The Dark OPERA

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- AEF, Marco Matone, PLB 450 (1999) 34; ... ; IJMPA 15 (2000) 1869.
- G. Bertoldi, AEF & M. Matone, CQG 17 (2000) 3925.
- AEF, EPJC 72 (2012) 1944; arXiv:1204.3185.

String Phenomenology 2012, Cambridge, 24–29 June 2012

Motivation General Relativity: Covariance & Equivalence Principle

$\longrightarrow$ fundamental geometrical principle

Quantum Mechanics: No Such Principle

Axiomatic formulation ... $\mathrm{P} \sim |\Psi|^2$

However Quantum $+$ Gravity Theory

not known

Main effort: quantize GR; quantize space–time: $e.g.$ superstring theory

The main successes of string theory:

1) Viable perturbative approach to quantum gravity

2) Unification of gravity, gauge & matter structures

$i.e.$ construction of phenomenologically realistic models

$\longrightarrow$ relevant for experimental observation

State of the art: MSSM from string theory

(AEF, Nanopoulos, Yuan, NPB 335 (1999) 347)

(Cleaver, AEF, Nanopoulos, PLB 455 (1999) 135)

Other approaches

Geometrical

Greene, Kirklin, Miron, Ross (1987)
Donagi, Ovrut, Pantev, Waldram (1999)
Blumenhagen, Moster, Reinbacher, Weigand (2006)
Heckman, Vafa (2008)
....

Orbifolds

Ibanez, Nilles, Quevedo (1987)
Bailin, Love, Thomas (1987)
Kobayashi, Raby, Zhang (2004)
Lebedev, Nilles, Raby, Ramos-Sanchez, Ratz, Vaudrevange, Wingerter (2007)
Blaszczyk, Groot–Nibbelink, Ruehle, Trapletti, Vaudrevange (2010)
....

Other CFTs

Gepner (1987)
Schellekens, Yankielowicz (1989)
Gato–Rivera, Schellekens (2009)
....

Orientifolds

Cvetic, Shiu, Uranga (2001)
Ibanez, Marchesano, Rabadan (2001)
Kiristis, Schellekens, Tsulaia (2008)
....

Adaptation of Hamilton–Jacobi theory

Hamilton's equations of motion $\dot{q} = \frac{\partial H}{\partial p} \; , \; \dot{p} = -\frac{\partial H}{\partial q}$

$$H(q,p) \longrightarrow K(Q,P) \equiv 0 \qquad \Longrightarrow \qquad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0 \, , \; \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0$$

The solution is the Classical Hamilton–Jacobi Equation

$$H(q,p) \longrightarrow K(Q,P) = H(q, p = \frac{\partial S}{\partial q}) + \frac{\partial S}{\partial t} = 0 \Rightarrow \mathrm{CHJE}$$

stationary case $\longrightarrow \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E = 0$

$(q,p) \rightarrow (Q,P)$ via canonical transformations

$q, \; p$ are independent. Solve. Then $p = \frac{\partial s}{\partial q}$

Quantum mechanics: $[q,p] = i\hbar \rightarrow q,\ p \rightarrow$ not independent

Assume $H \rightarrow K$ $i.e.\ W(Q) = V(Q) - E = 0$ always exists

But $q,\ p$ not independent. $p = \frac{\partial S}{\partial q}$.

Equivalence postulate:

Consider the transformations on

$$(q ,\ S_0(q) ,\ p = \frac{\partial S_0}{\partial q}) \longrightarrow (\tilde{q} ,\ \tilde{S}_0(\tilde{q}) ,\ \tilde{p} = \frac{\partial \tilde{S}_0}{\partial \tilde{q}})$$

Such that

$$W(q) \longrightarrow \tilde{W}(\tilde{q}) = 0$$

exist for all $W(q)$

$\Longrightarrow$ QHJE

$\longrightarrow$ Schrödinger equation

Implies: Covariance of HJE

But: $$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \;+\; V(q) \;-\; E \;=\; 0$$

Is not covariant under $q \;\longrightarrow\; Q(q)$.

Further: $W(q) \;\equiv\; 0$ is a fixed state under $q \;\longrightarrow\; \tilde{q}(q)$.

