Superstring Perturbation Theory Revisited

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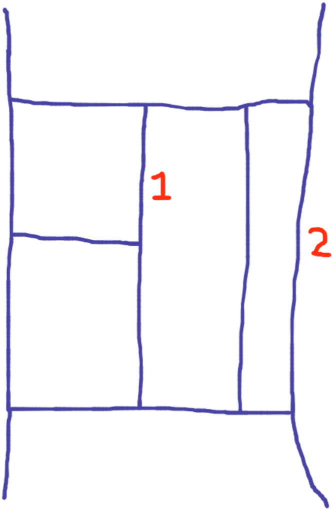
The role of modular invariance in string perturbation theory was discovered initially by J. Shapiro about forty years ago, after C. Lovelace had shown the special role of 26 dimensions. Although it took time for this to be fully appreciated, modular invariance eliminates the ultraviolet region from string and superstring perturbation theory, and consequently there is no issue of ultraviolet divergences. I will have nothing new to say about this today.

However, the literature from the 1980's has left some small unclarity about the infrared behavior of superstring perturbation theory, and this is what I want to revisit. First of all, the general statement one wants to establish is simply that the infrared behavior of superstring perturbation theory is the same as that of a field theory with the same massless particles and low energy interactions. There are some aspects of this that I want to reconsider. It is just a question of some details since at least 98% of the work was done 25 years ago.

I want to give a couple of examples of what I mean in saying that the infrared behavior of string theory is the same as that of a corresponding field theory. Let us consider a Feynman diagram. A very simple question of infrared behavior is to consider what happens when a single propagator goes on shell. First I'll consider a propagator whose "cutting" does not separate a diagram in two.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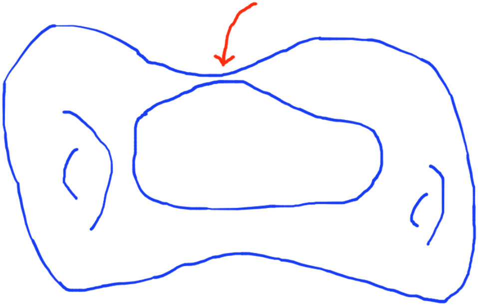
Let us assume our particles are massless so the propagator is $1/k^2$. In D noncompact dimensions, the infrared behavior when the momentum in a single generic propagator goes to zero is

$$\int \mathrm{d}^D k \, \frac{1}{k^2}$$

and this converges if $D > 2$. (For an exceptional internal line, such as the one labeled 2 in the diagram, the infrared behavior when a single momentum goes to zero is worse, because this forces other propagators to go on shell. In the case shown in the sketch, the condition to avoid a divergence is actually $D > 4$.)

All this has a close analog in string theory. First of all, a nonseparating line that goes to zero momentum is analogous to a nonseparating degeneration of a Riemann surface.



A degeneration of a Riemann surface – separating or not – can be described by an equation

$$xy = \varepsilon,$$

where x is a local parameter on one side, y is one on the other, and ε measure the narrowness of the neck – or, by a conformal transformation, the length of the tube separating the two sides.

The contribution of a massless string state propagating through the neck is

$$\int d^D k \int |d^2\varepsilon| \varepsilon^{L_0-1} \overline{\varepsilon}^{\overline{L}_0-1} = \int d^D k \int |d^2\varepsilon| |\varepsilon\overline{\varepsilon}|^{k^2/2-1}$$

where I use $L_0 = \overline{L}_0 = k^2/2$. Instead of doing the integral, let us introduce the analog of the Schwinger parameter by $\varepsilon = \exp(-(t+is))$ where s is an angle and t plays the same role as the Schwinger parameter of field theory. The integral over s just gives a factor of 2π, giving

$$2\pi \int d^D k \int^{\infty} \mathrm{d}t \exp(-tk^2).$$

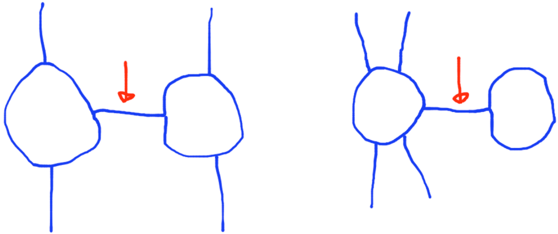
(Note that I indicated the upper limit of the t integral but not the lower limit, which is affected by modular invariance.)

