Effective Fractal Dimension in Computational Complexity and Algorithmic Information Theory

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Isaac Newton Institute, May 2012

Contents

Fractal dimensions

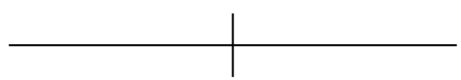
- I will talk specifically about Hausdorff dimension
- Widely used in fractal geometry

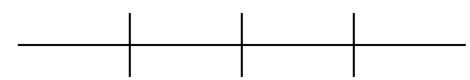
- It can be defined in any metric space. Today I will stick to Σ^{∞} the set of infinite sequences over finite Σ

Today we'll use $\Sigma=\{0,1\}$

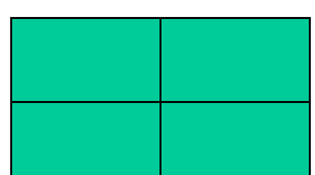
Fractal dimension

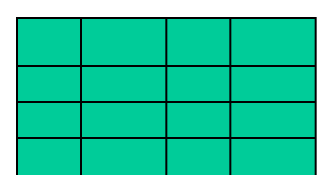
A line can be covered with N balls of diameter 1/N





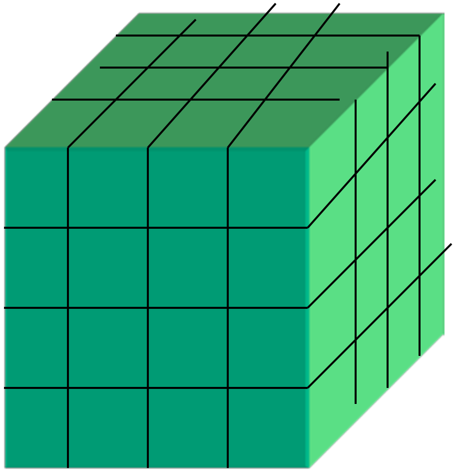
A surface can be covered with N^2 balls of diameter 1/N





A has dimension D

A can be covered with around N^D balls of diameter 1/N



A can be s-covered if

Number-of-balls * diameter^s

is bounded for arbitrarily small diameter

Hausdorff dimension

Let $s \in [0, \infty)$, $\delta>0$

- $H^s_\delta(A) = \inf \Sigma |U_i|^s$ (infimum over all covers of A, $\{U_i\}$ with diameter $|U_i| < \delta$)
- $H^s(A) = \lim_{\delta \to 0} H^s_\delta(A)$

dim(A)= $\inf \{s \mid H^s(A)=0\}$

New characterization

- Effective dimension is based in a characterization of Hausdorff dimension on Σ^{∞} given by Lutz (2000)

- The characterization is a very clever way to deal with a single covering using gambling

Hausdorff dimension (Lutz characterization)

A gambling game

A player bets on each element of an unknown sequence $x \in \Sigma^{\infty}$

- When betting on x[n] he knows x[1..n-1].
- A strategy in this game is a function $b: \Sigma^{*} x\ \Sigma \to [0,1]$, b(w, a)=$\alpha$ means "if x[1..n-1]=w, bet a fraction α of present capital to x[n]=a".
- If he wins he multiplies by | Σ |, if he loses he gets 0.
- But the house takes a fixed percentage before each bet.

Hausdorff dimension (Lutz characterization)

A **martingale** is $d: \Sigma^* \rightarrow [0,\infty)$ such that

$$\frac{\Sigma_{a \in \Sigma} d(wa)}{|\Sigma|} = d(w)$$

It is the capital corresponding to a fixed strategy and a fair game (the house takes nothing)

Hausdorff dimension (Lutz characterization)

An **s-gale** is $d: \Sigma^* \rightarrow [0,\infty)$ such that

$$\frac{\Sigma_{a \in \Sigma} d(wa)}{|\Sigma|^s} = d(w)$$

It is the capital corresponding to a fixed strategy and a the house taking a fraction of $1 - |\Sigma|^{s-1}$

The smaller the s the more the house takes

Hausdorff dimension (Lutz characterization)

- An s-gale d **succeeds** on $x \in \Sigma^{\infty}$ if
  $\limsup_{n \to \infty} d(x[1..n]) = \infty$
- d **succeeds** on $A \subseteq \Sigma^{\infty}$ if d succeeds
  on each $x \in A$
- G(A) = {s | there is an s-gale that succeeds on A}

The smaller the s the harder to succeed

Hausdorff dimension (Lutz characterization)

Let $A \subseteq \Sigma^{\infty}$

The Hausdorff dimension of A is the infimum of G(A).

