Space Complexity of Polynomial Calculus

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LOGICAL APPROACHES TO BARRIERS IN COMPLEXITY II
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(joint work with Y. Filmus, J. Nordström, N.Thapen and N. Zewi)

“SAT has been solved”

—the daring practitioner—

“SAT is clearly hard”

—the savvy theorist—

“SAT solving can get better. . .”

—the relentless coder—

Next step is to study **memory requirements** of **algebraic** sat-solvers/theorem provers.

Modern SAT solvers are based on **Resolution**

but

algebraic reasoning can be beneficial.

- shorter proofs

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- implementations (e.g PolyBori)

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- modern solvers are **very** optimized
- to change paradigm is bad for SAT races
- their main issue is **space**, not proof length

SPACE IN MODERN SAT-SOLVERS

- during a running, solvers **learn** clauses
- memory fills very quickly
- retaining the right clauses if fundamental

DOES ALGEBRA HELP?

- memory requirements
- relation between space and time

Our results

	width	variables	space	
PCR	n	n^2	$\Omega(n)$	**[Alekhnovich et al. 02]**
PC	3	n^2	$\Theta(n)$	pigeonhole principle
PC	$O(1)$	n	$O(n)$	any formula
PCR	$2\log n$	$n\log n$	$\Omega(n)$	bit-pigeonhole principle
PCR	4	n^2	$\Omega(n)$	xor-pigeonhole principle

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In this talk we sketch this proof!

Outline

1. Algebraic proof system PCR
2. Model space complexity in PCR refutations
3. We sketch the proof of a space lower bound for PCR

ALGEBRAIC PROOF SYSTEM

Proof Systems

Deterministic polynomial time $P(\cdot, \cdot)$

- if $F \in$ UNSAT then $P(F, \pi) = 1$ for some $\pi \in \{0,1\}^*$
- if $F \notin$ UNSAT then $P(F, \pi) = 0$ for all $\pi \in \{0,1\}^*$

Proof systems in complexity theory

There is P where any unsat formula has a "short" refutation in P

$$\Longleftrightarrow$$

NP=coNP

Cook-Reckhow program (1979)

Prove proof length lower bound for stronger and stronger system in order to prove NP$\neq$coNP

PROOF SYSTEMS AND SAT SOLVERS

The trace of "SAT-solver(F)=`unsat`" is a refutation for F.

DLL	$\longrightarrow$	tree-like resolution
Clause Learning	$\longrightarrow$	regular WRTL [BHJ '08]
CL + Restarts	$\longrightarrow$	resolution
CRYPTOMINISAT	$\longrightarrow$	fragments of PCR on $GF(2)$
POLYBORI	$\longrightarrow$	PC on $GF(2)$

Polynomial Calculus (PCR)

CNF formula	$\longrightarrow$	set of polynomials
SAT assignments	$\longrightarrow$	common roots
true	$\longrightarrow$	0
false	$\longrightarrow$	1
variable x	$\longrightarrow$	$x, \bar{x}$
$x \in \{\text{true}, \text{false}\}$	$\longrightarrow$	$x^2 - x$ $x + \bar{x} - 1$
$x \vee \neg y \vee \neg z \vee s \vee t$	$\longrightarrow$	$x \cdot \bar{y} \cdot \bar{z} \cdot s \cdot t$

Polynomial Calculus (PCR)

Linear combination

$$\frac{p \qquad q}{\alpha p + \beta q}$$

Multiplication

$$\frac{p}{xp}$$

$\boxed{F \vdash 1}$ iff $\boxed{F \in \text{UNSAT}}$

(SOUNDNESS) INFERENCE PRESERVES COMMON ROOTS

(COMPLETENESS) SIMULATES DECISION TREES

- PC defined in [CEI96] and PCR in [ABRW02];
- PCR strictly better than resolution in proof length;
- Size-Degree Trade-off [IPS99,GL10a];
- Exponential lower bounds on length are known [Raz98,AR03, BGIP01, BI10, IPS99, Raz98];
- Proof search is hard [GL10b] based on [AR08].