This agrees perfectly with field theory even before doing the k or t integral, bearing in mind that the Schwinger representation of the Feynman propagator is

$$\frac{1}{k^2} = \int_0^\infty \mathrm{d}t \exp(-tk^2).$$

Just as in field theory, we could also consider a situation in which one momentum going to zero puts other lines on-shell. This gives an infrared divergence if $D \leq 4$, whether in field theory or string theory.

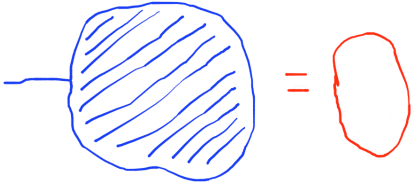
There are many other questions that match simply between string theory and field theory, for example "cutting" a diagram to probe unitarity. For something where the match is less straightforward, let us consider a separating line. Here are two cases in field theory.



The difference is that in the second case the external lines are all on one side.

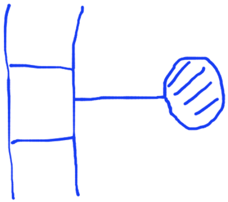
We don't integrate over the momentum that passes through the separating line; it is determined by momentum conservation. On the left, this momentum is generically nonzero so for typical external momenta, we don't sit on the $1/k^2$ singularity; when we vary the external momenta, the $1/k^2$ gives a pole in the S-matrix (at least in this approximation). This is physically sensible and we do not try to get rid of it. On the right, it is different. The momentum passing through the indicated line is 0 and hence we will get $1/0$ unless the matrix element on the right vanishes.

So a field theory with a massless scalar has a sensible perturbation expansion only if the "tadpole" or one point function of the scalar vanishes:



We have to impose this condition for all massless scalars. However, it is non-trivial only for the ones that are invariant under all (local or global) symmetries.

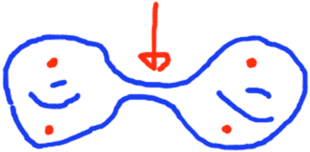
In field theory, if the tadpoles do vanish, we just throw away all the corresponding diagrams

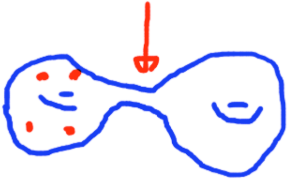


and evaluate the S-matrix by summing the others.

All this is relevant to perturbative string theory, since whenever we do have a perturbative string theory, there is always at least one massless neutral scalar field that might have a tadpole, namely the dilaton. So perturbative string theory will only make sense if the dilaton tadpole vanishes (along with other tadpoles, if there are more massless scalars). In either field theory or string theory, the usual way to show vanishing of the tadpole of a massless scalar (neutral under all symmetries) is to use supersymmetry. Indeed, without supersymmetry, it is unnatural to have a massless neutral scalar.

Just as in field theory, we can distinguish different kinds of diagrams with separating degenerations:





As one should anticipate from what I have said, it is the one on the right that causes trouble. There are two reasons that this problem is harder to deal with than in field theory:

1) Technically, it is harder to understand spacetime supersymmetry in string theory than in field theory, and to use it to show that the integrated massless tadpoles vanish.

2) In field theory, the tadpoles are the contributions of certain diagrams and if they vanish, one just throws those diagrams away. String theory is more subtle because it is more unified; the tadpole is part of a diagram that also has nonzero contributions. Vanishing tadpoles makes the diagrams of string perturbation theory infrared convergent but only conditionally so and so there is still some work to do to define them properly. (This is a point where I believe I've improved what was said in the 80's, but I won't explain it today.)

Since we can only hope for the tadpoles to vanish in the supersymmetric case, we have to do supersymmetric string theory. This means that our Riemann surfaces are really super Riemann surfaces. A super Riemann surface is a rather subtle sort of thing. It takes practice to get any intuition about them, and I can't really describe this topic today.

All I will say is that a super Riemann surface (with $N = 1$ SUSY) is a supermanifold Σ of dimension $(1|1)$, with some special structure – a superconformal structure. An NS vertex operator $\Phi(\widetilde{z}; z|\theta)$ is inserted at a generic point on Σ (my notation for worldsheet coordinates is adapted to the heterotic string), while a Ramond vertex operator is inserted at a point on Σ at which the superconformal structure of Σ has a certain kind of singularity.

