Stochastic calculus via regularization in Banach spaces and applications.

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Stochastic partial differential equations (spdes) follow-up meeting.

includes joint work with

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Financial support: ANR Project MASTERIE 2010 BLAN-0121-01.
Outline

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3. About infinite dimensional stochastic calculus.
5. Convolution type processes.
8. An infinite dimensional PDE.
Some related references


Available preprints: [http://uma.ensta.fr/~russo/](http://uma.ensta.fr/~russo/)
1 Motivations

Let $W$ be the real Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$. $\langle W \rangle_t = t$.

If $h \in L^2(\Omega)$, the martingale representation theorem states the existence of a predictable process $\xi \in L^2(\Omega \times [0, T])$ such that

$$h = \mathbb{E}[h] + \int_0^T \xi_s dW_s$$
If $h \in \mathbb{D}^{1,2}$ in the sense of Malliavin, the Clark-Ocone formula implies that $\xi_s = \mathbb{E} \left[ D_t^m h \mid \mathcal{F}_s \right]$, so that

$$h = \mathbb{E} [h] + \int_0^T \mathbb{E} \left[ D_s^m h \mid \mathcal{F}_s \right] dW_s$$

(1)

where $D^m$ is the Malliavin gradient.
We suppose that the law of $X = W$ is not anymore the Wiener measure but $X$ is still a finite quadratic variation process but not necessarily a semimartingale.

Are there reasonable classes of random variables which can be represented in the form

$$h = H_0 + " \int_0^T \xi_s dX_s "?$$

$H_0 \in \mathbb{R}, \xi$ adapted?
Examples of processes with finite quadratic variation

1) $S$ is an $(\mathcal{F}_t)$-\textbf{semimartingale} with decomposition
   $S = M + V$, $M$ ($\mathcal{F}_t$)-local martingale and $V$ bounded variation process. So $[S] = [M]$.

2) $D$ is a ($\mathcal{F}_t$)-\textbf{(Föllmer-)Dirichlet} process with decomposition
   $D = M + A$, $M$ ($\mathcal{F}_t$)-local martingale and $A$ an ($\mathcal{F}_t$)-adapted zero quadratic variation process. $[D] = [M]$. Föllmer (1981), Bertoin (1986).

3) $D$ is a ($\mathcal{F}_t$)-\textbf{weak-Dirichlet} process with decomposition
   $D = M + A$, $M$ ($\mathcal{F}_t$)-local martingale and $A$ such that
Definition 1 Let $T > 0$ and $X = (X_t)_{t \in [0,T]}$ be a real continuous process prolonged by continuity. Process $X(\cdot)$ defined by

$$X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-T, 0]\}$$

will be called window process.

- $X(\cdot)$ is a $C([-T, 0])$-valued stochastic process.
- $C([-T, 0])$ is a typical non-reflexive Banach space.
The representation problem
We suppose $X_0 = 0$ and $[X]_t = t$.

Which are the classes of functionals

$$H : C([-T, 0]) \to \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits a representation of the type

$$h = H_0 + " \int_0^T \xi_s dX_s "$$
In that case we look for an explicit expressions for

- $H_0 \in \mathbb{R}$
- $\xi$ adapted process with respect to the canonical filtration of $X$
Idea: **Representation of** $h = H(X_T(\cdot))$

We express $h = H(X_T(\cdot))$ as

$$h = H(X_T(\cdot)) = u(T, X_T(\cdot))$$

where $u \in C^{1,2}([0, T] \times C([−T, 0]))$ solves an infinite dimensional PDE.
We have

\[ h = u(0, X_0(\cdot)) + \int_0^T D\delta_0 u(s, X_s(\cdot)) d^- X_s \]  \hspace{1cm} (2)

where \( D\delta_0 u(s, \eta) = D u(s, \eta)(\{0\}) \). We recall that \( D u : [0, T] \times C([-T, 0]) \rightarrow C^*([-T, 0]) = \mathcal{M}([-T, 0]) \).
2 Finite dimensional calculus via regularization

Definition 2 Let $X$ (resp. $Y$) be a continuous (resp. locally integrable) process. Suppose that the random variables

$$
\int_0^t Y_s d^- X_s := \lim_{\epsilon \to 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} d\epsilon
$$

exists in probability for every $t \in [0, T]$. 

