Quasi-Poisson structures, Courant algebroids and Bianchi identities for non-geometric fluxes

Andreas Deser

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

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Non-geometric fluxes and gauged supergravity

Bosonic low energy effective field theory

$$S = \int d^n x \sqrt{-g} e^{-\Phi} \left(\mathfrak{R} + (\nabla\Phi)^2 - \tfrac{1}{12} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} \right)$$

- Scherk-Schwarz reduction, e.g on d-torus gives $\mathcal{O}(d,d)$-invariant action, containing $2d$ vectorfields V^a, B_a.
- Gauging a subgroup of $\mathcal{O}(d,d)$ leads to gauge generators X_a, corresponding to V^a and X^a corresponding to B_a.
- Most general commutation relations for the generators:

$$\begin{aligned} \left[X_a, X_b\right] &= \mathcal{F}_{ab}{}^c X_c + \mathcal{H}_{abc} X^c \\ \left[X_a, X^b\right] &= \mathcal{Q}_a{}^{bc} X_c - \mathcal{F}_{ac}{}^b X^c \\ \left[X^a, X^b\right] &= \mathcal{R}^{abc} X_c + \mathcal{Q}_c{}^{ab} X^c \end{aligned} \qquad (1)$$

- where $\mathcal{H}$ is the field strength of the NS-NS B-field and $\mathcal{F}, \mathcal{Q}$ and $\mathcal{R}$ contain geometric and non-geometric fluxes f, Q, R, arising in the conjectured chain of T-dualities:

$$\mathcal{H}_{abc} \ \rightarrow \ f_{ab}{}^c \ \rightarrow \ Q_a{}^{bc} \ \rightarrow \ R^{abc}$$

Representation on TM and Bianchi identities

(See also talk of R.Blumenhagen)

Representation on TM and Bianchi identities

- Basis of TM: $e_a = e_a{}^\mu \partial_\mu$, dual $e^a = e^a{}_\mu dx^\mu$
- Geometric flux: $\left[e_a\, , e_b\right] = f_{ab}{}^c e_c$
- Bi-vectorfield $\beta = \frac{1}{2}\beta^{ab} e_a \wedge e_b$ defines anchor map:

$$\beta^\# : T^\star M \to TM, \quad \beta^\#(e^a) = \beta^{ab} e_b =: e^a_\#$$

$\to$ compute the commutators

$$\begin{aligned} \left[e_a, e_b\right] &= \mathcal{F}_{ab}{}^c e_c + \mathcal{H}_{abc} e^c_\# \\ \left[e_a, e^b_\#\right] &= \mathcal{Q}_a{}^{bc} e_c - \mathcal{F}_{ac}{}^b e^c_\# \\ \left[e^a_\#, e^b_\#\right] &= \mathcal{R}^{abc} e_c + \mathcal{Q}_c{}^{ab} e^c_\# \end{aligned} \tag{2}$$

where we defined

$$\begin{aligned} \mathcal{F}_{ab}{}^c &:= f_{ab}{}^c - \mathcal{H}_{abm}\beta^{mc} \;, \\ \mathcal{Q}_a{}^{bc} &:= \partial_a \beta^{bc} + 2 f_{am}{}^{[\underline{b}}\beta^{m\underline{c}]} + \mathcal{H}_{amn}\beta^{mb}\beta^{nc} \;, \\ \mathcal{R}^{abc} &:= 3\left(\beta^{[\underline{a}m}\partial_m \beta^{\underline{bc}]} + f_{mn}{}^{[\underline{a}}\beta^{\underline{b}m}\beta^{\underline{c}]n}\right) - \mathcal{H}_{mnp}\beta^{ma}\beta^{nb}\beta^{pc} \end{aligned}$$

Bianchi identities

Blumenhagen,A.D.,Plauschinn,Rennecke, arXiv:1205.1522

Computing Jacobiators of $e_a, e^b_\#$ gives Bianchi-type identities:

