Numerical Algebraic Geometry:
A New Perspective on String and Gauge Theories

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Motivation

- String Phenomenology: Finding all the string vacua of a susy potential (specified by a Kaehler potential and superpotential); stabilizing moduli spaces; and many more …

- Potential Energy Landscape: Finding all the local/global minima and stationary points of a potential, etc.

- Systems of non-linear equations/minimization of multivariate functions

  Many, if not all, CAN BE VIEWED having polynomial-like non-linearity

- Can use Algebraic Geometry methods !

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Problem Definition

Solve

$$f_i(\vec{x}) = 0, i = 1, \ldots, N$$

where $\vec{x} = (x_1, x_2, \ldots, x_M)$, and M may or may not be equal to N.

Non-linear Equations!!!

If the equations are non-linear, then very difficult to solve them

But if they are polynomial equations, we are still in business …!

Only polynomials?!

Well, let's see ….

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$$\text{Take } \sin\theta_i = s_i, \cos\theta_i = c_i$$

$$f_i = s_{i+1}c_i - s_i c_{i+1} + s_i c_{i-1} - s_{i-1}c_i = 0, i = 1, \ldots, N$$

$$\text{and add } s_i^2 + c_i^2 - 1 = 0, \text{ for each } i = 1, \ldots, N$$

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$$f_i = \tan\left(\frac{y_{i+1}}{2}\right) x_{i-1} + z_i = 0$$

$$\text{Take } \tan(\tfrac{y_i}{2}) = t_i$$

$$f_i = t_{i+1} x_{i-1} + z_i = 0$$

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$$f_i = x^*_{i+1} x_{i-1} + z_i = 0$$

$$\text{Take } x_i = a_i + i\, b_i$$

$$f_i = (a_{i+1} - i\, b_{i+1})(a_{i-1} + i\, b_{i-1}) + z_i = 0$$

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$$f_i = F_{i+1}\, x_{i-1} + z_i = 0, \text{ where } F_i \text{ is some transcendetal function of } x_i$$

It is tricky!!! But sometimes can be transferred to polynomials [See Gray-He-Lukas (2006)]

An Example

$$x\,z - 3\,y + 1 = 0$$

$$x^2 - 2\,y = 0$$

$$x\,y - 5 = 0$$

Solve for x, y, z.

Two methods:

1. Groebner Basis Method (Used by the String Theorists/phenomenologists for a while now)

2. Numerical Algebraic Geometry (Recently introduced in particle theory, D Mehta Ph.D. Thesis 2009)

Groebner Basis

• Very roughly speaking, one can obtain another system of polynomial equations by performing a finite set of operations on the original system (the Buchberger algorithm with lexicographic monomial ordering)

• The new system is 'easier' to solve

• The new system has the same solutions as the original

• The new system is called the Groebner basis

• Packages like Singular, COCOA, McCAULEY2, Maple, Mathematica, etc.

• The first three are available for free !!

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Mathematica-interface of Singular STRINGVACUA by Oxford-Durham group

How is it useful?

For the running example, Mathematica gives (lexicographic monomial ordering)

$$x^3 - 10 = 0$$

$$-x^2 + 2\,y = 0$$

$$x^2 - 15\,x + 10\,z = 0$$

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There is a better way to use Groebner basis method (elimination+decomposition) [See, Gray et al. 2006]

Numerical Algebraic Geometry/ Homotopy Continuation Method

1. Estimate an upper bound of the number of solutions of the system to be solved.
   e.g.,
   Bezout bound = product of degrees of all the polynomials in the system.
   = 2x2x2 = 8, for our running example

$$\vec{f}(x,y,z) = (x\,z - 3\,y + 1, x^2 - 2\,y, x\,y - 5)^T$$

2. Construct a new system in the same variables
   (a) which has the same no. of solutions as the estimated upper bound,
   (b) easy to solve
   e.g., $$\vec{g}(x,y,z) = (x^2 - 1, y^2 - 1, z^2 - 1)^T$$

3. Track **each** solution of the new system using

$$\vec{H}((x,y,z),t) = (1-t)\vec{f}(x,y,z) + e^{i\gamma} t \vec{g}(x,y,z) = 0$$

from t=1 to t=0, using predictor-corrector or any other method.
If a solution of the new system converges to the original one at t=0, then it is a solution, otherwise not.
Note that 'gamma' is a generic real number, and is important here.