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If the limiting random function admits a continuous modification, it is denoted by $\int_0^\cdot Y \, d\bar{X}$ and called *(proper) forward integral of $Y$ with respect to $X*(FR-Vallois 1991)
Covariation of real valued processes

Definition 3 The **covariation of X and Y** is defined by

\[
[X, Y]_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) \, ds
\]

if the limit exists in the ucp sense with respect to \( t \).

Obviously \( [X, Y] = [Y, X] \).

If \( X = Y \), \( X \) is said to be **finite quadratic variation process** and \( [X] := [X, X] \).
Connections with semimartingales Let $S^1, S^2$ be $(\mathcal{F}_t)$-semimartingales with decomposition $S^i = M^i + V^i$, $i = 1, 2$ where $M^i$ $(\mathcal{F}_t)$-local continuous martingale and $V^i$ continuous bounded variation processes. Then

- $[S^i]$ classical bracket and $[S^i] = \langle M^i \rangle$.
- $[S^1, S^2]$ classical bracket and $[S^1, S^2] = \langle M^1, M^2 \rangle$.
- If $S$ semimartingale and $Y$ cadlag and predictable

$$\int_0^\cdot Y d^- S = \int_0^\cdot Y dS \quad (\text{Itô})$$
Itô formula for finite quadratic variation processes

**Theorem 4** Let $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $F \in C^{1,2}([0, T] \times \mathbb{R})$ and $X$ be a finite quadratic variation process. Then

$$\int_{0}^{t} \partial_{x} F(s, X_{s}) d^{-} X_{s}$$

exists and equals

$$F(t, X_{t}) - F(0, X_{0}) - \int_{0}^{t} \partial_{s} F(s, X_{s}) ds - \frac{1}{2} \int_{0}^{t} \partial_{xx} F(s, X_{s}) d[X]_{s}$$
Among the contributors to stochastic calculus via regularization:


and several others.
3 About infinite dimensional classical stochastic calculus

We fix now in a general (infinite dimensional) framework. Let

- $B$ separable Banach space
- $X$ a $B$-valued process
- $F : B \rightarrow \mathbb{R}$ be of class $C^2$ in Fréchet sense.
An Itô formula for $B$-valued processes.

**Aim:** An Itô type expansion of $F(X)$, available also for $B = C([-T, 0])$-valued processes, as window processes, i.e. when $X = X(\cdot)$. The literature does not apply: several problems appear even in the simple case $W(\cdot)$!
Fréchet derivative and tensor product of Banach spaces

$F : B \rightarrow \mathbb{R}$ be of class $C^2$ in Fréchet sense, then

$D F : B \rightarrow L(B; \mathbb{R}) := B^*$;

$D^2 F : B \rightarrow L(B; B^*) \cong \mathcal{B}(B, B) \cong (B \hat{\otimes}_\pi B)^*$

where

$\mathcal{B}(B, B)$ Banach space of real valued bounded bilinear forms on $B$

$(B \hat{\otimes}_\pi B)^*$ dual of the tensor projective tensor product of $B$ with $B$.

$B \hat{\otimes}_\pi B$ fails to be Hilbert even if $B$ is a Hilbert space (is not even a reflexive space).
A first attempt to an Itô type expansion of $F(X)$

\[
F(X_t) = F(X_0) + \int_0^t B^* \langle DF(X_s), dX_s \rangle_B \\
+ \frac{1}{2} \int_0^t (B \hat{\otimes}_\pi B)^* \langle D^2 F(X_s), d[X]_s \rangle_{B \hat{\otimes}_\pi B}
\]
A formal proof

\[ \int_0^t \frac{F(X_{s+\epsilon}) - F(X_s)}{\epsilon} ds \xrightarrow{ucp} \lim_{\epsilon \to 0} \frac{F(X_t) - F(X_0)}{\epsilon} \]

By a Taylor’s expansion the left-hand side equals the sum of

\[ \int_0^t B^* \langle DF(X_s), \frac{X_{s+\epsilon} - X_s}{\epsilon} \rangle_B ds + \]

\[ \int_0^t (B \odot \pi B)^* \langle D^2 F(X_s), \frac{(X_{s+\epsilon} - X_s) \otimes^2}{\epsilon} \rangle_{B \odot \pi B} ds + R(\epsilon, t) \]
Problems related to integration in Banach spaces

Let $B$ be a Banach space and $X$ a $B$-valued process.

