THE SUPPORT IN HÖLDER NORM OF A WAVE EQUATION IN DIMENSION THREE

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Based on joint work with Francisco Delgado

Stochastic PDEs
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Introduction
Objective

To prove a characterization of the topological support of the law of the solution of a stochastic wave equation in spatial dimension $d = 3$.

Definition For a random vector $X : \Omega \rightarrow M$, the topological support is the smallest closed $F \subset M$ such that $(P \circ X^{-1})(F) > 0$.

- What type of solution? Random field solution
- What topology? Hölder
- What method? Approximations
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Motivations

- The description of the support is an important ingredient to study irreducibility of the corresponding semigroups, and therefore of the uniqueness of invariant measure.
- Functional Itô calculus (Rama Cont and David Fournié): functional Kolmogorov equations.
- Curiosity.
Main ingredient

Approximation in probability and in Hölder norm of a stochastic wave equation by smoothing the driving noise. (Wong–Zakai’s type Theorem).

References


- For the background on the wave equation: Dalang 1999; Dalang–S.-S, 2009; Dalang–Quer-Sardanyons, 2011; ...
Plan of the work

- Vanishing initial conditions
- Non null initial conditions

Why such a distinction?

This is related to the property of spatial stationarity of the solution.
Description of The Equation
Stochastic wave equation in spatial dimension $d = 3$

\[
\left\{ \begin{aligned}
\left( \frac{\partial^2}{\partial t^2} - \Delta \right) u(t, x) &= \sigma(u(t, x)) \dot{M}(t, x) + b(u(t, x)), \\
u(0, x) &= u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = v_0(x),
\end{aligned} \right.
\]

$t \in [0, T]$, $x \in \mathbb{R}^3$.

**Description of the noise**

\{ $M(\varphi), \varphi \in C_0^\infty(\mathbb{R}^4)$ \} Gaussian process

- $E(M(\varphi)) = 0$,
- $E(M(\varphi)M(\psi)) = \int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi) F\varphi(s)\overline{F\psi(s)}(\xi)$, $\mu$ non-negative tempered symmetric measure on $\mathbb{R}^3$.

In non-rigorous terms

\[
E(\dot{M}(t, x)\dot{M}(s, y)) = \delta(t - s)f(x - y),
\]

$f = F\mu$. 

Formulation of the stochastic wave equation

\[ u(t, x) = [G(t) * v_0](x) + \frac{\partial}{\partial t} ([G(t) * u_0](x)) \]

\[ + \sum_{j \in \mathbb{N}} \int_0^t \langle G(t - s, x - \cdot) \sigma(u(s, \cdot)), e_j \rangle_{\mathcal{H}} W_j(ds) \]

\[ + \int_0^t G(t - s, \cdot) * b(u(s, \cdot))(x) ds, \]

(1)

\[ t \in [0, T], \ x \in \mathbb{R}^3, \ G(t) = \frac{1}{4\pi t} \sigma_t(dx). \]

We consider random field solutions \( \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\} \).
$M$ as a cylindrical Wiener process

$\mathcal{H}$ is the completion of the Schwartz space $S(\mathbb{R}^3)$ of test functions with the semi-inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \mu(d\xi) \mathcal{F} \varphi(\xi) \overline{\mathcal{F} \psi(\xi)}.$$

The process $B_t(\varphi) = M(1_{[0,t]} \varphi)$ is a cylindrical Wiener process: Gaussian, zero mean and

$$E(M_t(\varphi) M_s(\psi)) = \min(s, t) \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

In particular, for any CONS $(e_j)_{j \in \mathbb{N}} \subset S(\mathbb{R}^3)$,

$$(W_t^j = B_t(e_j), t \in [0, T])_{j \in \mathbb{N}}$$

defines a sequence of independent standard Brownian motions.
Hypotheses:

- \( u_0, v_0 \) vanish,
- \( \sigma, b : \mathbb{R} \rightarrow \mathbb{R} \) Lipschitz continuous,
- \( f(x) = |x|^{-\beta} \, dx, \beta \in ]0, 2[. \)

**Theorem** There exists a unique random field solution to (1).

This is an adapted process \( \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\} \) satisfying (1) for any \( (t, x) \in [0, T] \times \mathbb{R}^3\).