SPACE COMPLEXITY OF PCR

$$\cdots \to \begin{bmatrix} xz + yz \\ xz - 1 \\ \\ \end{bmatrix}$$

$$\cdots \rightarrow \begin{bmatrix} xz + yz \\ xz - 1 \\ \end{bmatrix} \rightarrow \begin{bmatrix} xz + yz \\ xz - 1 \\ 1 - yz \end{bmatrix}$$

- inference step from polynomials in memory

$$\cdots \rightarrow \begin{bmatrix} xz+yz \\ xz-1 \\ \end{bmatrix} \rightarrow \begin{bmatrix} xz+yz \\ xz-1 \\ 1-yz \end{bmatrix} \rightarrow \begin{bmatrix} xz+yz \\ \text{—} \\ 1-yz \end{bmatrix}$$

- inference step from polynomials in memory
- erasure of a polynomial

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- inference step from polynomials in memory
- erasure of a polynomial
- logical axiom/initial polynomial download

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Roads not taken

- $O(1)$ polynomials are always sufficient (#polynomials)
- too expensive compared to implementations (#symbols)

LITTLE IS KNOWN FOR PCR SPACE

- Lower bounds for wide CNFs [Alekhnovich et al. 2002]
- Length-Space trade-offs [Huynh, Nordström, 2012]
- Lower bounds for narrow CNFs [FLNTZ 2012]

Lower bound for narrow CNFs

BIT-PIGEONHOLE PRINCIPLE

Fix $n = 2^m$: there is no injective function $F : [n+1] \to \{0,1\}^m$.

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For each:

- two pigeons a and b
- hole $s \in \{0,1\}^m$

$$\underbrace{(f_{a,1} \neq s_1) \vee \cdots \vee (f_{a,m} \neq s_m)}_{F(a) \neq s} \bigvee \underbrace{(f_{b,1} \neq s_1) \vee \cdots \vee (f_{b,m} \neq s_m)}_{F(b) \neq s}$$

EXAMPLE

$$F(1) \neq \langle 1101 \rangle \text{ or } F(3) \neq \langle 1101 \rangle$$

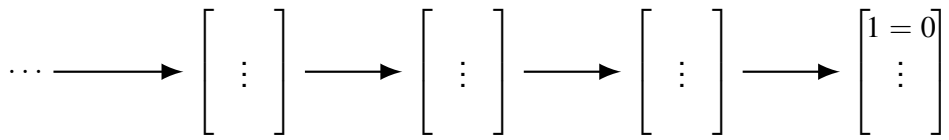
translates to

$$(\neg f_{1,1} \vee \neg f_{1,2} \vee f_{1,3} \vee \neg f_{1,4}) \bigvee (\neg f_{3,1} \vee \neg f_{3,2} \vee f_{3,3} \vee \neg f_{3,4})$$

THEOREM

Any PCR refutation of the “Bit” pigeohole principle has a configuration with at least $n/8$ monomials.

Proof Sketch



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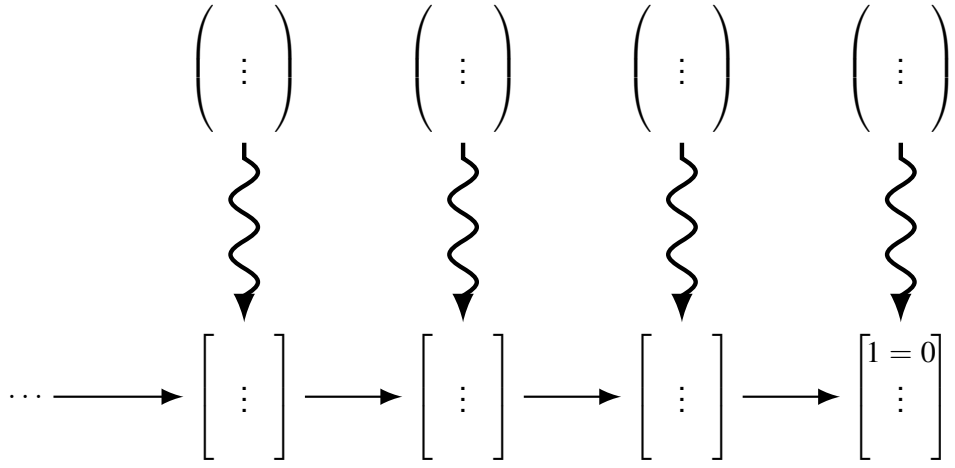