1. Stochastic integration with respect to an integrator $X$.
2. Quadratic variation of $X$. 
Among the known references about the subject:


5. Z. Brzezniak, J. Van Neerven, M. Riedle, M. Veraar, L. Weis ...
Da Prato - Zabczyk

- $B$ separable Hilbert space.
- $X$ $B$-valued Itô process.

but...

1. $C([-T, 0])$ is not a Hilbert space.
2. $W(\cdot)$ is not a a $C([-T, 0])$-valued semimartingale.

- In general $\langle \mu, W(\cdot) \rangle$ is not a real semimartingale for a large class of $\mu$ in $\mathcal{M}([-\tau, 0]) = C^*([-\tau, 0])$.
- There are necessary and sufficient conditions for this.
Metivier-Pellaumail and Dinculeanu

6 $B$ is a separable Banach space.

6 $\mathbb{X}$ essentially semimartingale.

6 The natural generalization concept of quadratic variation for Banach valued processes $\mathbb{X}$ is a $(B \widehat{\otimes}_\pi B)$-valued process denoted by $[\mathbb{X}]^\otimes$ and called tensor quadratic variation. It becomes operational as soon as the so-called scalar quadratic variation exists.
But... $W(\cdot)$ does not admit a scalar quadratic variation. In fact the quantity

$$\frac{1}{\epsilon} \int_0^t \| W_{s+\epsilon}(\cdot) - W_s(\cdot) \|_{C([-T,0])}^2 ds.$$ 

diverges when $\epsilon$ goes to zero.
4 Stochastic calculus via regularization in Banach spaces

A stochastic integral for $B^*$-valued integrand with respect to $B$-valued integrators, which are not necessarily semimartingales.

$\chi$-quadratic variation of $X$
A new concept of quadratic variation which generalizes the tensor quadratic variation and which involves a Banach subspace $\chi$ of $(B \hat{\otimes}_\pi B)^*$.
Definition 5  Let $X$ (resp. $Y$) be a $B$-valued (resp. a $B^*$-valued) continuous stochastic process. Suppose that the random function defined for every fixed $t \in [0, T]$ by

$$\int_0^t B^* \langle Y_s, d^* X_s \rangle_B := \lim_{\epsilon \to 0} \int_0^t B^* \langle Y_s, \frac{X_{s+\epsilon} - X_s}{\epsilon} \rangle_B dS$$

in probability exists and admits a continuous version. Then, the corresponding process will be called forward stochastic integral of $Y$ with respect to $X$. 
Connection with Da Prato-Zabczyk integral

Let $B = H$ be a separable Hilbert space.

**Theorem 6** Let $\mathbb{W}$ be a $H$-valued $Q$-Brownian motion with $Q \in L^1(H)$ and $Y$ be $H^*$-valued process such that

$$\int_0^t \|Y_s\|_{H^*}^2 \, ds < \infty \text{ a.s. Then, for every } t \in [0, T],$$

$$\int_0^t H^* \langle Y_s, d\mathbb{W}_s \rangle_H = \int_0^t Y_s \cdot d\mathbb{W}_s dz$$

**(Da Prato-Zabczyk integral)**
Notion of Chi-subspace

**Definition 7** A Banach subspace $\chi$ continuously injected into $(B \hat{\otimes}_\pi B)^*$ will be called **Chi-subspace** (of $(B \hat{\otimes}_\pi B)^*$). In particular it holds

$$\| \cdot \|_\chi \geq \| \cdot \| (B \hat{\otimes}_\pi B)^*.$$
Notion of Chi-quadratic variation