Friedan, Martinec, and Shenker in 1985 explained what kind of vertex operators are inserted at such superconformal singularities – they are often called spin fields – and how to compute their operator product expansions. In particular, the operators that generate spacetime supersymmetry are of this kind, so their work made it possible to see spacetime supersymmetry in a covariant way in superstring theory. As regards practical calculations, their work also made it possible to compute in a covariant way arbitrary tree amplitudes with bosons and fermions, and many loop amplitudes of low order. Moreover, in the intense period of effort in the 1980's, the main ingredients of a systematic, all-orders algorithm were assembled. My reconsideration of the problem has aimed at simplifying and extending the understanding of a few details.

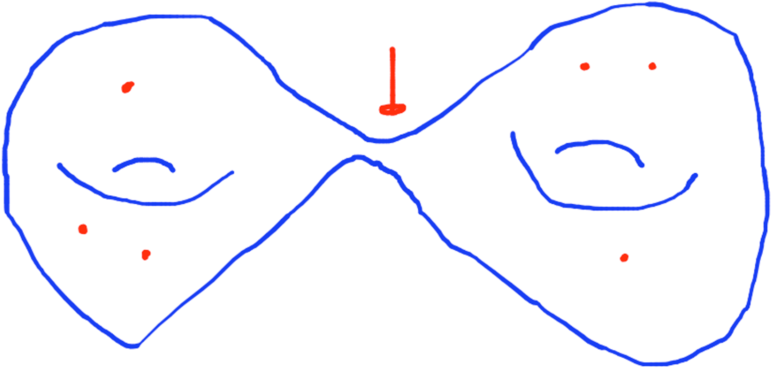
It turns out that this problem requires greater sophistication in understanding supermanifolds and how to integrate over them than is needed in any other problem that I know of in supersymmetry and supergravity. That is probably the main reason for any unclarity that surrounds it.

Some low order cases are deceptively simple and really don't give a good idea of a general algorithm for superstring perturbation theory. For example, in genus $g = 1$, the dilaton tadpole vanishes in $\mathbb{R}^{10}$ by summing over spin structures, but the fact that this makes sense depends upon the fact that in $g = 1$ (with no punctures) there are no fermionic moduli. As soon as there are odd moduli, there is no meaningful notion of two super Riemann surfaces being the same but with different spin structures. In particular, in genus $g > 1$, there is no meaningful operation of summing over spin structures without integrating over supermoduli. In genus $g = 2$, E. D'Hoker and D. Phong found an effective and very beautiful way to integrate over fermionic moduli first (after which the sum over spin structures makes sense and could be used to show the vanishing of the dilaton tadpole) and then integrate over bosonic moduli. This calculation is currently the gold standard, but it is more or less clear that for generic g their procedure has no analog and the only natural operation is the combined integral over all bosonic and fermionic moduli.

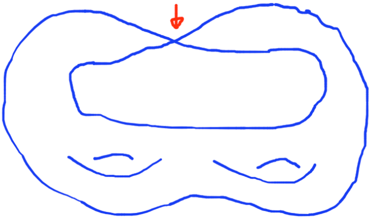
Instead of talking more about what doesn't work in general, let us discuss what does work. First of all, there is a natural measure on supermoduli space, which I will call $\widetilde{\mathcal{M}}_{g,n}$. This was constructed in the 1980's via conformal field theory (in varied approaches by G. Moore, P. C. Nelson, and J. Polchinski; E. & H. Verlinde; L. Alvarez-Gaumé, C. Gomez, P. C. Nelson, G. Sierra, and C. Vafa; and D'Hoker and Phong) by adapting the analogous formulas for the bosonic string. Also, though less well known, there is for the important case of strings in $\mathbb{R}^{10}$ a slightly abstract but very elegant – and mathematically completely rigorous – construction of the measure by A. Rosly, A. Schwarz and A. Voronov (1985) via algebraic geometry.

Another key point is that integration of a bounded function on a compact supermanifold is a well-defined operation just as on an ordinary manifold. We will say a little about integration later.

Supermoduli space is not compact – or if we take its Deligne-Mumford compactification, then the function we want to integrate has singularities – precisely because of the infrared effects that we have been talking about.



