Microscopic derivation of the Ginzburg-Landau model[1]

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[1] *Frank, Hainzl, Seiringer, Solovej,* J. Amer. Math. Soc. **25** *(2012), 667–713*

Abstract of Talk

I will discuss how the **Ginzburg-Landau** (GL) model of **superconductivity** arises as an **asymptotic limit** of the microscopic **Bardeen-Cooper-Schrieffer** (BCS) model.

The asymptotic limit may be seen as a **semiclassical limit**.

One can deduce BCS from QM on a heuristic basis. But, it is not rigorously understood how the BCS model approximates the underlying **many-body quantum system**.

Ginzburg-Landau (GL) model

For superconducting materials on $3D$ box Λ (Could be 1D or 2D):
W potential, $\mathbf{A}$ magnetic vector potential:
GL functional: For constants $B_1, B_3, D > 0$ and $B_2 \in \mathbb{R}$:

$$\mathcal{E}^{\mathrm{GL}}(\psi) = \int_\Lambda B_1|(-i\nabla + 2\mathbf{A}(x))\psi(x)|^2 + B_2 W(x)|\psi(x)|^2 + B_3 D(|\psi(x)|^4 - 2|\psi(x)|^2)dx\,, \quad \psi \in H^1(\Lambda)$$

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We will derive GL from BCS in an appropriate limit:
$T \approx T_c$, $\mathbf{A}$, W small and slowly varying on microscopic scale.

BCS Free energy functional

BCS free energy functional: Temperature $T \geq 0$

$$\mathcal{F}(\Gamma) = \mathrm{Tr}\left[\left((-i\nabla + \mathbf{A}(x))^2 - \mu + W(x)\right)\gamma\right] - T\,S(\Gamma) + \int_{\Lambda\times\Lambda} V(|x-y|)|\alpha(x,y)|^2\,dx\,dy\,.$$

The BCS states can be expressed as 2×2-block matrix-operator

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \overline{\alpha} & 1-\overline{\gamma} \end{pmatrix}$$

$$0 \leq \Gamma \leq 1.$$

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This non-vanishing Cooper pairs imply macroscopic correlation of the particles, hence superfluidity (superconductivity)

BCS free energy in non-interacting case

This question can be answered in two simplified cases.

• **Non-interacting case** $V = 0$: BCS minimizer is **normal** state Γ_0: $\alpha = 0$,

$$\gamma_0 = (1 + \exp(\mathfrak{h}/T))^{-1},$$

with

$$\mathfrak{h} = (-i\nabla + \mathbf{A}(x))^2 + W(x) - \mu$$

Reduced BCS free energy

$\mathbf{A} = \mathbf{0}$, $W = 0$: For $\boldsymbol{\gamma} = \boldsymbol{\gamma}(x-y), \boldsymbol{\alpha} = \boldsymbol{\alpha}(x-y)$ the **Free energy per volume** functional simplifies.

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It was shown by H-Hamza-Seiringer-Solovej (CMP '08) that there exists **critical temperature** $T_c(V) \geq 0$ such that

- $T \geq T_c$: Minimizer is normal (as above with $\mathbf{A} = \mathbf{0}$, $W = 0$)
- $T < T_c$: Minimizer has $\alpha \neq 0$.

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The critical temperature may be characterized by the operator

$$K_{\mathrm{T_c}}(-\nabla^2 - \mu) + V(|x|), \qquad K_T(\eta) = \frac{\eta}{\tanh(\eta/2T)}$$

having 0 as the lowest eigenvalue. Note $\sigma(K_T(-\nabla^2 - \mu)) = [2T, \infty)$.

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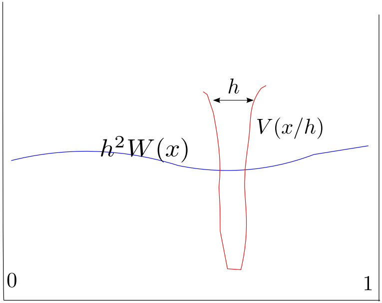
$\sigma(K_T(-\nabla^2 - \mu)) = [2T, \infty)$.
We assume $T_c > 0$ and **eigenfunction** α_0 unique (e.g. $\widehat{V} < 0$ OK).

Separation of scales

Introduce **small parameter** $h > 0$. h describes the relative scale between the microscopic length scale and the macroscopic length scale.

$\mathbf{A}$, W occurring in GL functional are small and live on the **macroscopic** scale. The particles interacting via V live on the microscopic scale.



x is the macroscopic variable. x/h is microscopic variable.