Let

6 $X$ be a $B$-valued continuous process,

6 $\chi$ a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$,

6 $C([0, T])$ space of real continuous processes equipped with the ucp topology.
Let $[X]^\epsilon$ be the application

$$[X]^\epsilon : \chi \rightarrow C([0, T])$$

defined by

$$\phi \mapsto \left( \int_0^t \chi\langle \phi, \frac{(X_{s+\epsilon} - X_s) \otimes^2}{\epsilon} \rangle \chi^* \, ds \right)_{t \in [0, T]}.$$
Definition 8 \( \mathbb{X} \) admits a \( \chi \)-quadratic variation if

**H1** For all \( (\epsilon_n) \downarrow 0 \) there exists a subsequence \( (\epsilon_{n_k}) \) such that

\[
\sup_k \int_0^T \left\| \left( X_s + \epsilon_{n_k} - X_s \right) \otimes^2 \right\|_{\chi^*} \frac{d\epsilon_{n_k}}{\epsilon_{n_k}} < \infty \quad \text{a.s.}
\]

**H2** There exists \( [X] : \chi \longrightarrow \mathcal{C}([0, T]) \) such that

\[
[X]^{\epsilon}(\phi) \xrightarrow{\text{ucp} \ \epsilon \rightarrow 0} [X](\phi) \quad \forall \ \phi \in \chi
\]
H3 There is a $\chi^*$-valued bounded variation process $\widetilde{[X]}$, such that $[\widetilde{X}]_t(\phi) = [X](\phi)_t$ a.s. for all $\phi \in \chi$.

For every fixed $\phi \in \chi$, processes $[\widetilde{X}]_t(\phi)$ and $[X](\phi)_t$ are indistinguishable.
Definition 9  When \( X \) admits a \( \chi \)-quadratic variation, the \( \chi^* \)-valued process \( \tilde{[X]} \) (and even the application \( [X] \)) will be called \( \chi \)-quadratic variation of \( X \).
Remark 10  

1. $\tilde{[X]}$ is the quadratic variation intervening in the second order derivative term of Itô’s formula.

2. For every fixed $\phi \in \chi$, processes $\tilde{[X]}_t(\phi)$ and $[X](\phi)_t$ are indistinguishable.

3. The $\chi^*$-valued process $\tilde{[X]}$ is weakly star continuous, i.e. $\tilde{[X]}(\phi)$ is continuous for every fixed $\phi \in \chi$.

4. Condition (H3) is always verified if $\chi$ is separable.
Definition 11  We say that $\mathbb{X}$ admits a \textit{global quadratic variation (g.q.v.)} if it admits a $\chi$-quadratic variation with $\chi = (B \hat{\otimes}_{\pi} B)^*$. 
When $\chi = (B \hat{\otimes}_\pi B)^*$

6. **H2** is related to weak* convergence in $(B \hat{\otimes}_\pi B)^{**}$. If $\Omega$ were a singleton then (H2) would imply

$$\widetilde{[X]}_t(\Phi) \xrightarrow{\epsilon \to 0} \widetilde{[X]}_t(\Phi), \quad \forall \Phi \in (B \hat{\otimes}_\pi B)^*, \ t \in [0, T].$$

6. The g.q.v. $\widetilde{[X]}$ is $(B \hat{\otimes}_\pi B)^{**}$-valued.
Connection with other concepts of quadratic variation

Global quadratic variation permit us to recover quadratic variation concepts in literature.

1. $B = \mathbb{R}^n$. $X = (X^1, \ldots, X^n)$ admits a covariations matrix $[X^*, X] = ([X^i, X^j])_{1 \leq i, j \leq n}$ if and only if $X$ admits g.q.v. $\tilde{[X]} = [X^*, X]$. 
2. *Da Prato-Zabczyk quadratic variation.*

\[ B = H \] Hilbert separable. \( X = M \) is a local martingale. 
\[ [M]^{dz} \] is a \( H \hat{\otimes}_{\pi} H \)-valued process such that if \( a, b \in H \), 
\[ \Phi = a^* \otimes b^* , \]

\[ \langle [M]^{dz}, \Phi \rangle = [\langle M, a \rangle_H; \langle M, b \rangle_H]. \]

Also \( M \) admits g.q.v. \( \widehat{[M]} = [M]^{dz} \) (equality in \( (H \hat{\otimes}_{\pi} H)^{**} \)).