Numerical Algebraic Geometry/ Homotopy Continuation Method

There are well-written packages available for free:

Bertini, HOM4PS2, PHCPack.

Groebner Basis	Numerical Algebraic Geometry
1. Exact solutions	Numerical, but ALL solutions/extrema
2. Exponential space complexity	No such scaling problems
3. Highly sequential	‘Embarrassingly’ parallelizable
4. Non-integer coefficients a problem	Any floating point coefficients are fine

Ex: M-theory Compactified on $\frac{SU(3)\times U(1)}{U(1)\times U(1)}$

M-theory on 7-dimensional manifold with SU(3) structure, Micu, Palti, Saffin, JHEP 2006.

$$K = -4\log\big(-i(U-\bar{U})\big) - \log\big(-i(T_1-\bar{T}_1)(T_2-\bar{T}_2)(T_3-\bar{T}_3)\big)$$

$$W = \frac{1}{\sqrt{8}}\big(4U(T_1+T_2+T_3)+2T_2T_3-T_1T_3-T_1T_2+200\big)$$

$$T_i = -i\,t_i + \tau_i,\ \text{for } i=1,2,3,\ \text{and } U = -i\,x + y$$

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$$\begin{aligned}V = \frac{1}{256t_1t_2t_3x^4}(&40000+t_3^2\tau_1^2-400\tau_1\tau_2-4t_3^2\tau_1\tau_2+4t_3^2\tau_2^2+\tau_1^2\tau_2^2-400\tau_1\tau_3\\
&+800\tau_2\tau_3+2\tau_1^2\tau_2\tau_3-4\tau_1\tau_2^2\tau_3+\tau_1^2\tau_3^2-4\tau_1\tau_2\tau_3^2+4\tau_2^2\tau_3^2-24t_2t_3x^2\\
&+4t_3^2x^2-24t_1(t_2+t_3)x^2+4\tau_1^2x^2+8\tau_1\tau_2x^2+4\tau_2^2x^2+8\tau_1\tau_3x^2+8\tau_2\tau_3x^2\\
&+4\tau_3^2x^2+1600\tau_1y-8t_3^2\tau_1y+1600\tau_2y+16t_3^2\tau_2y-8\tau_1^2\tau_2y-8\tau_1\tau_2^2y\\
&+1600\tau_3y-8\tau_1^2\tau_3y+16\tau_2^2\tau_3y-8\tau_1\tau_3^2y+16\tau_2\tau_3^2y+16t_3^2y^2+16\tau_1^2y^2\\
&+32\tau_1\tau_2y^2+16\tau_2^2y^2+32\tau_1\tau_3y^2+32\tau_2\tau_3y^2+16\tau_3^2y^2+t_1^2(t_2^2+t_3^2+\tau_2^2+2\tau_2\tau_3+\tau_3^2+4x^2-8\tau_2y-8\tau_3y+16y^2)\\
&+t_2^2(4t_3^2+\tau_1^2-4\tau_1(\tau_3+2y)+4(\tau_3^2+x^2+4\tau_3y+4y^2))\end{aligned}$$

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$$W = \frac{1}{\sqrt{8}}\big(4U(T_1+T_2+T_3)+2T_2T_3-T_1T_3-T_1T_2+200\big)$$

- 9 equations in 9 variables
- Stringvacua (i.e., Singular) could not solve it completely
- Classical Bezout bound = 103680
- Numerical Algebraic Geometry: solved it in ~590 seconds on a regular desktop

More from Numerical Algebraic Geometry

- Irreducible decomposition of moduli spaces: dimensions and number components [Dhagash Mehta, Yang-Hui He, Jonathan Huenstein, JHEP2012]
- Numerical elimination, Numerical Hilbert Series [Jonathan Hauenstein, Yang-Hui He, Dhagash Mehta, To Appear]
- Numerical Enumerative Geometry [Dhagash Mehta, To Appear]
- Potential Energy Landscape in Statistical Mechanics/Theoretical Chemistry [Dhagash Mehta, 2011-12]
- PEL in the String Theory scenario [Dhagash Mehta + Liam McAllister et al., Frederik Denef et al.]

Conclusions

- Many models in complex systems/theoretical physics are non-linear with polynomial-like non-linearity
- Symbolic algebraic geometry methods based on the Groebner basis technique seem to be running out of the steam
- Numerical Algebraic Geometry, due to its parallelizability, can take it over.