Although supermoduli space is very subtle, if one asks precisely the questions whose answers one needs, those particular questions tend to have simple answers. For instance, although a sum over spin structures (independent of the integration over supermoduli) does not make sense in general, a very small piece of it makes sense when a node develops



and this leads to the GSO projection on the physical states that propagate through the node.

For another example, the description of the moduli space near a node is just as simple as for a bosonic Riemann surface. In the bosonic case, the gluing of a surface with local parameter x to one with local parameter y is by

$$xy = \varepsilon'.$$

For the super case, the gluing of local parameters x, θ to y, ψ is by an almost equally simple formula

$$xy = \varepsilon^2, \quad y\theta = \varepsilon\psi, \ x\psi = \varepsilon\theta.$$

Importantly, the gluing depends in both cases on only one bosonic parameter ε or ε'. In the super case, there are no odd moduli for the gluing. The locus $\varepsilon = 0$ in $\widetilde{\mathcal{M}}_{g,n}$ is a product of spaces of the same type $\widetilde{\mathcal{M}}_{g_1,n_1+1} \times \widetilde{\mathcal{M}}_{g_2,n_2+1}$ with $g_1 + g_2 = g$, $n_1 + n_2 = n$.

This is the fundamental reason that there is no integration ambiguity in superstring theory. There is a good parameter at infinity. If one replaces x and y by other local parameters, one transforms ε by $\varepsilon \to e^{\phi}\varepsilon$ but not by $\varepsilon \to \varepsilon + \alpha\beta$, which could have led to an integration ambiguity, as was explained in the literature of the 1980's.

It is impossible to explain everything today and I decided to explain the origin of the picture-changing phenomenon. Let us start with an ordinary manifold M with bosonic coordinates $t^1 \dots t^n$. For each coordinate t^i, we introduce a corresponding fermionic variable $\mathrm{d}t^i$, which we consider to have degree 1. A function $F(t^1 \dots t^n | \mathrm{d}t^1 \dots \mathrm{d}t^n)$ can be expanded as a polynomial in the fermionic variables $\mathrm{d}t^i$. For example, the term of degree p is

$$\sum_{i_1 < \cdots < i_p} F_{i_1 \dots i_p}(t^1 \dots t^n) \mathrm{d}t^{i_1} \dots \mathrm{d}t^{i_p}$$

and is called a p-form. We define the operator

$$\mathrm{d} = \sum_{i=1}^{n} \mathrm{d}t^i \frac{\partial}{\partial t^i}$$

that maps p-forms to $p+1$-forms.

We define integration just by following ordinary rules to integrate over bosons and fermions:

$$\int \mathcal{D}(t, \mathrm{d}t)\, F(t^1 \dots | \dots \mathrm{d}t^n).$$

The integral over the $\mathrm{d}t$'s is a Berezin integral. By the rules of Berezin integration, we only get a nonzero integral if all $\mathrm{d}t$'s are present in F. This means that if M is n-dimensional, a p-form can be integrated only if $p = n$. An important fact to remember is that $F(t^1 \dots | \dots \mathrm{d}t^n)$ is automatically a polynomial in the $\mathrm{d}t^i$, just because they are fermionic. Stokes's theorem says that if M has no boundary, then

$$\int \mathcal{D}(t|\mathrm{d}t)\, \mathrm{d}G(t|\mathrm{d}t) = 0$$

for any G.

Now suppose that M is a supermanifold, with even and odd coordinates $t^1 \dots t^n | \theta^1 \dots \theta^m$. Aiming to do the same thing, we introduce new variables $\mathrm{d}t^1 \dots \mathrm{d}t^n$ and $\mathrm{d}\theta^1 \dots \mathrm{d}\theta^m$ such that for any even or odd x, $\mathrm{d}x$ has opposite statistics from x. We define a scaling degree under which $\mathrm{d}t$ and $\mathrm{d}\theta$ have degree 1, while t and θ have degree 0. So the exterior derivative

$$\mathrm{d} = \sum_{i=1}^{n} \mathrm{d}t^i \frac{\partial}{\partial t^i} + \sum_{j=1}^{m} \mathrm{d}\theta^j \frac{\partial}{\partial \theta^j}$$

increases the degree by 1.