- \( [M]^{dz} \) is uniquely determined.
- \( H \hat{\otimes}_{\pi} H \) can be identified with \( L^1(H) \).
3. $B$ general. If $X$ admits a scalar and a tensor quadratic variation $[X]^\otimes$ then $X$ admits g.q.v. $\widehat{[X]} = [X]^\otimes$. 
Infinite dimensional Itô’s formula

Let $B$ a separable Banach space

**Theorem 12** Let $X$ a $B$-valued continuous process admitting a $\chi$-quadratic variation.

Let $F : [0, T] \times B \rightarrow \mathbb{R}$ be $C^{1,2}$ Fréchet such that

$$D^2 F : [0, T] \times B \rightarrow \chi \subset (B \hat{\otimes}_\pi B)^* \quad \text{continuously}$$

Then for every $t \in [0, T]$ the forward integral

$$\int_0^t B^* \langle DF(s, X_s), d^- X_s \rangle_B$$

exists and the following formula holds.
\[ F(t, \mathbf{X}_t) = F(0, \mathbf{X}_0) + \int_0^t \partial_s F(s, \mathbf{X}_s) ds + \]
\[ + \int_0^t B^* \left\langle DF(s, \mathbf{X}_s), \mathbf{d}^- \mathbf{X}_s \right\rangle_B + \]
\[ + \frac{1}{2} \int_0^t \chi \left\langle D^2 F(s, \mathbf{X}_s), \mathbf{d}[\mathbf{X}]_s \right\rangle_{\chi^*} \]
5 Convolution processes

Typical example of $\chi$-subspace
Let $\nu_0, \nu_1$ be Banach subspaces of $B^*$. Then $\nu_0 \hat{\otimes}_\pi \nu_1$ is a Chi-subspace.

Let $A$ be the generator of a $C_0$-semigroup $(S_t)_{t \in [0,T]}$ on a separable Hilbert space $B = H$.
Let $U$ be another separable Hilbert space.
Let $W$ be a $Q$-Wiener process, $U_0 = Q^{\frac{1}{2}} U$. 

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Let $\sigma : \Omega \times [0, T] \to \mathcal{L}^2(U_0, H)$, $b : \Omega \times [0, T] \to H$ predictable processes such that

$$\int_0^T \left( \|\sigma_s\|_{\mathcal{L}^2(U_0, H)}^2 + |b_s|^2_H \right) ds < \infty, \text{ a.s.}$$

Let $x_0 \in H$. 

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Definition 13  A convolution type process is of the type

\[ X_t = x_0 + \int_0^t S_{t-s} \sigma_s dW_s + \int_0^t S_{t-s} b_s ds. \]

We set \( \nu_0 = D(A^*) \), \( \chi = \nu_0 \hat{\otimes}_\pi \nu_0 \).

Motivation: Mild solution of SPDEs.
Proposition 14  1. $X$ is a $\chi$-Dirichlet process, i.e. it admits a decomposition of the type $X = M + V$ where

$$M_t = x_0 + \int_0^t \sigma_s dW_s, \quad V = \int_0^t b_s ds + A,$$

$$\langle A_t, \varphi \rangle = \int_0^t \langle X_s, A^* \varphi \rangle ds, \quad \forall \varphi \in D(A^*).$$

2. $V$ has zero $\chi$-quadratic variation.

3. $X$ has a $\chi$-quadratic variation which is given by the "restriction" of $[M]^{dz}$ to $\chi$. 