1. a parallel sequence of $(\ldots)$ such that $(\ldots) \rightsquigarrow [\ldots]$;

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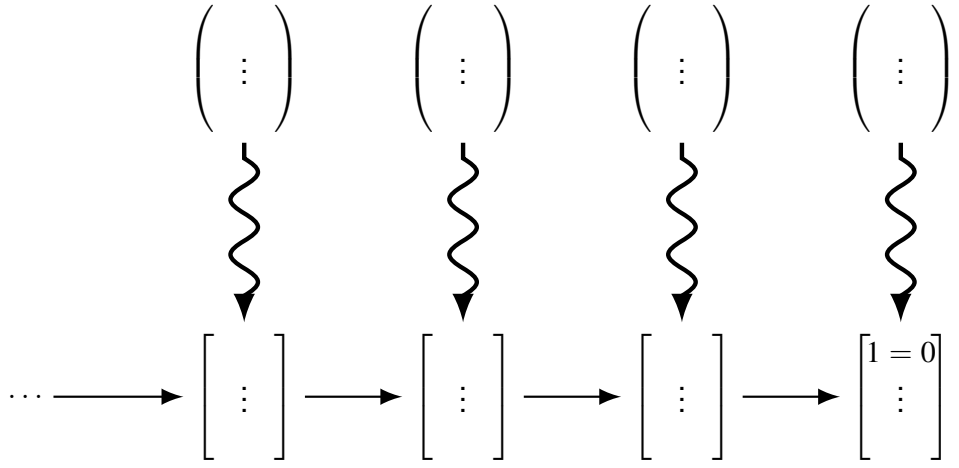

1. a parallel sequence of $(\ldots)$ such that $(\ldots) \rightsquigarrow [\ldots]$;
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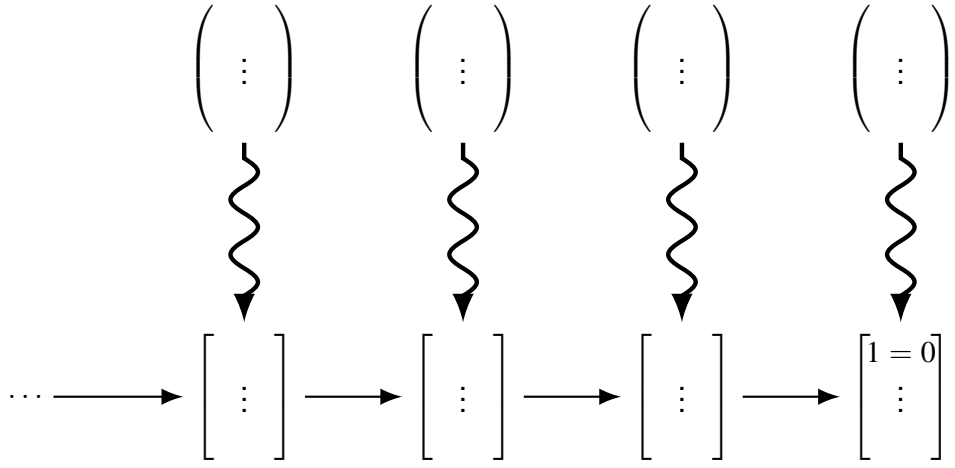