Assume: $$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \;+\; W(q) \;+\; Q(q) \;=\; 0$$

The most general transformations $$\tilde{W}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} W(q) + (\tilde{q}; q),$$
$$\tilde{Q}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} Q(q) - (\tilde{q}; q),$$

with $\tilde{S}_0(\tilde{q}) \;=\; S_0(q)$ under $q \;\longrightarrow\; \tilde{q} \;=\; \tilde{q}(q)$

With: $W^0(q^0) = 0 \rightarrow$ All: $W(q) = (q; q^0)$

Cocycle Condition: $$(q^a; q^c) = \left(\frac{\partial q^b}{\partial q^c}\right)^2 \Big[(q^a; q^b) - (q^c; q^b) \Big]$$

$\Rightarrow$ Theorem $(q^a; q^c)$ invariant under Möbius transformations $\gamma(q^a)$

In 1D: $(q^a; q^c) \sim \{q^a; q^c\}$ Uniquely

Schwarzian derivative $$\{h(x); x(y)\} = \left(\frac{\partial y}{\partial x}\right)^2 \{h(x); y\} - \left(\frac{\partial y}{\partial x}\right)^2 \{x; y\}.$$

$$U(q) = \{h(q); q\} = \{\frac{Ah + B}{Ch + D}; q\}$$

Invariant under Möbius transformations

Identity $$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} - \{S_0; q\}\right)$$

Make the following identifications

$$W(q) = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} = V(q) - E$$

$$Q(q) = \frac{\beta^2}{4m}\{S_0; q\}$$

The Modified Hamilton–Jacobi Equation becomes

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0; q\} = 0$$

in the limit $\beta \rightarrow 0$ we get back the CSHJE and $S_0^{cl} = \lim_{\beta \rightarrow 0} S_0$

From the properties of the SD $\{;\}$

$$V(q) - E = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\}$$

is a potential of the 2^{nd}–order diff. Eq.

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0 \qquad \Rightarrow \qquad \beta \; = \; \hbar$$

The general solution

$$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$$

and

$$\mathrm{e}^{+\frac{i2S_0}{\hbar}} \; = \; \mathrm{e}^{i\alpha}\,\frac{w + i\bar{\ell}}{w - i\ell} \qquad\qquad w \; = \; \frac{\psi_1}{\psi_2}$$

$$\ell \; = \; \ell_1 \; + \; i\;\ell_2 \qquad \ell_1 \; \neq \; 0 \qquad \alpha \; \in \; R$$

Generalizations:

Cocycle condition $\longrightarrow$ D-dimensional E&M metrics

invariant under D–dimensional

Mobiüs (conformal) trans.

Quadratic identity:

$$\alpha^2(\nabla S_0)\cdot(\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \alpha\left(2\frac{\nabla R\cdot\nabla S_0}{R} + \Delta S_0\right),$$

or

$$\alpha^2(\partial S)\cdot(\partial S) = \frac{\partial^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \alpha\left(2\frac{\partial R\cdot\partial S}{R} + \partial^2 S\right),$$

or

$$\alpha^2(\partial S - eA)\cdot(\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2}\partial\cdot\left(R^2(\partial S - eA)\right),$$

$$D^\mu \ = \ \partial^\mu - \alpha e A^\mu$$

Relativistic case in $1+1$ dimensions: Start with KG equation:

$$(-\hbar^2 c^2 \Delta + m^2 c^4 - E^2)\psi = 0 , \quad \text{with} \quad \psi = R e^{\frac{1}{\hbar}S_0}.$$

$$\Longrightarrow \quad (\nabla S_0)^2 + m^2 c^2 - \frac{E^2}{c^2} - \hbar^2 \frac{\Delta R}{R} = 0 , \quad \text{RQHJE}$$

$$\nabla \cdot (R^2 \nabla S_0) = 0 . \quad \text{continuity equation}$$

$$\text{with} \quad Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R} , \quad \text{and} \quad p = \nabla S_0$$

$$\Longrightarrow \quad E^2 = p^2 c^2 + m^2 c^4 + 2mQc^2 .$$

In $1+1$ D: $\quad R = \frac{1}{\sqrt{S_0'}}, \quad \Longrightarrow \quad Q = \frac{\hbar^2}{4m}\{S_0; q\} ,$