By a “form” we mean a function $F(t^1 \dots \mathrm{d}\theta^m | \theta^1 \dots \mathrm{d}t^n)$ of all the even and odd variables. Unlike the previous case, F is not automatically polynomial in all of the $\mathrm{d}x^i$, since some of them are bosonic. We call F a differential form if it is polynomial. But in that case, F can't be integrated. We'd like to define the integral

$$\int \mathcal{D}(x, \mathrm{d}x)\, F(x, \mathrm{d}x)$$

where x runs over all of $t^1 \dots t^n | \theta^1 \dots \theta^m$. But if F has a polynomial dependence on the even variables $\mathrm{d}\theta^i$, then the integral over those variables will certainly diverge.

A typical example of a form that can be integrated is

$$F(x, \mathrm{d}x) = f(t^1 \dots | \dots \mathrm{d}\theta^m)\mathrm{d}t^1 \dots \mathrm{d}t^n \delta(\mathrm{d}\theta^1) \dots \delta(\mathrm{d}\theta^m).$$

This is called an integral form (of top degree). For such an $F(x, \mathrm{d}x)$, one can integrate out the $\mathrm{d}t$'s and $\mathrm{d}\theta$'s and reduce to

$$\int \mathcal{D}(x, \mathrm{d}x)\, F(x, \mathrm{d}x) = \int \mathcal{D}(t^1 \dots | \dots \theta^m)\, f(t^1 \dots | \dots \theta^m),$$

where the integral on the right hand side is an ordinary Berezin integral.

More generally, if $F(x, \mathrm{d}x)$ has definite degree (under scaling of $\mathrm{d}t$ and $\mathrm{d}\theta$) and can be integrated over a submanifold $N \subset M$ of dimension $p|q$, then F must obey two conditions: (1) It must have degree $p - q$. (2) It must have picture number $-q$. The picture number is defined as minus the number of $\mathrm{d}\theta$'s with respect to which $F(x, \mathrm{d}x)$ has delta function localization. Superstring perturbation theory naturally leads to this sort of formalism, with M being the moduli space of super Riemann surfaces. (The meaning of pictures was elucidated in the 1990's by A. Belopolsky, following work of E. and H. Verlinde in the late 1980's.)

The next topic that I want to discuss is integration by parts. We need this to prove the decoupling of pure gauge degrees of freedom and also to prove spacetime supersymmetry and vanishing of tadpoles. This is actually one place where what was done in the 1980's can be improved (but again, see Belopolsky). Traditonally, arguments involving integration by parts have been made by first integrating over odd moduli and then using the bosonic version of Stokes's theorem to integrate by parts on a purely bosonic manifold. However, this introduces many technicalities and complications. There is a perfectly good super-analog of Stokes's theorem and it is best to use this.

You probably all know the basic idea of fermionic integration by parts, which is that for an odd variable α and any function $f(\alpha)$, one has

$$\int \mathrm{d}\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} f = 0.$$

Indeed the Berezin integral

$$\int \mathrm{d}\alpha \cdot 1 = 0, \int \mathrm{d}\alpha \cdot \alpha = 1$$

is defined to make this true.

Integration of forms over supermanifolds has an equivalent relation (attributed to Bernstein and Leites, 1979, also with origins in the supergravity literature):

$$\int_X \mathrm{d}\Lambda = \int_{\partial X} \Lambda.$$

Here $\mathrm{d}\Lambda$ is the analog of a "volume form" and Λ is the analog of a "form of codimension 1." (They are integral forms and have delta function – or derivative of delta function – localization with respect to all $\mathrm{d}\theta$'s.)

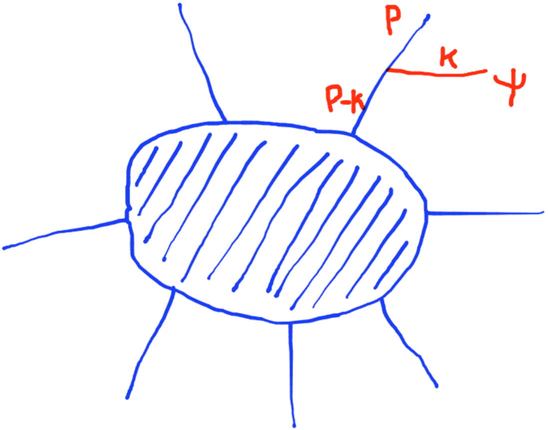
Now a scattering amplitude $\langle V_1 V_2 \dots V_n \rangle$ is associated with a "volume form" Υ that must be integrated over, roughly speaking, supermoduli space. Just as for the bosonic string, if, say, $V_1 = \{Q, W\}$ for some W, then the volume form Υ is $\mathrm{d}\Lambda$ for some Λ. Then in checking decoupling of $\{Q, W\}$, we get

$$\langle \{Q, W\} V_2 \dots V_n \rangle = \int_\Gamma \Upsilon = \int_{\partial\Gamma} \Lambda.$$

