Electron/Positron Pair Creation
in a Nonlinear Dirac model

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The Dirac operator

\[ D^0 = -\sum_{k=1}^{3} i\alpha_k \partial_k + \beta \quad \text{on} \quad L^2(\mathbb{R}^3, \mathbb{C}^4) =: \mathcal{H}, \]

where \( \alpha_k, \beta \) are \( 4 \times 4 \) Hermitian matrices (Pauli matrices).

### Spectrum of the Dirac operator

\[ \sigma(D^0) = (-\infty, -1] \cup [1, +\infty). \]

**Dirac interpretation:** All negative kinetic energy states are already occupied by “virtual” electrons.

⇒ Vacuum (:= state with no **real** particle) represented by “\( \Omega = \varphi_1 \wedge \cdots \wedge \varphi_j \wedge \cdots \)”, where \( (\varphi_j)_{j \geq 1} \) orthonormal basis of

\[ \mathcal{H}_- := P_- \mathcal{H}, \quad P_- := \chi_{(-\infty,0)}(D^0). \]
Consequences of the Dirac interpretation

- External (electric) field in vacuum $\Rightarrow$ **Break** uniformity of virtual electric charges distrib. $\Rightarrow$ **Vacuum Polarization**.

- Energy given to vacuum can **lift** virtual particles to positive kinetic energy space $\Rightarrow$ **Electron/Positron Pair Creation**.

If triggered by external (classical) electric potential $\Rightarrow$ **Schwinger effect** [Sch51].

Mathematical Formulation of Pair Creation

Hilbert space = $\mathcal{F} = (\bigoplus_{N \in \mathbb{N}} \wedge^N \mathcal{H}_-) \otimes (\bigoplus_{M \in \mathbb{N}} \wedge^M \mathcal{H}_+)$;

$\mathcal{H}_- := P_0^\mathcal{H}; \quad \mathcal{H}_+ := (1 - P_0^\mathcal{H}) \mathcal{H} = p_0^\mathcal{H}$; \quad $(P_0^\mathcal{H} = \chi(-\infty, 0)(D^0))$

$\wedge^0 \mathcal{H}_\pm := \mathbb{C} \Omega_\pm; \quad \Omega = \Omega_- \otimes \Omega_+ \text{ free vacuum;}

$V$ = external electric potential; $\mathcal{H}(V)$ = Hamiltonian on $\mathcal{F}$ with $V$ (e.g. QED).

Two choices to study Polarization of the Vacuum by $V$:

- $\Omega(V) \in \mathcal{F}$ “Absolute ground state” of $\mathcal{H}(V)$;
- $\Omega_t \in \mathcal{F}$ solution of the equation (here $V$ can be time dependent)

$$\begin{align*}
\begin{cases}
    i \partial_t \Omega_t &= \mathcal{H}(V)\Omega_t, \\
    \Omega|_{t=0} &= \Omega.
\end{cases}
\end{align*}$$

Probability that $V$ creates a pair

$$p(V) := \begin{cases} 
1 - |\langle \Omega, \Omega(V) \rangle_{\mathcal{F}}|^2 & \text{“Static”;} \\
1 - \lim_{t \to +\infty} |\langle \Omega, \Omega_t \rangle_{\mathcal{F}}|^2 & \text{“Dynamical”}.
\end{cases}$$
The Linear Case

No interactions between the particles.

- Absolute ground state \( \Omega(V) \leftrightarrow \chi_{(-\infty,0)}(D^0 + V) \sim \) Proba. \( p(V) \).