$$\Longrightarrow \quad \left(\frac{\partial S_0}{\partial q}\right)^2 + m^2 c^2 - \frac{E^2}{c^2} + \frac{\hbar^2}{2}\{S_0; q\} = 0 .$$

$$\Longrightarrow \quad \left\{ e^{\frac{2iS_0}{\hbar}} ; q \right\} = -\left(m^2 c^2 - \frac{E^2}{c^2} \right)$$

$$e^{\frac{2i}{\hbar}S_0} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell} , \quad \text{where} \quad w = \psi^D/\psi$$

Use Jacobi theorem to define time

$$t - t_0 = \frac{\partial S_0}{\partial E} \;=\; \frac{\partial}{\partial E}\int_{q_0}^{q} dx \frac{\partial S_0}{\partial x} \;=\; \int_{q_0}^{q} dx \frac{E/c^2 - mc^2 \partial_E Q}{(E^2/c^2 - m^2c^2 - 2mc^2 Q)^{\frac{1}{2}}}$$

$$\Longrightarrow \quad \frac{dq}{dt} = \left(\frac{dt}{dq}\right)^{-1} = \frac{\partial_q S_0}{E/c^2 - mc^2 \partial_E Q}$$

$$\Longrightarrow \quad \dot{q} \;=\; \frac{p}{(E/c^2 - mc^2 \partial_E Q)} \;=\; \frac{pc^2}{E\left(1 - \frac{m}{E}\frac{\partial Q}{\partial E}c^4\right)}$$

take $\frac{m}{E}\frac{\partial Q}{\partial E}c^4 << 1$,

$$\Longrightarrow \quad \dot{q} \;=\; \frac{p}{E}c^2\left(1 + \frac{m}{E}\frac{\partial Q}{\partial E}c^4\right).$$

$\Longrightarrow$ modified disperssion relations

For KGE $\psi = \sin(kq) \quad \psi_D = \cos(kq)$

$$\frac{4m}{\hbar^2} Q(q) = \frac{k^2}{4\left(\cos^2(kq) + (\ell_1^2 + \ell_2^2)\sin^2(kq) + \ell_2 \sin(2kq)\right)^2} \cdot$$

$$\Big(3 - 6\ell_1^2 + 3\ell_1^4 + 6\ell_2^2 + 6\ell_1^2\ell_2^2 + 3\ell_2^4$$

$$-4(-1 + \ell_1^4 + 2\ell_1^2\ell_2^2 + \ell_2^4)\cos(2kq)$$

$$+(1 + \ell_1^4 - 6\ell_2^2 + \ell_2^4 + 2\ell_1^2(-1 + \ell_2^2))\cos(4kq)$$

$$+8\ell_2 \sin(2kq) + 8\ell_1^2\ell_2 \sin(2kq) + 8\ell_2^3 \sin(2kq)$$

$$+4\ell_2 \sin(4kq) - 4\ell_1^2\ell_2 \sin(4kq) - 4\ell_2^3 \sin(4kq)\Big)$$

$\Longrightarrow$ Modified dispersion relations, but not necessarily superluminal

Quantum potential as a curvature term:

Using the property of the Schwarzian derivative

$$\{S_0;q\} = -\left(\frac{\partial S_0}{\partial q}\right)^2 \{q;S_0\},$$

We can rewrite the Quantum Stationary Hamilton Jacobi Equation as

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial \hat{q}}\right)^2 + V(\hat{q}) - E = 0,$$

where

$$\hat{q} = \int^q \frac{dx}{\sqrt{1-\frac{\hbar^2}{2}\{q;S_0\}}}.$$

Flanders: J. Diff.Geom. 1970, 575 $\longrightarrow$ $\{\ ;\ \}$ $\longrightarrow$ a curvature term

In higher dimensions $Q(q) \sim \frac{\Delta R(q)}{R}$ $\longrightarrow$ curvature of $R(q)$

Length Scale

For $W^0(q^0) = 0$)

$$\frac{\partial^2 \Psi}{\partial q^2} = 0 \Rightarrow \qquad \psi_1 = q^0 \quad ; \quad \psi_2 = const$$

