Lifshitz Holographic Renormalization from AdS

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Introduction

- Many systems in nature exhibit critical points with non-relativistic scale invariance. Such systems typically have Lifshitz symmetries:

$$
\begin{aligned}
D_z &: \quad \vec{x} \to \lambda \vec{x} \qquad t \to \lambda^z t\,, \\
H &: \quad t \to t + c\,, \\
P_i &: \quad x^i \to x^i + a^i\,, \\
M_{ij} &: \quad x^i \to R^i{}_j x^j\,.
\end{aligned}
$$

- Lifshitz algebra (only nonzero commutators shown, left out M_{ij} and $z \neq 1$):

$$
[D_z, H] = zH\,, \qquad [D_z, P_i] = P_i\,.
$$

- Universality: dissimilar systems behave effectively the same at the critical point.
- Assumption: one of these systems has a holographic dual, i.e. a gravitational theory admitting as a solution a space-time whose isometry group is the Lifshitz group.
- This idea is obviously inspired by the AdS/CFT correspondence. The Lifshitz symmetry group is a subgroup of the relativistic conformal symmetry group.
- Some review papers: [Hartnoll, 2009], [McGreevy, 2009] and [Sachdev, 2011].

- Aim: to construct holographic techniques for (strongly coupled) systems with NR symmetries.
- From a different perspective, Lifshitz space-times form interesting examples of non-AdS space-times for which it appears to be possible to construct explicit holographic techniques.
- Lifshitz holography initiated by: [Kachru, Liu, Mulligan, 2008].

Outline Talk

- Summary of the approach
- Holographic renormalization of 5D AdS gravity coupled to an axion-dilaton system
- 4D Asymptotically locally $z = 2$ Lifshitz space-times from 5D asymptotically locally AdS space-times
- The Lifshitz scale anomaly(ies)

Basic observations underlying the approach

- A 4D $z=2$ Lifshitz space-time is the reduction of a 5D $z=0$ Schrödinger space-time [Balasubramanian, Narayan, 2010], [Donos, Gauntlett, 2010]:

$$\begin{aligned} d\hat{s}^2 &= \frac{1}{r^2}\left(2dudt + d\vec{x}^2 + dr^2\right) + du^2 \\ &= -\frac{dt^2}{r^4} + \frac{1}{r^2}\left(d\vec{x}^2 + dr^2\right) + (du + \frac{1}{r^2}dt)^2 \,. \end{aligned}$$

- A 5D $z=0$ Schrödinger space-time is asymptotically AdS [Costa, Taylor, 2011] and a solution of

$$\mathcal{L} = \sqrt{-\hat{g}}\left(\hat{R} + 12 - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2}e^{2\hat{\phi}}(\partial\hat{\chi})^2\right)$$

with $\hat{\phi} = $ cst and $\hat{\chi} \propto u$ (Scherk–Schwarz reduction, [Cassani, Faedo, 2011], [Chemissany, J.H., 2011]).

The method

- Perform holographic renormalization for AdS gravity coupled to an axion-dilaton system using the methods of [de Haro, Solodukhin, Skenderis, 2001].
- Identify the subset of asymptotically locally AdS solutions of AdS gravity coupled to an axion-dilaton system that are also asymptotically locally $z = 0$ Schrödinger space-times and that satisfy the reduction Ansatz.
- Scherk–Schwarz reduce.
- This provides us with the Fefferman–Graham expansions for 4D AlLif$_{z=2}$ space-times as well as with the Lifshitz counterterms.

$$S_{\text{ren}} = \int d^5x\sqrt{-\hat{g}}\left(\hat{R} + 12 - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2}e^{2\hat{\phi}}(\partial\hat{\chi})^2\right) + 2\int d^4x\sqrt{-\hat{h}}\hat{K} + S_{\text{ct}}$$

Fefferman–Graham expansions for AlAdS solutions with axion-dilaton [Papadimitriou, 2011]: $d\hat{s}^2 = \frac{dr^2}{r^2} + \hat{h}_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}}$

$$\begin{aligned}
\hat{h}_{\hat{a}\hat{b}} &= \frac{1}{r^2}\left[\hat{h}_{(0)\hat{a}\hat{b}} + r^2\hat{h}_{(2)\hat{a}\hat{b}} + \mathcal{O}(r^4\log r)\right] \\
\hat{\phi} &= \hat{\phi}_{(0)} + r^2\hat{\phi}_{(2)} + \mathcal{O}(r^4\log r) \\
\hat{\chi} &= \hat{\chi}_{(0)} + r^2\hat{\chi}_{(2)} + \mathcal{O}(r^4\log r)
\end{aligned}$$

The fourth order coefficients are needed for the first variation of the on-shell action w.r.t. the boundary fields $\hat{h}_{(0)\hat{a}\hat{b}}$, $\hat{\phi}_{(0)}$, $\hat{\chi}_{(0)}$, i.e. for the 1-point functions.