**Theorem (KlaSch-77)**

\[
p(V) = 1 \iff \exists \lambda(t) \in \sigma(D^0 + tV) \text{ crosses } 0 \text{ when } t \text{ increases from } 0 \text{ to } 1.
\]

- \( \Omega_t = \bigwedge_i \varphi_{i,t} \Rightarrow i\partial_t \varphi_t = (D^0 + V(t,x))\varphi_t, \qquad \varphi|_{t=0} := \varphi_i \in \mathcal{F}_-.
\]

**Theorem (PicDür-08)**

If \( V(t,x) = \mu(t)V(x) \) with \( V(x) \geq 0 \), \( \mu \) compactly supported such that \( \mu(0) = 1 \), \( \mu'(0) > 0 \), and \( 1 \in \sigma(D^0 + V(x)) \) with finite multiplicity. Then, if \( \varphi|_{t=0} \in \ker(D^0 + V(x) - 1) \), we have (in the adiabatic limit)

\[
\lim_{t \to -\infty} \| P^-_0 \varphi_t \|_{L^2} = 1, \quad \lim_{t \to +\infty} \| P^0_+ \varphi_t \|_{L^2} = 1.
\]

In particular, \( p(t) \to 1 \) as \( t \to +\infty \).


Goal

- Study the **interacting** case, in the **time-independent** setting (i.e. study of the absolute ground state).

- One body approach of interacting systems $\Rightarrow$ **Mean-field** theory of QED.

- Two results:
  
  1. Estimate on the probability $p(Z)$ as $Z \to +\infty$ (strong fields);
  
  2. Compare the interacting/non-interacting cases by letting coupling constant $\alpha \ll 1$. 
**The Mean-Field Approximation**

**Idea:** Assume $\Omega(V)$ quasi-free $\leftrightarrow$ Uniquely described by orthogonal projection $P(V) : \mathcal{H} \to \mathcal{H}$ (one-body density matrix).

**Example:** $\Omega(V) = \bigwedge_i \varphi_i \Rightarrow P(V) = \sum_i |\varphi_i\rangle \langle \varphi_i|.$

$\Omega \in \mathcal{F}$ (1-body density matrix)

$\Omega(V) \in F \quad \sim \langle \Omega, \Omega(V) \rangle_F$

$P^0_\perp \in \mathcal{B}(\mathcal{H})$ (minimization)

$P \in \mathcal{B}(\mathcal{H})$

**Rk:** $\Omega(V)$ not necessarily pure (ie $\Omega(V) \notin \mathcal{F}$), but still normal.

**Hartree-Fock Approximation of QED:** Chaix, Iracane, Lions '89, Bach, Barbaroux, Helffer, Siedentop '99, Gravejat, Hainzl, Lewin, Séré, Solovej '05-'11.


The Bogoliubov-Dirac-Fock Energy

**Variable:** \( Q = P - P_0 \), with \( 0 \leq P = P^* \leq 1 \), \( P : \mathcal{H} \to \mathcal{H} \) (hence free vacuum \( \leftrightarrow Q = 0 \)).

Assume \( V = Z \nu \ast |\cdot|^{-1} \), \( Z > 0 \), \( \nu : \mathbb{R}^3 \to \mathbb{R} \) “shape” of nucleus.

**BDF Energy**

\[
\mathcal{E}_{BDF}^{Z \nu}(Q) = \text{Tr}_0(D^0 Q) - \alpha Z \iint \frac{\rho_Q(x)\nu(y)}{|x - y|} + \frac{\alpha}{2} \iint \frac{\rho_Q(x)\rho_Q(y)}{|x - y|} + X(Q),
\]

- \( \rho_Q(x) := \text{Tr}_{\mathbb{C}^4} Q(x, x) \) **density of charge** of \( Q \),
- \( X(Q) = 0 \Rightarrow \text{reduced} \) model \( (\mathcal{E}_{rBDF}^{Z \nu}) \),
- \( X(Q) = -\frac{\alpha}{2} \iint \frac{|Q(x, y)|^2}{|x - y|} \Rightarrow \text{full} \) model.