[Lutz 00]

- $\dim(A) = \inf \{ s \mid \forall \delta, \varepsilon\ \exists \{U_i\}$

  $A \subseteq \cup U_i\ , \mid U_i \mid < \delta$

  $\Sigma |U_i|^s < \varepsilon \quad \}$

- $\dim(A) = \inf \{ s \mid \exists$ d an s-gale

  for each $x \in A$,

  $\limsup_n d\,(x[1..n]) = \infty \ \}$

Efectivizing Hausdorff dimension

- We restrict to constructive or effective strategies and get the corresponding “dimensions” that are meaningful in subsets of Σ^{∞} we are interested in

Constructive dimension [Lutz00]

- A function f: $\Sigma^* \rightarrow R$ is **constructive** if it is lower semi-computable

(there exists a non-decreasing computable function f': $\Sigma^* \times N \rightarrow R$ such that for every w

$$\lim_{m\rightarrow\infty} f'(w,m)=f(x) \text{)}$$

- $G_{constr}(A)$ = {s | there is a constructive s-gale that succeeds on A}

- $\mathrm{cdim}(A) = \inf G_{constr}(A)$

Effective dimensions

- Restricting to effectively computable strategies we have:
  - computable by a finite automata $\dim_{FS}$
  - computable in polynomial time $\dim_{p}$
  - computable in polynomial space $\dim_{pspace}$
- Each of this effective dimensions is “the right one” for a set of sequences (complexity class)

[Lutz00, DaiLatLutMay04]

Contents

1. Fractal dimensions: a new characterization
2. **r. b. dimension in Computational Complexity**
3. Information theory and compression algorithms

Uses of effective dimension in complexity

1. <u>To prove abundance results</u>

Example: How dense are hard sets for exponential time?

E= DTIME(2^n)

$\dim_p(E)=1$

DENSE= { L | $\exists\varepsilon\ \forall n\ |L^{\le n}|>2^{n^\varepsilon}$ }

Density of hard sets

The p-dimension of sets that reduce to nondense sets (under p-majority reduction) is 0

Same for more general reductions

[Harkins-Hitchcock 2011]

Density of hard sets

1. Abundance result (dim$_p$(E)=1)

Most sets in E do not reduce to nondense sets

2. Existence result (prob. method)

There is a set in E that does not reduce to nondense sets

- Consequence:

All hard sets for E are dense

Uses of effective dimension in complexity

3. New hypothesis

If $\dim_p(NP) > 0$ then all hard sets for NP are dense

Uses of effective dimension in complexity

3. New insights

Small span theorems: given a reduction, either the upper or the lower span is small

Small span theorem

- For every A in E, either

$$\dim'_p(P_m(A) \cap E) = 0$$

or

$$\dim'_p(P_m^{-1}(A) \cap E) = 0$$

[Hitchcock]

Open questions

- What is the p-dimension of NP?
- Is it possible that $0 < \dim_p(NP) < 1$?
- **Oracle results?**

Open questions

- Finding sources for BPP

Can I have $P^A = BPP$ when $\dim_?(A)>0$?

Open questions

- What kind of

  strong natural proofs against P/poly

  can we get from $\dim_p(\text{P/poly}) = 0$?

- Known: how to get natural proofs from a weaker assumption, therefore

Strong ps.generators ➔ $\dim_p(\text{P/poly}) \neq 0$

Open questions

- Disjoint pairs: (A, B) such that $A \cap B = \emptyset$
- P-separable: S in P s.t. $A \subseteq S$, $B \subseteq S^c$

- Can you get inseparable pairs from $\dim_p(NP)$?
- Strongly inseparable?