Counterterms

$$S_{\text{ct}} = \int d^4x\sqrt{-\hat{h}}\left[-3 - \frac{1}{4}\left(\hat{R}_{(\hat{h})} - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2}e^{2\hat{\phi}}(\partial\hat{\chi})^2\right) + \log r\mathcal{A}\right]$$

$$\mathcal{A} = \frac{1}{8}\left[\hat{Q}^{\hat{a}\hat{b}}\hat{Q}_{\hat{a}\hat{b}} - \frac{1}{3}\hat{Q}^2 + \frac{1}{2}\left(\Box^{(\hat{h})}\hat{\phi} - e^{2\hat{\phi}}(\partial\hat{\chi})^2\right)^2 + \frac{1}{2}e^{2\hat{\phi}}\left(\Box^{(\hat{h})}\hat{\chi} + 2\partial_{\hat{a}}\hat{\chi}\partial^{\hat{a}}\hat{\phi}\right)^2\right]$$

$$\hat{Q}_{\hat{a}\hat{b}} = \hat{R}_{(\hat{h})\hat{a}\hat{b}} - \frac{1}{2}\partial_{\hat{a}}\hat{\phi}\partial_{\hat{b}}\hat{\phi} - \frac{1}{2}e^{2\hat{\phi}}\partial_{\hat{a}}\hat{\chi}\partial_{\hat{b}}\hat{\chi}\,, \qquad \hat{Q} = \hat{h}^{\hat{a}\hat{b}}\hat{Q}_{\hat{a}\hat{b}}\,.$$

The boundary stress tensor $\hat{t}_{\hat{a}\hat{b}}$ satisfies:

$$\hat{t}_{\hat{a}\hat{b}} = -\frac{2}{\sqrt{-\hat{h}_{(0)}}}\frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta\hat{h}_{(0)}^{\hat{a}\hat{b}}}$$

$$\hat{t}^{\hat{a}}{}_{\hat{a}} = \mathcal{A}_{(0)}$$

$$\mathcal{A}^{\text{on-shell}} = r^{-4}\mathcal{A}_{(0)} + \text{higher order corrections}$$

$$S = \int d^5x \sqrt{-\hat{g}} \left(\hat{R} + 12 - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2} e^{2\hat{\phi}} \left(\partial\hat{\chi}\right)^2 \right) + 2 \int d^4x \sqrt{-\hat{h}} \hat{K}$$

- Scherk–Schwarz reduction on S^1 preserving radial gauge, but not in Einstein frame (similar to [Kanitscheider, Skenderis, Taylor, 2008]):

$$\begin{aligned} d\hat{s}^2 &= ds_4^2 + e^{2\Phi}(du + A)^2 \\ \hat{\chi} &= ku + \chi \,, \qquad \hat{\phi} = \phi \end{aligned}$$

$$S = \int d^4x \sqrt{-g} \left(e^{\Phi} R - \frac{1}{4} e^{3\Phi} F^2 - \frac{1}{2} e^{\Phi} \left(\partial\phi\right)^2 - \frac{k^2}{2} e^{2\phi+\Phi} B^2 - e^{2\Phi} V \right)$$
$$+2 \int d^3x \sqrt{-h} e^{\Phi} K$$

where $\quad F = dB \,, \quad B = A - \frac{1}{k} d\chi \,, \quad V = \frac{k^2}{2} e^{-3\Phi+2\phi} - 12 e^{-\Phi} \,.$

AlLif$_{z=2}$ space-times

In order to find out how to obtain/define AlLif$_{z=2}$ space-times via dimensional reduction consider again the 5D pure $z=0$ Schrödinger space-time:

$$d\hat{s}^2 = \frac{dr^2}{r^2} + \hat{h}_{\hat{a}\hat{b}} dx^{\hat{a}} dx^{\hat{b}} = ds^2_{\text{AdS}_5} + \frac{k^2}{4} e^{2\hat{\phi}_{(0)}} du^2 = ds^2_{\text{Lif}_4} + e^{2\Phi} \left(du + A\right)^2$$

$$\hat{\chi} = ku + \text{cst}\,, \qquad \hat{\phi} = \hat{\phi}_{(0)} = \text{cst}\,.$$

In general in FG coordinates we have

$$e^{2\Phi} = \hat{h}_{uu} = \frac{1}{r^2}\hat{h}_{(0)uu} + \hat{h}_{(2)uu} + \mathcal{O}(r^2 \log r)$$

Constant Φ implies that

$$\hat{h}_{(0)uu} = 0\,, \qquad \hat{R}_{(0)uu} = 0 \quad (\text{follows from} \quad \hat{h}_{(2)uu} = \frac{k^2}{4} e^{2\hat{\phi}_{(0)}})$$