Example:

$\left[e^a_\#, [e^b_\#, e^c_\#]\right] + \mathrm{cycl}$ leads to

$$0 = \Big(\beta^{[\underline{c}m}\,\partial_m \mathcal{R}^{\underline{ab}]d} - 2\mathcal{R}^{[\underline{ab}m}\,\mathcal{Q}_m{}^{\underline{cd}]}\Big) + \Big(\beta^{[\underline{c}m}\,\partial_m \mathcal{Q}_n{}^{\underline{ab}]} + \mathcal{R}^{[\underline{ab}m}\,\mathcal{F}_{mn}{}^{\underline{c}]} + \mathcal{Q}_m{}^{[\underline{ab}}\,\mathcal{Q}_n{}^{\underline{c}]m}\Big)\beta^{nd}$$

Representation on $TM \bigoplus T^{\star}M$

Lie algebroids

Definition

A vector bundle $E \to M$ is called a Lie algebroid, if it has the following additional structure:

- *A bracket* $[,]_E : E \times E \to E$
- *A bundle homomorphism (called anchor)* $\rho : E \to TM$
- *Leibniz rule: For sections* s_1, s_2 *of* E *and* $f \in \mathcal{C}^\infty(M)$*:*

$$[s_1, fs_2]_E = f[s_1, s_2]_E + \rho(s_1)(f)\, s_2$$

$\to$ immediately: $\Gamma(\wedge^\bullet E^\star)$ is graded differential, i.e.

$$(d_E\,\omega)(s_0, \ldots, s_k) = \sum_{i=0}^{k} (-1)^i \rho(s_i)\,(\omega(s_0, \ldots, \hat{s}_i, \ldots, s_k)) + \sum_{i<j} (-1)^{i+j} \omega\,([s_i, s_j]_E, s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_K)$$

Standard examples

- $(TM, [,], \rho = \mathrm{id})$
  $d_{TM}\omega = d\omega$, (Physics: $\mathcal{H} = dB$)

- If (M, β) is a (quasi-)Poisson manifold, we have $(T^\star M, [,]_{KS(\beta)}, \rho = \beta^\#)$, where

$$[\alpha, \omega]_{KS(\beta)} = \mathcal{L}_{\beta^\#(\alpha)}\omega - \mathcal{L}_{\beta^\#(\omega)}\alpha - d\left(\beta(\alpha, \omega)\right)$$

  Differential on TM? $\rightarrow$ need an extension of the Lie-bracket to multi-vectorfields: *Shouten-Nijenhuis* bracket.

$$d_{T^\star M}X = d_\beta X := [\beta, X]_{SN}$$

Inclusion of $\mathcal{H}$: Quasi-Lie algebroids

- $(TM, [,], \mathrm{id})$

$$[X,Y] \to [X,Y]^{\mathcal{H}} := [X,Y] + \beta^{\#}(\iota_X \iota_Y \mathcal{H})$$
$$d \to d^{\mathcal{H}}$$

evaluation on basis

$$[e_a, e_b]^{\mathcal{H}} = \mathcal{F}_{ab}{}^{c} e_c$$

- $(T^{\star}M, [,]_{KS(\beta)}, \beta^{\#})$

$$[\alpha,\omega]_{KS(\beta)} \to [\alpha,\omega]^{\mathcal{H}}_{KS(\beta)} := [\alpha,\omega]_{KS(\beta)} + \iota_{\beta^{\#}(\alpha)} \iota_{\beta^{\#}(\omega)} \mathcal{H}$$
$$d_\beta \to d^{\mathcal{H}}_\beta$$

evaluation on basis

$$[e^a, e^b]^{\mathcal{H}}_{KS(\beta)} = \mathcal{Q}_p{}^{ab} e^p$$

Courant algebroid structure on $TM \oplus T^\star M$

Result in Mathematics (Roytenberg, arXiv:math/9910078): Two quasi-Lie algebroids E,$E^\star$ together with compatibility conditions give rise to an associated Courant algebroid structure on $E \oplus E^\star$.