$\Rightarrow$ duality implies a length scale

$$\Longrightarrow \quad e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}},$$

$$p_0 = \frac{\partial S_0^0}{\partial q^0} = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}.$$

$\text{Max}|p_0| = \frac{\hbar}{\text{Re}\ell_0} \quad \longrightarrow \quad \text{Re}\ell_0 \neq 0 \longrightarrow$ ultraviolet cutoff

$$\lim_{\hbar \to 0} p_0 = 0 \Rightarrow \mathrm{Re}\,\ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}.$$

$\ell_0 = \lambda_p \quad \longrightarrow$ choice consistent with the classical limit

$$Q^0 = \frac{\hbar^2}{4m}\{S_0^0, q^0\} = -\frac{\hbar^2 (\mathrm{Re}\,\ell_0)^2}{2m} \frac{1}{|q^0 - i\ell_0|^4}.$$

Equivalence postulate $\Longrightarrow \quad q \longrightarrow q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi}$

Consistency $\Longrightarrow \quad q^0 = \psi^D/\psi$ is continuous on $\hat{R} = R \cup \{\infty\}$

$\Longrightarrow$ Energy quantisation

Taking $m \sim 100GeV$;

$$\mathrm{Re}\,\ell_0 = \lambda_p \approx 10^{-35}m;$$

$$q^0 \sim 93Ly,$$

$$\Longrightarrow \quad |Q| \sim 10^{-202}eV.$$

For $q^0 \sim 1m \quad |Q| \sim 10^{-96}eV$.

The multiparticle case:

$$\frac{1}{2m_1}(\nabla_1 S_0)^2 + \frac{1}{2m_2}(\nabla_2 S_0)^2 - E - \frac{\hbar^2}{2m_1}\frac{\Delta_1 R}{R} - \frac{\hbar^2}{2m_2}\frac{\Delta_2 R}{R} = 0.$$

$$\frac{1}{m_1}\nabla_1 \cdot (R^2 \nabla_1 S_0) + \frac{1}{m_2}\nabla_2 \cdot (R^2 \nabla_2 S_0) = 0.$$

set $r = r_1 - r_2, \qquad r_{c.m.} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}, \qquad \mu = \frac{m_1 \, m_2}{m_1 + m_2},$

$$\frac{1}{2(m_1+m_2)}(\nabla_{r_{c.m.}} S_0)^2 + \frac{1}{2\mu}(\nabla S_0)^2 - E - \frac{\hbar^2}{2(m_1+m_2)}\frac{\Delta_{r_{c.m.}} R}{R} - \frac{\hbar^2}{2\mu}\frac{\Delta R}{R} = 0,$$

$$\frac{1}{m_1+m_2}\nabla_{r_{c.m.}} \cdot (R^2 \nabla_{r_{c.m.}} S_0) + \frac{1}{\mu}\nabla \cdot (R^2 \nabla S_0) = 0,$$

The centre of mass motion

$$\frac{1}{2(m_1+m_2)}(\nabla_{r_{c.m.}} S_0)^2 - \tilde{E} - \frac{\hbar^2}{2(m_1+m_2)}\frac{\Delta_{r_{c.m.}} R}{R} = 0,$$

$$\frac{1}{(m_1+m_2)}\nabla_{r_{c.m.}} \cdot (R^2 \nabla_{r_{c.m.}} S_0) = 0.$$

$\Longrightarrow \quad m$ is commulative $i.e. \; m \sim \sum_i m_i$

conclusions :

The equivalence postulate $\Longrightarrow$ $S_0 \neq const \Leftrightarrow \hbar \neq 0$

$$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(Ae^{+\frac{i}{\hbar}S_0} + Be^{-\frac{i}{\hbar}S_0}\right)$$

$\mathrm{Re}\ell_0 = \lambda_P \rightarrow$ fundamental length scale

$$Q(q) = -\frac{\hbar^2}{4m}\{S_0; q\} \quad \text{and} \quad Q(q) = -\frac{\hbar^2}{2m}\frac{\Delta R(q)}{R(q)}$$

Intrinsic curvature terms of elementary particles

$S_0 \neq Aq + B \Longrightarrow Q(q) \neq 0$ Always

$\Longrightarrow$ Intrinsic energy of elementary particles which is never vanishing