**Rk:** \( \mathcal{E}_{BDF}^{Z \nu} \) bounded-below but no minimizer \( \to \) cut-off \( \Lambda > 0 \) needed.
The Polarized Vacuum

\[ \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4) \] replaced by \( \mathcal{H}_\Lambda := \{ f \in L^2(\mathbb{R}^3, \mathbb{C}^4), \quad \text{supp}\hat{f} \subset B(0, \Lambda) \}. \)

**Coulomb space:** \( \mathcal{C} := \{ f, \quad \int |k|^{-2} |\hat{f}(k)|^2 < \infty \}. \)

**Theorem (HaiLewSér-05)**

Let \( 0 \leq \alpha < 4/\pi, \Lambda > 0 \) and \( \nu \in \mathcal{C} \). Then, there exists a minimizer \( Q_{\text{vac}} \) for the \( \mathcal{E}^\nu_{\text{BDF/rBDF}} \) energies, on the set of all the \( Q \)'s verifying \( -P_0^- \leq Q = Q^* \leq 1 - P_0^- \), \( P_\pm Q P_\pm \in \mathcal{S}_1(\mathcal{H}_\Lambda) \).

**Rk:** \( \mathcal{S}_1 = \) trace-class operators.

In the BDF case, polarized vacuum satisfies the **mean-field** equation

\[
\begin{align*}
Q_{\text{vac}} &= \chi_{(-\infty,0]}(D Q_{\text{vac}}) - P_-, \\
D Q_{\text{vac}} &= \Pi_\Lambda (D^0 + \alpha (\rho Q_{\text{vac}} - \nu) + | \cdot |^{-1} - \alpha \frac{Q_{\text{vac}}(x,y)}{|x-y|}) \Pi_\Lambda,
\end{align*}
\]

where \( \Pi_\Lambda = \) Fourier multiplier by \( \chi_{B(0,\Lambda)}. \)

A Global Estimate

- **Reduced** model (without exchange term).

**Regime:** \( \alpha > 0, \Lambda > 0 \) fixed, \( Z \to +\infty \)

\( p_Z := \text{prob. the vacuum polarized by } Z \nu \ast | \cdot |^{-1} \) creates (at least) a pair.

**Theorem (S-11)**

Let \( \nu \in L^1(\mathbb{R}^3, \mathbb{R}) \cap C, \alpha > 0, \text{ and } \Lambda > 0. \) Assume \( \int_{\mathbb{R}^3} |x||\nu(x)| \, dx < \infty, \)
\( \int_{\mathbb{R}^3} \nu \neq 0. \) Then, there exists \( Z_0 = Z_0(\nu, \alpha, \Lambda) \geq 0 \) and \( c = c(\nu, \alpha, \Lambda) > 0 \) such that \( \forall Z \geq Z_0, \)

\[
p_Z \geq 1 - e^{-cZ^{2/3}}.
\]

**Rk:** Power \( Z^{2/3} \) due to cut-off, without it \( \to Z. \)

**Rk:** In some cases, estimate on the proba. to create a least \( k \) pairs.

Idea of the Proof

- If $\Omega_Z \in \mathcal{F}$ = state representing polarized vacuum,
  
  $$p_Z = 1 - |\langle \Omega, \Omega_Z \rangle_{\mathcal{F}}|^2.$$ 

- General estimate for a quasi-free $\Omega' \in \mathcal{F}$:
  
  $$|\langle \Omega, \Omega' \rangle_{\mathcal{F}}|^2 \leq e^{-a\langle N \rangle_{\Omega'}},$$

where $\langle N \rangle_{\Omega'} = $ average particle number of $\Omega'$, universal constant $a$.

**Rk:** Stated here for pure states, also true for mixed states.

- If $E(Z) = \min\{E_{rBDF}^Z(Q), Q\} =$ rBDF energy of polarized vacuum $\Omega_Z$,
  
  $$|E(Z)| \leq CZ\langle N \rangle_{\Omega_Z}.$$

- $(-c_1 Z^{5/3} \leq E(Z) \leq -c_0 Z^{5/3})$. 