If Λ has a good behavior on $\partial\Gamma$, then the right hand side vanishes and so therefore does the left hand side. For vanishing of $\int_{\partial\Gamma} \Lambda$, one needs to know (i) vanishing of tadpoles, otherwise none of the integrals converge and (ii) a certain condition about mass renormalization that I have been suppressing though we will incorporate it shortly. (This condition has a field theory analog: the condition on which modes are supposed to decouple can depend on the particle masses so it can be affected by mass renormalization.)

This argument is much simpler than any argument using the bosonic version of Stokes's theorem. It has an important corollary. If one knows that the massless tadpoles vanish, then spacetime supersymmetry is a special case of the decoupling of pure gauge modes. This may be deduced from the following standard argument.

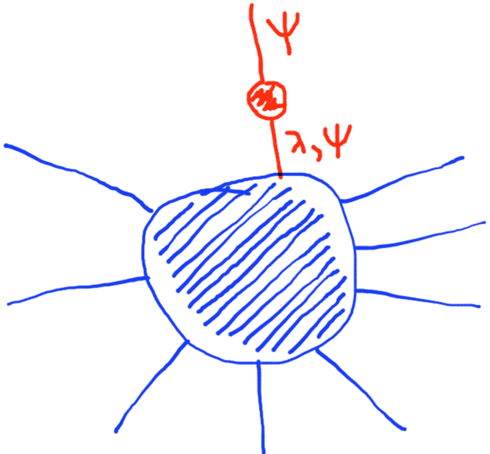
We consider a scattering amplitude involving a soft gravitino. We take its wavefunction to be $\Psi_{I\alpha} = \exp(ik \cdot x)\eta_{I\alpha}$ where I is a vector index and α is a spinor index. A matrix element for emission of a soft gravitino has singular terms where the gravitino is attached to an external leg:



I've drawn this as a field theory picture, but I hope you all understand that there is an analogous string theory picture. The line that emits the gravitino is just slightly off shell, with momentum $P-k$. If $P^2 = M^2$ and $k^2 = 0$, then $(P-k)^2 = M^2 - 2P\cdot k$, so the propagator of this line is $1/((P-k)^2 - M^2) = -1/2P\cdot k$ (or something similar if the line represents a particle with spin). This is singular at $k \to 0$. The amplitude also comes with a numerator which is a matrix element of the supercurrent S, via which the gravitino couples, between the two states $\langle (P-k)'|S|P\rangle$ (the prime in $\langle (P-k)'|$ is meant to remind us that S has acted on the particle spin). In all, this soft emission amplitude is essentially $\langle (P-k)'|S|P\rangle/(-2P\cdot k)$ times an amplitude with the external gravitino and particle $|P\rangle$ replaced by an external state $|(P-k)'\rangle$.

Now if we set the gravitino polarization vector-spinor $\eta_{I\alpha}$ to be $k_I\zeta_\alpha$ (for some spinor ζ_α), then the whole amplitude must vanish. This is a special case of the decoupling of states $\{Q, W\}$ for any W. It is hard to evaluate this condition exactly, but its leading behavior as $k \to 0$ can be evaluated, and is the sum of terms of the form $\langle (P-k)'|k\cdot S|P\rangle/(-2P\cdot k)$ times an amplitude with one of the external particles $|P\rangle$ replaced by $|(P-k)'\rangle$. The sum of all these terms must vanish and this is the Ward identity of spacetime supersymmetry.