- ∂_u is a boundary Killing vector that is null ($\hat{h}_{(0)uu} = 0$) and hypersurface orthogonal (null Killing vector with $\hat{R}_{(0)uu} = 0$).
- The fact that $\hat{h}_{(0)\hat{a}\hat{b}}$ admits a hypersurface orthogonal null Killing vector turns out to be true for all uplifted AlLif$_{z=2}$ space-times.
- This follows from the boundary conditions that for AlLif$_{z=2}$ space-times, Φ can go at most as r^0 near $r = 0$ and that $\Phi - \hat{\phi}$ is a fixed constant equal to $\log \frac{k}{2}$.

- So what do we get upon reduction if we impose that $\hat{h}_{(0)\hat{a}\hat{b}}$ admits ∂_u as a hypersurface orthogonal (HSO) null Killing vector (NKV)?
- We introduce a special coordinate system on the boundary of the 5D AlAdS space-time that is adapted to the existence of a HSO NKV.
- Choose a double null split by writing

$$\hat{h}_{(0)\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = -\hat{H}_{(0)\hat{a}}\hat{N}_{(0)\hat{b}} - \hat{H}_{(0)\hat{b}}\hat{N}_{(0)\hat{a}} + \hat{\Pi}_{(0)\hat{a}\hat{b}}$$

where $\hat{H}_{(0)} = \partial_u$ and $\hat{N}^{\hat{a}}_{(0)}$ is any null vector satisfying $\hat{N}^{\hat{a}}_{(0)}\hat{H}_{(0)\hat{a}} = -1$ and $\hat{\Pi}_{(0)\hat{a}\hat{b}}$ projects onto the space orthogonal to both $\hat{H}^{\hat{a}}_{(0)}$ and $\hat{N}^{\hat{a}}_{(0)}$.

- We can WLOG choose

$$\hat{H}_{(0)\hat{a}} = H_{(0)}\partial_{\hat{a}}t\,, \quad \hat{N}_{(0)\hat{a}} = -\partial_{\hat{a}}u$$

so that $\hat{N}_{(0)\hat{a}}\hat{H}^{\hat{a}}_{(0)} = -1$ so that

$$\hat{h}_{(0)\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = H_{(0)}dudt + \Pi_{(0)ij}\left(dx^i + N^i_{(0)}dt\right)\left(dx^j + N^j_{(0)}dt\right)$$

in which ∂_u is manifestly a HSO NKV.

- We can now straightforwardly reduce to 4D using that

$$d\hat{s}^2 = \frac{dr^2}{r^2} + \hat{h}_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = \frac{dr^2}{r^2} + h_{ab}dx^a dx^b + e^{2\Phi}\left(du + A\right)^2$$

$$h_{ab} = \hat{h}_{ab} - \frac{\hat{h}_{ua}\hat{h}_{ub}}{\hat{h}_{uu}}$$

- The 4D metric $\frac{dr^2}{r^2} + h_{ab}dx^a dx^b$ and KK scalar have the following expansions

$$
\begin{aligned}
h_{tt} &= -\frac{1}{r^4} H^2_{(0)} e^{-2\Phi_{(0)}} + \mathcal{O}\left(\frac{\log r}{r^2}\right) \\
h_{ti} &= \frac{1}{r^2} h_{(0)ti} + \mathcal{O}\left(\log r\right) \\
h_{ij} &= \frac{1}{r^2} \Pi_{(0)ij} + \mathcal{O}\left(r^0\right) \\
\Phi &= \Phi_{(0)} + \mathcal{O}(r^2 \log r)
\end{aligned}
$$

- We define AlLif$_{z=2}$ such that in radial gauge and ADM decomposition for h_{ab} we have the above boundary conditions. Hence in this frame the space-time is AlLif$_{z=2}$ (also according to the definition of [Ross, 2011]).

- However in Einstein frame $e^{\Phi}\left(\frac{dr^2}{r^2}+h_{ab}dx^a dx^b\right)$ going to radial gauge for non constant $\Phi_{(0)}$ does not respect the AlLif$_{z=2}$ boundary conditions.
- We therefore also require that $\Phi_{(0)}$ is constant. Since $\Phi_{(0)}-\hat{\phi}_{(0)}=\log\frac{k}{2}$ this implies that $\hat{\phi}_{(0)}$ is constant.
- Hence we require from a 5D point of view that
    - $\hat{h}_{(0)\hat{a}\hat{b}}$ admits a HSO NKV
    - $\hat{\phi}_{(0)}$ is constant
    - $\hat{\chi}_{(0)}=ku+\chi_{(0)}(x)$

  in order that the SS reduced space-times is a 4D AlLif$_{z=2}$ space-time.

