ,\quad P_i\ ,\quad M_{ij}\ ,\quad G_i \sim M_{-i}\ ,\quad K \sim K_-\ ,\quad D = \Delta - M_{05}$$

can be extended to Osp(6,2|4): 32 SUSY reduced to 24

The non-relativistic superconformal index

- One can count BPS local operators in NR quantum mechanics.
- Equivalently, one can count BPS states (non-relativistic operator-state map):

$$M^{-1}(iD)M = H + K \quad = L_0 \qquad\qquad M = e^{H/2}e^{-K}$$

$$M^{-1}QM = Q - iS \equiv \hat{Q}\ , \quad M^{-1}SM = -i/2(Q + iS) = -\frac{i}{2}\hat{S}$$

$$\{\hat{Q}, \hat{S}\} = L_0 \mp (4J_{2R} + 2J_{1R})$$

- Our index is a **nonrelativistic superconformal index** [Nakayama]

$$I_{SC} = \mathrm{Tr}\left[(-1)^F e^{-\beta\{\hat{Q},\hat{S}\}} e^{-2i\gamma_R J_R} e^{-2i\gamma_1 J_{1L} - 2i\gamma_2 J_{2L}} e^{-i\alpha_i \Pi_i}\right]$$

- $SU(2)_R$ charge ~ BPS dilatation eigenvalues
- U(N) chemical potentials: project to U(N) singlets (gauge invariant operators)

Some tests

- Tests: U(1) instantons vs. DLCQ index of single M5 (rather trivial)
- Large N index at k=1: SUGRA index on $AdS_7 \times S^4$ (DLCQ basis in AdS)

$$\boxed{I_{k=1}} = \frac{e^{i\gamma_2} + e^{-i\gamma_2} - e^{i\gamma_1} - e^{-i\gamma_1}}{(1 - te^{i\gamma_1})(1 - te^{-i\gamma_1})} \left[\sum_{n=0}^{N-1} \chi_{\frac{n}{2}}(\gamma_2) t^{n+1} - \sum_{n=1}^{N-1} \chi_{\frac{n-1}{2}}(\gamma_2) t^{n+2} \right] \qquad t \equiv e^{-i\gamma_R}$$

$$I_{N\to\infty,k=1} = \frac{e^{i\gamma_2} + e^{-i\gamma_2} - e^{i\gamma_1} - e^{-i\gamma_1}}{(1 - te^{i\gamma_1})(1 - te^{-i\gamma_1})} \frac{t - t^3}{(1 - te^{i\gamma_2})(1 - te^{-i\gamma_2})} = I_{\text{SUGRA , k=1}}$$

projected to
SU(N) singlets

	D	J_{1L}	J_{2L}	$2(J_{1R} + J_{2R})$	boson/fermion
$p \geq 1$	$2p$	0	$\frac{p}{2}$	p	b
$p \geq 1$	$2p+1$	0	$\frac{p-1}{2}$	$p+1$	f
$p \geq 1$	$2p$	$\frac{1}{2}$	$\frac{p-1}{2}$	p	f
$p \geq 2$	$2p+1$	$\frac{1}{2}$	$\frac{p-2}{2}$	$p+1$	b
$p \geq 2$	$2p$	0	$\frac{p-2}{2}$	p	b
$p \geq 3$	$2p+1$	0	$\frac{p-3}{2}$	$p+1$	f
·	3	0	0	2	b (fermionic constraint)

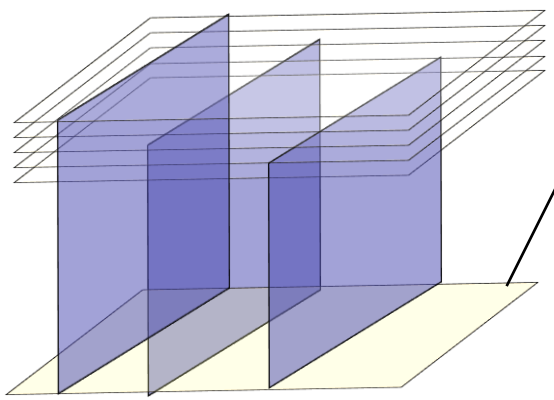
- Similar agreement at k=2 (& presumably true with higher momentum)

Concluding remarks

- Instantons reveal interesting information on (2,0) theory on circle via index: in both Coulomb & symmetric phases

- Microscopic studies of conformal 6d (2,0): 6d limit (large instanton number k) & beyond low E. For instance, N^3 is beyond SUGRA & maybe at large k.

- Self-dual string anomaly ($SO(4)_N$ in SO(5) from Coulomb branch) [Berman,Harvey]



G broken to H x U(1)

$$\alpha_e \int_{\Sigma_6} h_3 \wedge \Omega_3(\hat{\phi}, A)$$

$$\alpha_e = \frac{1}{4}(|G| - |H| - 1)$$

$$d\Omega_3 = \frac{1}{64\pi^2}\epsilon_{a_1..a_5}[(D\hat{\phi})^{a_1} \wedge (D\hat{\phi})^{a_2} \wedge (D\hat{\phi})^{a_3} \wedge (D\hat{\phi})^{a_4} - 2F^{a_1a_2} \wedge (D\hat{\phi})^{a_3} \wedge (D\hat{\phi})^{a_4} + F^{a_1a_2} \wedge F^{a_3a_4}]\hat{\phi}^{a_5}$$

For G = SU(N), H = $U(1)^{N-2}$,

$$c = \frac{\alpha_e}{2} = \frac{N^2 - N}{8} \text{ from } N - 1 \text{ different strings}$$