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References

- Effective Fractal Dimension Bibliography
  Maintained by John Hitchcock
  http://www.cs.uwyo.edu/~jhitchco/

The dimension of a point

- If we take a single $x \in \Sigma^{\infty}$ we can have cdim({x})>0

- In fact constructive dimension can be defined in a pointwise way [Lutz03]

$$\mathrm{cdim}(A)=\sup_{x \in A} \mathrm{cdim}(\{x\})$$

I will write cdim(x) for cdim({x})

Martin-Löf randomness & information content

- x is Martin-Löf random iff $(\exists c)$

  $\forall n \quad K(x[1..n]) \geq n \log|\Sigma| - c$

  (x is c-incompressible for a constant c)

$$K(w) = \min\{|p| \mid p \in \{0,1\}^*, U(p) = w\}$$

Dimension is Kolmogorov complexity

- The constructive dimension of a sequence is related to its information content
- [Mayordomo 02] For each $x \in \Sigma^{\infty}$

$$\mathrm{cdim}(x) = \liminf_{n\to\infty} \frac{K(x[1..n])}{n \log|\Sigma|}$$

A sequence x has dimension bigger than s iff

$(\exists\varepsilon)\ \forall n \quad K(x[1..n]) \geq (s+\varepsilon)\, n \log|\Sigma|$

In resource-bounded setting

$KS^f(w) = \min\{|p| \mid U(p)=w \text{ in space } f(|w|)\}$

- For each $x \in \Sigma^\infty$

$$\dim_{comp}(x) = \inf_{f \in comp} \liminf_{n \to \infty} \frac{KS^f(x[1..n])}{n \log|\Sigma|}$$

In resource-bounded setting

$KS^q(w) = \min\{|p| \ | \ U(p)=w \text{ in space } q(|w|)\}$

- For each $x \in \Sigma^\infty$

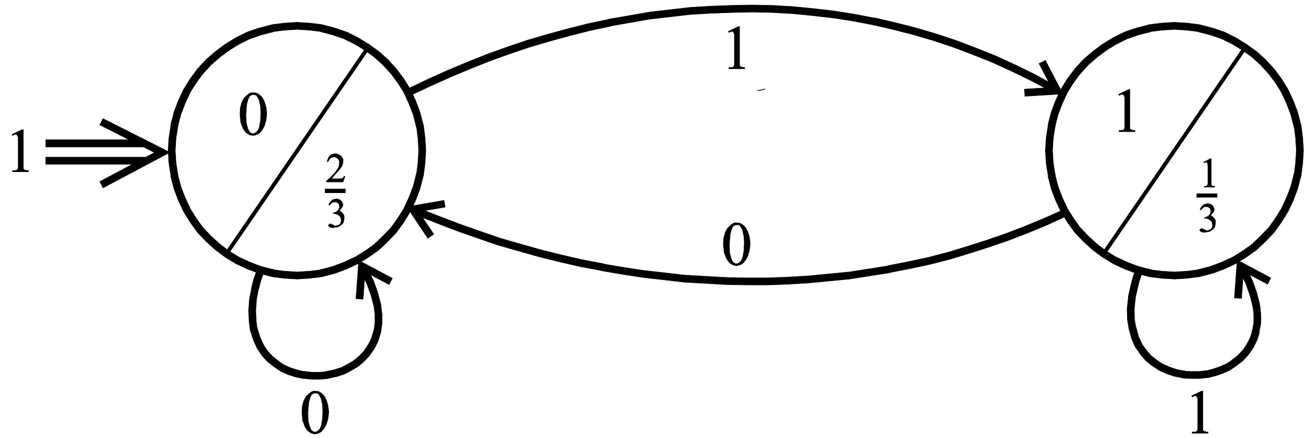
$$\dim_{pspace}(x) = \inf_{\substack{q \\ \text{polynomial}}} \ \liminf_{n\to\infty} \frac{KS^q(x[1..n])}{n \log|\Sigma|}$$