Rescaled BCS functional

The rescaled BCS functional is

$$\mathcal{F}(\Gamma) = \operatorname{Tr}\left[\left((-ih\nabla + h\mathbf{A}(x))^2 - \mu + h^2 W(x)\right)\gamma\right] - T\,S(\Gamma) + \int_{\Lambda\times\Lambda} V(h^{-1}|x-y|)|\alpha(x,y)|^2\,dx\,dy\,.$$

Here we assume

$$T = T_c(1 - Dh^2), \quad D > 0.$$

Note the **semiclassical nature** of the asymptotics.
The **order of the free energy** is h^{-3}.

Main result

Theorem

If $T = T_c(1 - Dh^2)$ *there exist* B_1, B_2, B_3, D *in GL functional giving*

$$\inf_{\Gamma} \mathcal{F}(\Gamma) = \mathcal{F}(\Gamma_0) + h^{-3+4} D(\inf_{\psi} \mathcal{E}^{\mathrm{GL}}(\psi) + o(1)),$$

as $h \to 0$, *where* Γ_0 *is the* **normal state**. *Moreover, if* $\mathcal{F}(\Gamma) \leq \mathcal{F}(\Gamma_0) + h\left(\inf_{\psi} \mathcal{E}^{\mathrm{GL}}(\psi) + o(1)\right)$ *then the corresponding* **Cooper pair wave function** α *satisfies:*

$$\|\alpha - \alpha_{\mathrm{GL}}\|_{L^2}^2 \leq o(h)\|\alpha_{\mathrm{GL}}\|_{L^2}^2 = o(h)h^{2-3}$$

$$\alpha_{\mathrm{GL}}(x,y) = h^{-3+1}\psi_0\left(\frac{x+y}{2}\right)\alpha_0\left(\frac{x-y}{h}\right) = \mathit{Op}(h\psi_0(x)\widehat{\alpha}_0(p))$$

(α_0 *appropriately normalized) and* $\mathcal{E}^{\mathrm{GL}}(\psi_0) \leq \inf_{\psi} \mathcal{E}(\psi) + o(1)$.

This answers what ψ is in Ginzburg-Landau:

The **center-of-mass** motion of the Cooper pair wavefunction $\boldsymbol{\alpha}(x,y)$,

$$\boldsymbol{\alpha}(x,y) = h^{-2}\psi_0\left(\frac{x+y}{2}\right)\alpha_0\left(\frac{x-y}{h}\right)$$

Sketch of proof

Upper bound:
We use the trial state

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and estimate using Cauchy's integral theorem.

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$$\Gamma_\Delta = \frac{1}{1 + e^{\frac{1}{T} H_\Delta}},$$

with

$$\Delta(x,y) = 2V(|x-y|/h)\alpha_{\mathrm{GL}}(x,y) = 2h\mathsf{Op}(\psi_0(x)(\widehat{\alpha_0 V})(p))$$

$$H_\Delta = \begin{pmatrix} \mathfrak{h} & \Delta \\ \overline{\Delta} & -\overline{\mathfrak{h}} \end{pmatrix}, \quad \mathfrak{h} = (-ih\nabla + h\mathbf{A}(x))^2 - \mu + h^2 W(x).$$

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Lower Bound:

Abuse the bound on the relative entropy: For general Γ and $\Gamma_\Delta = \frac{1}{1+e^{\frac{1}{T}H_\Delta}}$ it is true that

$$\mathcal{H}(\Gamma, \Gamma_\Delta) = \operatorname{Tr} \Gamma(\ln \Gamma - \ln \Gamma_\Delta) \geq \operatorname{Tr} \left[\frac{\frac{1}{T} H_\Delta}{\tanh\left(\frac{H_\Delta}{2T}\right)} (\Gamma - \Gamma_\Delta)^2 \right] + \operatorname{Tr} \left[\Gamma(1-\Gamma) - \Gamma_\Delta(1-\Gamma_\Delta) \right]^2$$

Critical temperature

Remember, the critical temperature T_c is given by the parameter T_c, such that the linear operator

$$K_{T_c}(p^2 - \mu) + V, \quad K_T(\eta) = \frac{\eta}{\tanh(\eta/2T)},$$

has 0 as lowest eigenvalue. Now define D to be the lowest eigenvalue of the linear operator

$$\frac{1}{2B_3}\left((-i\nabla + 2\mathbf{A}(x))^2 + B_2 W(x)\right).$$

Then we can recover the critical temperature T_c^{BCS} of the full functional $\mathcal{F}(\Gamma)$

Theorem (critical temperature)

$$T_c^{\mathrm{BCS}} = T_c(1 - Dh^2) + o(h^2).$$

Hence, the magnetic potential $\mathbf{A}(x)$ **lowers** the critical temperature and the external field $W(x)$ can **raise** the critical temperature.

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