This type of argument may be familiar from field theory. It works the same way in string theory, except that we have to know that the massless tadpoles vanish (or none of the amplitudes are defined). However, in either field theory or string theory, I have left something out so far. Potentially, the supersymmetric Ward identity can contain another term if the coupling of a soft gravitino has a singular contribution like this:



This happens if loops generate a term in the effective action that is of the form $\overline{\Psi}_I \Gamma^I \lambda$, with some previously massless fermion λ, or a term $\overline{\Psi}_I \Gamma^{IJ} \Psi_J$. In the first case, supersymmetry is spontaneously broken, with λ as a Goldstone fermion; in the second case, we land in AdS space with unbroken supersymmetry. (An example of the first type was described by Dine, Ichinose, Seiberg and by Atick, Dixon, Sen in the 1980's. No example of the second type seems to be known in any dimension, though it seems to me that it might be possible for this to happen in three spacetime dimensions.)

In many classes of string vacua, it is straightforward to prove that $\overline{\Psi}_I \Gamma^I \lambda$ and $\overline{\Psi}_I \Gamma^{IJ} \Psi_J$ terms are not generated by loops. For example in all of the ten-dimensional superstring theories except Type IIA, this follows from considerations of spacetime chirality which make it impossible to write the interactions in question. For Type IIA, the result follows if one also uses the fact that perturbation theory has $(-1)^{F_L}$ as a symmetry. (This excludes the Romans mass term.)

So all we need in order to land in a happy place is an extension of this type of reasoning to show that the massless tadpoles vanish. Though this is expected to follow from spacetime supersymmetry, I believe that the type of argument I have given is not quite powerful enough to prove it.

Given the experience from the old literature (see for example E. Martinec (1986), Atick, Moore and Sen (1988)), one expects that what one should do is to make a similar argument but with k set to 0 at the beginning. We used the fact that the vertex operator $V_{\Psi,k}$ for a gravitino of polarization $\eta_{I\alpha} = k_I \zeta_\alpha$ is $\{Q, W_k\}$ for some W_k. If we set $k = 0$, then $V_{\Psi,k} = 0$ and the relation becomes $0 = \{Q, S\}$ where S, which is the limit of W_k for $k = 0$, is the fundamental spin field. S has ghost number 1 (while a vertex operator for particle emission such as V has ghost number 2) so by analogy with more simple cases, the condition $\{Q, S\} = 0$ should mean that S generates a symmetry in spacetime.

For practice, let us look at a correlation function $\langle SV_1 \dots V_n \rangle$. This can't be integrated over the usual integration cycle Γ, since the ghost number is too small by 1. But it can be integrated over the codimension 1 cycle $\partial\Gamma$. Schematically, we have

$$0 = \int_\Gamma \langle \{Q, S\} V_1 \dots V_n \rangle = \int_{\partial\Gamma} \langle SV_1 \dots V_n \rangle.$$

This vanishing relation can be written as a sum of contributions from the many components of $\partial\Gamma$. Many of them don't contribute because the momentum flowing through the node is off-shell.

The following contributions do have on-shell momentum flowing through the node and definitely can contribute:

If these are the only nonzero boundary contributions, then again we get the supersymmetric Ward identity, much as before.

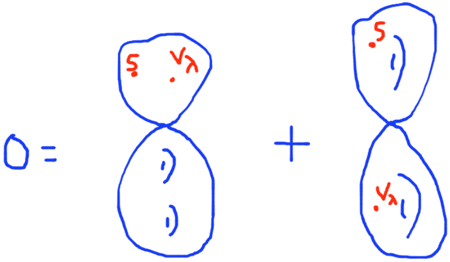
The other contributions that might appear (because they involve on-shell momentum flowing through the node) correspond to supersymmetry breaking (or a cosmological constant) or a massless tadpole. We'll draw them in a moment, in a slightly simpler situation.

To finally address the question of whether there is a massless tadpole, let us replace the product $V_1 \cdots V_n$ with a single vertex operator V_λ of a massless fermion that is a superpartner of a scalar ϕ whose tadpole we want to understand. The relation

$$0 = \int_{\partial \Gamma} \langle S V_\lambda \rangle$$

is now simple because $\partial \Gamma$ has only two types of components.

The relation is explicitly then



The first term is the dilaton tadpole, and the second – but this still needs to be clarified – should be nonzero precisely when supersymmetry is spontaneously broken (or a cosmological constant is being generated).

When one can show that the gravitino cannot gain a mass in perturbation theory – for instance in $\mathbb{R}^{10}$ – this relation should (when combined with what was discovered in the 80's and a few details that we haven't had time for today) – remove the very slight unclarity that has surrounded superstring perturbation theory.