Finite state dimension

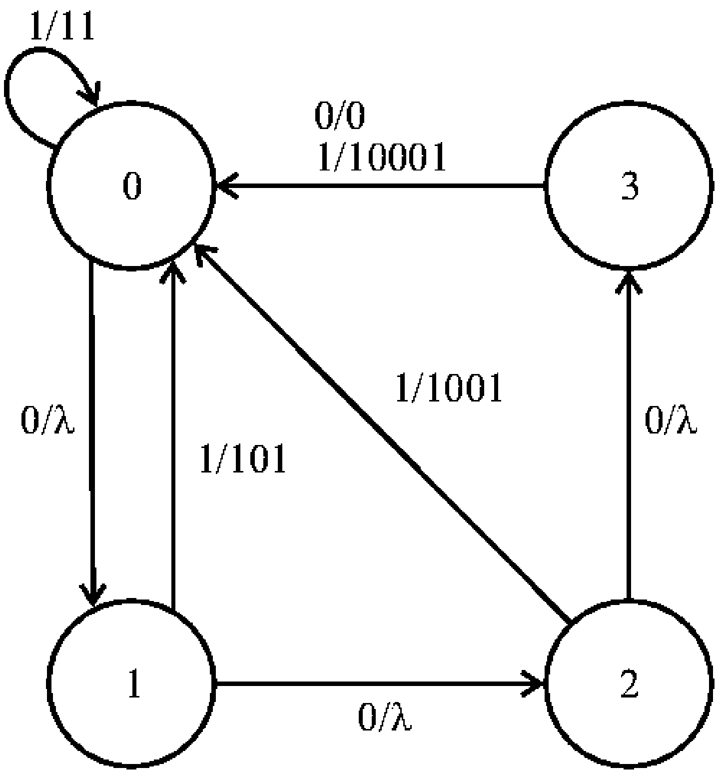
- We consider the case of Finite-State computation

  $G_{FS}(A)$ = {s | there is a Finite State s-gale that succeeds on A}

  $\dim_{FS}(A) = \inf G_{FS}(A)$



Finite state compression



The input can be recovered given the output and the final state

Finite state compression

- Theorem Finite-state dimension is exactly the best compression rate achievable through finite-state compressors [Dai et al 04]

$$\dim_{FS}(x) = \inf_{C\ \text{FS-comp}}\ \liminf_{n\to\infty} \frac{|\text{output}_C(x[1..n])|}{n \log|\Sigma|}$$

Any interest for non computer scientist?

- A real $\alpha \in [0,1]$ is **b-normal** if its base-b represent-ation $x \in \{0,\ldots, b-1\}^\infty$ is a sequence such that for every $w \in \{0,\ldots, b-1\}^*$

$$\lim_{n\to\infty} \frac{\# \{i \leq n \mid x[i+1-|w|..i]=w\}}{n} = b^{-|w|}$$

- Other characterizations are based on the distribution of $\{b^k\alpha\}_k$ (Weyl criterion, etc)

Dimension characterization of normality

- A real $\alpha \in [0,1]$ is b-normal iff its base-b representation $x \in \{0,\ldots, b\text{-}1\}^\infty$ has $\dim_{FS}(x)=1$
- A real $\alpha \in [0,1]$ is b-normal iff its base-b representation $x \in \{0,\ldots, b\text{-}1\}^\infty$ is FS-random

[several papers]

FS-dimension is base dependent

What we know so far

- Several levels of effective dimension for which dimension and compression coincide

  (constructive, pspace, finite state)

- Some intermediate levels for which the question of dimension vs compression has the known limitations of prediction vs compression algorithms

  (polynomial time)

Polynomial time dimension

- If we consider only a class of reasonable compressors (those that don’t start from scratch) then

p-compression is exactly p-dimension

[Lopez-Valdes, May]

pushdown dimension

- We consider BPD the set of pushdown machines that work with a bounded number of λ-transitions per input symbol

- **Theorem** [Albert et al 07]

  BPD-dimension is exactly the best compression rate achievable through BPD-compressors

Still open for general PD-computation

Comparison with other levels

- PD-randomness is not normality
- Still open: Is PD-dimension 1 exactly normality?
- Is PD-dimension base dependent?

Comparison with other levels

- **Theorem** [Albert et al 08]

  PD-compression is incomparable with the Lempel-Ziv compression algorithm:

  - There are sequences for which PD-compression is better
  - There are sequences for which Lempel-Ziv compression is better

Lempel-Ziv compressor is widely-used and universal for FS-compressors

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