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Another Regime

**Full** model (with exchange term),

**Regime:** \( \alpha \rightarrow 0, \kappa := \alpha Z \) fixed, \( \alpha \log \Lambda \) fixed.

Equation satisfied by the (density matrix of the) polarized vacuum:

\[
\begin{align*}
P &= \chi(-\infty,0] (D_P), \\
D_P &= \Pi_\Lambda (D^0 - \alpha Z V_\nu + \alpha (V_{\rho, p_0} - R_{p_0} - P_0)) \Pi_\Lambda,
\end{align*}
\]

\( V_\rho := \rho \star | \cdot |^{-1}, \ R_Q := \frac{Q(x,y)}{|x-y|}. \)

\( \alpha \ll 1 \leftrightarrow \) switching on the interactions/the non-linearity.

Comparison \( D_P \leftrightarrow D_\kappa := D^0 - \kappa V_\nu. \)

Addition of a **chemical potential** \( \mu \) (i.e. \( \chi(-\infty,0] \rightsquigarrow \chi(-\infty,\mu] \)).
A Fixed-Point Result

**Theorem (S-12)**

Let \( \nu \in L^2 \cap C \) and \( \mu \in (-1,1) \) s.t. \( \mu \notin \sigma(D^0 - V_\nu) \). Then, \( \exists \alpha_0 = \alpha_0(\mu, \nu) > 0 \), \( \exists \Lambda_0 = \Lambda_0(\mu, \nu) \geq 0 \), and \( \exists L_0 = L_0(\mu, \nu) > 0 \), s.t. \( \forall 0 \leq \alpha \leq \alpha_0, \forall \Lambda \geq \Lambda_0 \) satisfying \( \alpha \log \Lambda \leq L_0 \), the equation

\[
P = \chi_{(-\infty, \mu)}(\Pi_\Lambda(D^0 - V_\nu + \alpha(V_{\rho P - \rho_0} - R_{P - P_0})))\Pi_\Lambda)
\]

admits a unique solution (in a certain ball in the adequate Banach space). Furthermore, as \( \alpha \to 0 \) with \( \alpha \log \Lambda = L \) is fixed, we have (for some norm)

\[
P - \chi_{(-\infty, \mu)}(\Pi_\Lambda(D^0 - (1 + 2(3\pi)^{-1}L)^{-1}V_\nu))\Pi_\Lambda) \to 0.
\]

- The limit \( \alpha \to 0 \) is singular.
- Charge renormalization with exchange term.

Assume:

\[
\begin{align*}
\sigma(D^0 - \kappa V_\nu) &= \lambda_1(\kappa \nu) \\
\end{align*}
\]

\[
\left \{ 
\begin{array}{ll}
Q_+(\alpha) &= \chi(\infty, \mu_+)(D_Q(\alpha)) - P_0^-,
\\
Q_-(\alpha) &= \chi(\infty, \mu_-)(D_Q(\alpha)) - P_0^-,
\end{array}
\right.
D_Q := \Pi_\Lambda(D^0 - \kappa V_\nu + \alpha(V_{\rho Q} - R_Q))\Pi_\Lambda.
\]

\[
F(\kappa, \alpha) := \mathcal{E}^{\kappa \nu}_{\text{BDF}}(Q_+(\alpha)) - \mathcal{E}^{\kappa \nu}_{\text{BDF}}(Q_-(\alpha)).
\]

\(F < 0 \Rightarrow\) pair created, \(F > 0 \Rightarrow\) no pair created.

**Theorem (S-12)**

*In the limit* \(\alpha \to 0\), with \(\kappa = \alpha Z\) and \(\alpha \log \Lambda = L\) fixed, we have

\[
F(\kappa, \alpha) = \lambda_1 \left( \frac{\kappa}{1 + 2(3\pi)^{-1}L} \nu \right) + O_{\alpha \to 0}(\alpha).
\]
Conclusions & Perspectives

Conclusions:
- Electron/Positron Pair Creation in the **interacting** case;
- Estimate of the probability to create pairs in strong fields;
- Comparison with the linear case.

Perspectives:
- Next order: $F(\kappa, \alpha) = \lambda_1(\alpha_{ph}Z\nu) + c\alpha + o_{\alpha \to 0}(\alpha)$;
- Estimation on the number of pairs created;
- Time dependent setting.