- The leading behavior of the 4D massive vector field $B_\mu = A_\mu - \frac{1}{k}\partial_\mu \chi$ is

$$
\begin{aligned}
B_t &= \frac{1}{r^2} H_{(0)} e^{-2\Phi_{(0)}} + \mathcal{O}(\log r) \\
B_i &= \mathcal{O}(r^2 \log r) \\
B_r &= r B_{(0)r} + \mathcal{O}(r^3 \log r)
\end{aligned}
$$

- We now have all the 5D counterterms (local as well as the conformal anomaly) and the 5-dimensional restriction such that upon SS reduction we obtain an AlLif$_{z=2}$ space-time.
- As an immediate application of this we compute the counterterms for AlLif$_{z=2}$ space-times with special attention to the structure of the anomalies.

Lifshitz counterterms

- For 5D AlAdS space-times the conformal anomaly is induced by those diffeomorphisms that act as conformal rescalings on $\hat{h}_{(0)\hat{a}\hat{b}}$. In 4D this leads to the anisotropic conformal rescalings of [Horava, Melby-Thompson, 2009]:

$$h_{(0)tt} \to \Omega^4 h_{(0)tt}\,, \quad h_{(0)ti} \to \Omega^2 h_{(0)ti}\,, \quad \Pi_{(0)ij} \to \Omega^2 \Pi_{(0)ij}$$

- Reduction of the 5D counterterm

$$S_{\text{ct}} = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}e^{\Phi}\left[-3 - \frac{1}{4}\left(R_{(h)} - \frac{1}{4}e^{2\Phi}F^2 - \frac{1}{2}(\partial\phi)^2 - \frac{k^2}{2}e^{2\phi}B^2 - \frac{k^2}{2}e^{2\phi-2\Phi}\right)\right] + \log r\int_{\partial\mathcal{M}} d^3x\sqrt{-h}e^{\Phi}\left(\mathcal{A}^{(0)} + \mathcal{A}^{(2)} + \mathcal{A}^{(4)}\right)\,,$$

$\mathcal{A}^{(n)}$ is nth order in derivatives

For AlLif$_{z=2}$ space-times (with the 4D $\chi_{(0)} = $ cst)

$$\int_{\partial\mathcal{M}} d^3x\sqrt{-h}e^{\Phi}\left(\mathcal{A}^{(0)}+\mathcal{A}^{(2)}+\mathcal{A}^{(4)}\right)|_{\text{on-shell}} = \int_{\partial\mathcal{M}} dt d^2x H_{(0)}\sqrt{\Pi_{(0)}}\times$$
$$\times\left[C_1\left(4K_{(0)ij}K_{(0)}^{ij}-2K_{(0)}^2\right)+C_2\left(\mathcal{R}_{(0)}+D^{(0)i}\partial_i\log H_{(0)}\right)^2\right]$$

where $K_{(0)ij}$ is the extrinsic curvature (in ADM decomposition) and $K_{(0)}$ its trace and where $\mathcal{R}_{(0)}$ is the Ricci scalar of $\Pi_{(0)ij}$ and $D_i^{(0)}$ its covariant derivative. The constants C_1 and C_2 are the central charges defined in [Baggio, de Boer, Holsheimer, 2011] and are given by

$$C_1 = \frac{le^{3\Phi_{(0)}}}{64\pi G_4}, \quad C_2 = \frac{le^{\Phi_{(0)}}}{384\pi G_4} = \frac{1}{6}e^{-2\Phi_{(0)}}C_1 .$$

The on-shell anomaly is of the Horava–Lifshitz type with nonzero potential for $z=2$ conformal gravity.

- Except for $C_2 \neq 0$ we get the same results (i.e. the same local counterterms as well as the same C_1 anomaly) as have been obtained for the massive vector model without scalars [Ross, 2011], [Mann, McNees, 2011], [Griffin, Horava, Melby-Thompson, 2011], [Baggio, de Boer, Holsheimer, 2011] .
- From a boundary perspective u parametrizes a lightlike circle [Costa, Taylor, 2011]. Yet in the bulk the reduction is always along a spacelike circle.
- The dual field theory is expected to be some DLCQ of $\mathcal{N} = 4$ SYM in the background of a linear theta angle (dual of the axion). This gives rise to Lifshitz–Chern–Simons gauge theory [Balasubramanian, McGreevy, 2011]

Conclusions and future work

- The computation of 1-point functions for AlLif$_{z=2}$.
- Relation between the stress tensor complex of [Ross, Saremi, 2009] and the 5D boundary stress tensor.
- Coordinate independent definition of AlLif$_{z=2}$.
- The Lifshitz asymptotic symmetry group and the role of the two central charges.
- It would be interesting to understand the DLCQ that leads to the Lifshitz–Chern–Simons gauge theory.