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4. Let \( u : [0, T] \times H \to \mathbb{R} \) of class \( C^{0,1} \). The real process \( Z_t = u(t, X_t) \) is a real weak Dirichlet process with martingale component

\[
M_t = u(0, X_0) + \int_0^t \partial_x u(s, X_s) \sigma_s d\mathbb{W}_s, \quad t \in [0, T].
\]
6 Window processes

- We fix now the attention on \( B = C([-T, 0]) \)-valued window processes.

- \( X \) continuous real valued process and \( X(\cdot) \) its window process.

- \( \bar{X} = X(\cdot) \)
If $X$ has Hölder continuous paths of parameter $\gamma > 1/2$, then $X(\cdot)$ has a zero g.q.v. For instance:

- $X = B^H$ fractional Brownian motion with parameter $H > 1/2$.
- $X = B^{H,K}$ bifractional Brownian motion with parameters $H \in ]0, 1[ , K \in ]0, 1]$ s.t. $HK > 1/2$.

$W(\cdot)$ does not admit a g.q.v.
Some examples of Chi-subspaces

\( \chi \) Chi-subspace of \( (B \hat{\otimes}_\pi B)^* \). For instance:

- \( \mathcal{M}([-T, 0]^2) \) equipped with the total variation norm.
- \( L^2([-T, 0]^2) \).
- \( D_{0,0} = \{ \mu(dx, dy) = \lambda \delta_0(dx) \otimes \delta_0(dy) \} \).
- \( (D_0 \oplus L^2) \hat{\otimes}_h^2 = D_{0,0} \oplus L^2([-T, 0]) \hat{\otimes}_h D_0 \oplus D_0 \hat{\otimes}_h L^2([-T, 0]) \oplus L^2([-T, 0]^2) \).
- \( \text{Diag} := \{ \mu(dx, dy) = g(x)\delta_y(dx)dy; g \in L^\infty([-T, 0]) \} \).
Evaluations of $\chi$-quadratic variation for window processes

1. $W(\cdot)$ does not admit a $\mathcal{M}([-T, 0]^2)$-quadratic variation.

2. If $X$ is a real finite quadratic variation process, then
   - $X(\cdot)$ has zero $L^2([-T, 0]^2)$-quadratic variation.
   - $X(\cdot)$ has $\mathcal{D}_{0,0}$-quadratic variation
     \[ [X(\cdot)] : \mathcal{D}_{0,0} \longrightarrow C[0, T], \quad [X(\cdot)]_t(\mu) = \mu(\{0, 0\})[X]_t \]
   - $X(\cdot)$ has $(\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^{2}$-quadratic variation
     \[ [X(\cdot)] : (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^{2} \longrightarrow C[0, T] \]
     \[ [X(\cdot)]_t(\mu) = \mu(\{0, 0\})[X]_t \]
$X(\cdot)$ has $Diag$-quadratic variation

$$[X(\cdot)] : Diag \rightarrow C[0, T]$$

$$[X(\cdot)]_t(\mu) = \int_0^t g(-x)[X]_{t-x} \, dx$$

where $\mu(dx, dy) = g(x)\delta_y(dx)dy$. 
7 A generalized Clark-Ocone type formula

We set \( B = C([-T, 0]) \).

6 \( X \) real continuous stochastic process with values in \( B \).
6 \( X_0 = 0 \),
6 \( [X]_t = \sigma^2 t, \sigma \geq 0 \).
Main task: to look for classes of functionals

\[ H : B \rightarrow \mathbb{R} \]

such that the r.v.

\[ h := H(X_T(\cdot)) \]

admits representation

\[ h = H_0 + \int_0^T \xi_s d^- X_s \]
Moreover we look for an explicit expression for

1. $H_0 \in \mathbb{R}$
2. $\xi$ (adapted) process with respect to the canonical filtration of $X$
Idea

Obtain the representation formula by expressing
\[ h = H(X_T(\cdot)) \]
as
\[ h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot)) \]

where \( u \in C^{1,2}([0, T] \times B) \) solves an infinite dimensional PDE, if previous limit exists.
8 An infinite dimensional PDE