→ Take the following Courant algebroid structure on $TM \oplus T^\star M$:

- For sections $X+\xi, Y+\eta \in \Gamma(TM \oplus T^\star M)$, $\langle X+\xi, Y+\eta\rangle_\pm = \xi(Y) \pm \eta(X)$
- Bracket $[\![,]\!]$ on $\Gamma(TM \oplus T^\star M)$:

$$\begin{aligned}
[\![X,Y]\!] &= [X,Y]^{\mathcal{H}} + \iota_Y \iota_X \mathcal{H} \,, \\
[\![X,\xi]\!] &= [\iota_X, d^{\mathcal{H}}]_+ \xi - [\iota_\xi, d_\beta^{\mathcal{H}}]_+ X + \tfrac{1}{2}\left(d^{\mathcal{H}} - d_\beta^{\mathcal{H}}\right) \langle X,\xi\rangle_- \,, \\
[\![\xi,X]\!] &= [\iota_\xi, d_\beta^{\mathcal{H}}]_+ X - [\iota_X, d^{\mathcal{H}}]_+ \xi + \tfrac{1}{2}\left(d^{\mathcal{H}} - d_\beta^{\mathcal{H}}\right) \langle \xi,X\rangle_- \,, \\
[\![\xi,\nu]\!] &= [\xi,\nu]_K^{\mathcal{H}} + \iota_\nu \iota_\xi \mathcal{R} \,, \qquad (3)
\end{aligned}$$

Results

Blumenhagen,A.D.,Plauschinn,Rennecke, arXiv:1205.1522

- **This is indeed a Courant algebroid.**

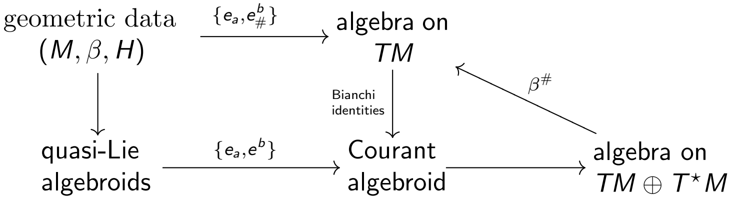
Results

Blumenhagen,A.D.,Plauschinn,Rennecke, arXiv:1205.1522

- **This is indeed a Courant algebroid.**
- **With the bracket $[\![,]\!]$, we again get the right commutation relations, but now realized on** $\Gamma(TM \oplus T^\star M)$:

$$[\![e_a, e_b]\!] = \mathcal{F}_{ab}{}^c\, e_c + \mathcal{H}_{abc}\, e^c \ ,$$
$$[\![e_a, e^b]\!] = \mathcal{Q}_a{}^{bc}\, e_c - \mathcal{F}_{ac}{}^b\, e^c \ ,$$
$$[\![e^a, e^b]\!] = \mathcal{Q}_c{}^{ab}\, e^c + \mathcal{R}^{abc}\, e_c \ ,$$

Conclusion


geometric data
(M, β, H)
$\{e_a, e_\#^b\}$
algebra on
TM
Bianchi
identities
$\beta^\#$
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Courant
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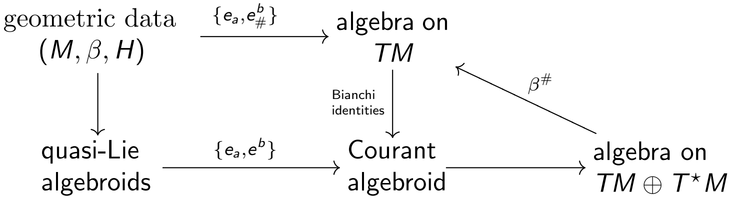
Conclusion



Thank you!