Outlook

EP and phase space duality $\leftrightarrow$ T–duality

EP $\rightarrow$ axiomatic approach to quantum gravity

Energy quantization:

Probability: $\Longrightarrow$ (Ψ, Ψ') continuous ; $\Psi \in L^2(R)$

$\Longrightarrow$ quantization, bound states

What are the conditions on the trivializing transformations?

$$q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi}$$

we have $$\{w, q\} = -\frac{4m}{\hbar^2}(V(q) - E)$$

$\Rightarrow$ $w \neq const$; $w \in C^2(R)$ and w'' differentiable on R

In addition from the properties of $\{,\}$ $\rightarrow$ $\{w, q^{-1}\} = q^4\{w, q\}$

$\Rightarrow$ $w \neq const$; $w \in C^2(\hat{R})$ and w'' differentiable on $\hat{R}$

where $\hat{R} = R \cup \{\infty\}$

$$\Rightarrow \quad w(-\infty) = \begin{cases} w(+\infty), & for \; w(-\infty) \neq \pm\infty, \\ -w(+\infty), & for \; w(-\infty) = \pm\infty \end{cases}$$

$$\Longrightarrow$$

Equivalence postulate $\Longrightarrow$ continuity of (ψ^D, ψ) and $(\psi^{D'}, \psi')$

Theorem:

$$if \qquad V(q) - E = \begin{cases} P_-^2 > 0 & for \;\; q < q_- \\ P_+^2 > 0 & for \;\; q > q_+ \end{cases}$$

then the ratio $w = \psi^D/\psi$ *is continuous on* $\hat{R}$ *iff*

the corresponding Schrödinger equation admits

an $L^2(R)$ *solution*

1) $\psi \in L^2(R) \;\Rightarrow\; \psi^D \notin L^2(R)$

$$w = \frac{A\psi_D + B\psi}{C\psi_D + D\psi} \;\Rightarrow\; \lim_{q\to\pm\infty} = \frac{A}{C}$$

$$\Rightarrow\; w(-\infty) = w(+\infty)$$

2) ...

Potential Well:

$$V(q) = \begin{cases} 0 & |q| \leq L \\ V_0 & |q| > L \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad K = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$|q| \leq L \qquad \Psi_1^1 = \cos kq \qquad \Psi_2^1 = \sin kq$$

$$q > L \qquad \Psi_1^2 = \mathrm{e}^{-Kq} \qquad \Psi_2^2 = \mathrm{e}^{Kq}$$

The solution at $q < L$ is fixed by parity

four possibilities $(1,1)\ (2,1)\ /\ (1,2)\ (2,2)$

take $(1,1)$: $\Psi,\ \Psi'$ continuous $\Rightarrow$ $k\ \tan kL = K$

Use $\Psi^D = c\ \Psi\ \int_{q_0}^{q} dx\ \Psi^{-2}(x) + d\ \Psi$

$$\Rightarrow \quad w = \frac{1}{[k \sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q < -L \\ \sin(2kL)\tan(kq) & |q| \leq L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi} = \pm\infty \qquad \Longrightarrow \qquad E_n(k\ \tan kL\ =\ K)$$ are admissible solutions

take $(1,2)$: $\quad \Psi,\ \Psi'$ continuous $\quad \Rightarrow \quad k\ \tan(kl)\ =\ -K$

$$\Rightarrow \quad w = \frac{1}{[k\sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q\ <\ -L \\ \sin(2kL)\tan(kq) & |q|\ \leq\ L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q\ >\ L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi}\ =\ \mp\frac{1}{k}\cot(2kL) \qquad \Longrightarrow \qquad w(-\infty)\ \neq\ w(+\infty)$$

($k^{-1}(\cot 2kL) = 0$ is not compatible with $k\tan(kL) = -K$)

$\Rightarrow \quad E_n(k\ \tan kL\ =\ -K)$ are not admissible solutions

$\Longrightarrow$

We can understand the

$$\Psi \in L^2(R)$$

condition

$+$ existence of bound states

with quantized energy eigenvalues

as a consequence of the

postulated equivalence principle.