Let $H : B \to \mathbb{R}$. We suppose the existence of a function $u : [0, T] \times B \to \mathbb{R}$ of class $C^{1,2} (\mathbb{R}^2) \cap C^0 ([0, T] \times B)$ solving the **Infinite dimensional PDE**

\[
\begin{cases}
\partial_t u(t, \eta) + \int_{-t}^{0} D^{ac} u(t, \eta) \, d\eta + \frac{\sigma^2}{2} \langle D^2 u(t, \eta) , 1_{D_t} \rangle = 0 \\
\quad \quad \quad u(T, \eta) = H(\eta)
\end{cases}
\]

(3)
where

\[ 1_{D_t}(x, y) := \begin{cases} 
1 & \text{if } x = y, \ x, y \in [-t, 0] \\
0 & \text{otherwise} 
\end{cases} \]

\[ D^{ac} u(t, \eta) \] absolute continuous part of measure \( D u(t, \eta) \)

If \( x \mapsto D^{ac}_x u(t, \eta) \) has bounded variation, previous integral is defined by an integration by parts.
Then

\[ h = H_0 + \int_0^T \xi_s d^- X_s \]  

(4)

with

\[ H_0 = u(0, X_0(\cdot)) \]

\[ \xi_s = D^{\delta_0} u(s, X_s(\cdot)) \]
Methodology: two steps

1. We will choose a functional $u : [0, T] \times B \rightarrow \mathbb{R}$ which solves the infinite dimensional PDE (3) with final condition $H$.

2. Using Itô formula we establish a representation form (4).
Particular cases

1. \( H(\eta) = f(\eta(0)) \) where \( f : \mathbb{R} \to \mathbb{R} \) continuous and polynomial growth \( \Rightarrow \) \( u \) such that \( D^2 u(t, \eta) \in \mathcal{D}_{0,0} \)

2. \( H(\eta) = \left( \int_{-T}^{0} \eta(s) ds \right)^2 \) \( \Rightarrow \) \( u \) such that
   \[
   D^2 u(t, \eta) \in (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2
   \]

3. \( H(\eta) = \int_{-T}^{0} \eta(s)^2 ds \) \( \Rightarrow \) \( u \) such that
   \[
   D^2 u(t, \eta) \in (\text{Diag} \oplus \mathcal{D}_{0,0})
   \]
Representation theorem.

**Theorem 15**  \( H : B \rightarrow \mathbb{R} \)

- \( u \in C^{1,2} ([0, T] \times B) \cap C^0 ([0, T] \times B) \)
- \( x \mapsto D^a_{x} u (t, \eta) \) has bounded variation for all \( t \in [0, T], \eta \in B \).
- \( D^2 u (t, \eta) \in (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2 \oplus \text{Diag}, \) for all \( t \in [0, T], \eta \in B \).
\[ u \text{ solves } \]
\[
\begin{cases}
\partial_t u(t, \eta) &+ \int_{-t,0} D^{ac}u(t, \eta) \, d\eta \\
+ \frac{\sigma^2}{2} D^2 u(t, \eta) (D_t) &= 0 \\
\quad u(T, \eta) &= H(\eta), \quad \eta \in B.
\end{cases}
\] (5)

Then \( h \) has representation (4).

**Proof.** Application of Itô’s formula.
Sufficient conditions to solve (5)

1. When $X$ general process such that $[X]_t = t$.
   - $H$ has a smooth Fréchet dependence on $L^2([−T, 0])$.
   - $h := H(X_T(·)) = f \left( \int_0^T \varphi_1(s)d^-X_s, \ldots, \int_0^T \varphi_n(s)d^-X_s \right)$,
   - $f : \mathbb{R}^n \to \mathbb{R}$ measurable and with linear growth
   - $(\varphi_i) \in C^2([0, T]; \mathbb{R})$

2. When $X = W$ if Clark-Ocone formula does not apply. For instance when $h \notin \mathbb{D}^{1,2}$, or $h \notin L^2(\Omega)$ (even not in $L^1(\Omega)$).
Alternative approach via “generalized PDE”:

Functional Itô’s calculus

of R. Cont and D. Fournié.