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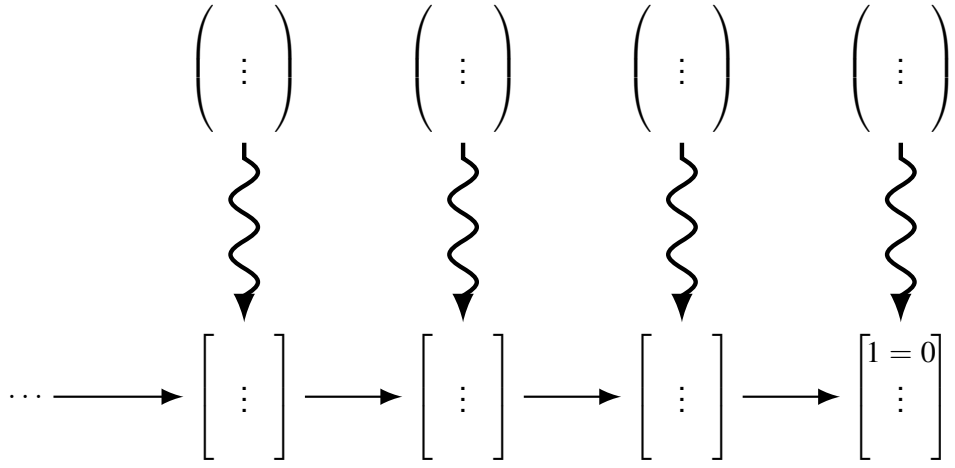

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2. size of $(\ldots)$ if at most twice the size of $[\ldots]$; **(assuming monomial space $\leq n/8$:)**
3. $(\ldots)$ of size $\leq n/4$ are all satisfiable;
4. contradiction since $(\ldots) \rightsquigarrow [1 = 0]$

SPECIAL CONFIGURATIONS

(2-CNFs where no pigeon is mentioned twice)

$$\begin{pmatrix} \neg f_{1,3} & \vee & f_{4,2} \\ f_{5,1} & \vee & f_{7,3} \\ & \vdots & \\ \neg f_{6,5} & \vee & \neg f_{2,3} \end{pmatrix}$$

A satisfying assignment:

- satisfies the 2-CNFs
- no collision on the **occurring** pigeons

Observation

Any special configuration with at most $n/4$ clauses is satisfiable.

Proof.

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1. at most $n/2$ pigeons;
2. at least $n/2+1$ free holes per pigeon;
3. at least one free hole per satisfied literal.

□

Input: $(M_0, M_1, \ldots, M_l)$ with $|M_i| \leq n/8$

Output: $(S_0, S_1, \ldots, S_l)$ such that $|S_i| \leq 2|M_i|$ and S_i implies M_i

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$S_0 := \emptyset$ [Initial configuration]

$S_{i+1} := S_i$ [Inference]

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[Download of $F(a) \neq s \vee F(b) \neq s$]

$S_{i+1} := S_i$ $[a, b \in S_i]$

$S_{i+1} := S_i \cup \{f_{a,1} \vee f_{b,1}\}$ $[a, b \notin S_i]$

$S_{i+1} := S_i \cup \{f_{a,1} \vee f_{c,1}\}$ for some $c \notin S_i$ $[a \notin S_i, b \in S_i]$

[Erasure step] is the hard case

How may clauses in 2-CNF influence the value of a monomial?

≤ 2 space complexity preserved;

> 2 weak influence, most clauses can be removed.

Proof recap

Assuming monomial space $\leq n/8$:

- a corresponding sequence of small 2-CNFs;
- all such small 2-CNFs are “satisfiable”;
- last memory configuration is satisfiable. **(contradiction)**

WRAPPING UP

- PCR is a candidate model of future SAT solvers;
- we need to study space to discover if it is the case;
- we have sketched a space lower bounds for $2 \log n$-CNFs.

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- A linear PCR space lower bounds for (random) 3-CNFs
- space bounds for other proof systems (e.g. cutting planes)
- trading space for time

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- theoretical space bounds vs memory usage
- improve PCR implementations

Thank you

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