The solution is \( L^2 \)-continuous and bounded in \( L^p \):

\[
\sup_{(t,x)\in[0,T] \times \mathbb{R}^3} E(|u(t, x)|^p) < \infty.
\]
Support Theorem
Sample path properties of the wave equation

Notation

▶ For \( t_0 \in [0, T] \), \( K \subset \mathbb{R}^3 \) compact, \( \rho \in ]0, 1[ \),

\[
\|g\|_{\rho,t_0,K} := \sup_{(t,x) \in [t_0, T] \times K} |g(t,x)| + \sup_{(t,x),(\bar{t},\bar{x}) \in [t_0, T] \times K, t \neq \bar{t}, x \neq \bar{x}} \frac{|g(t,x) - g(\bar{t},\bar{x})|}{(|t - \bar{t}| + |x - \bar{x}|)\rho},
\]

▶ \( C^\rho([t_0, T] \times K) \) is the space of real functions \( g \) such that

\( \|g\|_{\rho,t_0,K} \leq \infty \).

Theorem (Dalang–S.-S., 2009)

Almost surely, the sample paths of the random field solution of (1) belong to the space \( C^\rho([t_0, T] \times K) \) with \( \rho \in \left[0, \frac{2-\beta}{2}\right] \).
Support theorem (null initial conditions)

For $t \in ]0, T]$, set $\mathcal{H}_t := L^2([0, t]; \mathcal{H})$. Let

$$
\Phi^h(t, x) = \left\langle G(t - \cdot, x - \cdot)\sigma(\Phi^h), h \right\rangle_{\mathcal{H}_t}
+ \int_0^t ds[G(t - s, \cdot) * b(\Phi^h(s, \cdot))](x),
$$

$h \in \mathcal{H}_T$.

Theorem (Delgado–S.-S., 2012)

Let $u = \{u(t, x), (t, x) \in [t_0, T] \times K\}$, $t_0 > 0$, be the random field solution to (1). Fix $\rho \in \left]0, \frac{2-\beta}{2}\right]$. Then the topological support of the law of $u$ in the space $C^\rho([t_0, T] \times K)$ is the closure in $C^\rho([t_0, T] \times K)$ of the set of functions $\{\Phi^h, h \in \mathcal{H}_T\}$. 
A method to prove the support theorem
Part I

- $\Phi : \mathcal{H}_T \rightarrow C^\rho([t_0, T] \times K)$,
- Assume that there exist $w^n : \Omega \rightarrow \mathcal{H}_T$,
such that for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \| u - \Phi w^n \|_{\rho,t_0,K} > \epsilon \right\} = 0.$$ 

Then $\text{supp}(\mathbb{P} \circ u^{-1}) \subset \{ \Phi^h, h \in \mathcal{H}_T \}$.

Remarks
- This follows from Portmanteau’s theorem.
- The closure refers to the Hölder norm $\| \cdot \|_{\rho,t_0,K}$. 
Part II

Assume that:

- for any $h \in \mathcal{H}_T$, there exists a sequence $T^h_n : \Omega \to \Omega$ such that $\mathbb{P} \circ (T^h_n)^{-1} \ll \mathbb{P},$
- the following convergence holds

$$\lim_{n \to \infty} \mathbb{P} \left\{ \|u(T^h_n) - \Phi^h\|_{\rho, t_0, K} > \epsilon \right\} = 0.$$

Then $\text{supp}(\mathbb{P} \circ u^{-1}) \supset \{\Phi^h, h \in \mathcal{H}_T\}.$

This follows from Girsanov’s theorem.
**Choice for \( w^n \)**

Let \( \Delta_i = \left[ \frac{iT}{2^n}, \frac{(i+1)T}{2^n} \right] \). For \( 1 \leq j \leq n \), let

\[
\dot{W}_j^n(t) = \begin{cases} 
\sum_{i=0}^{2^n-1} 2^n \theta_1 T^{-1} W_j(\Delta_i) 1_{\Delta_{i+1}}(t), & t \in [2^{-n}T, T], \\
0, & t \in [0, 2^{-n}T].
\end{cases}
\]

\( \theta_1 \in ]0, \infty[. \)

For \( j > n \), put \( \dot{W}_j^n = 0. \) Set

\[
w^n(t, x) = \sum_{j \in \mathbb{N}} \dot{W}_j^n(t) e_j(x).
\]

**Remark:**

\[
M(ds) = \sum_{j \in \mathbb{N}} W_j(ds) \sim w^n(s)ds.
\]
Choice for $T^h_n$

$$T^h_n(\omega) = \omega - w^n + h.$$ 

For the rigorous setting: abstract Wiener space associated with $\{W_j, j \in \mathbb{N}\}$. 
Approximation result

\[ X(t, x) = \sum_{j \in \mathbb{N}} \int_0^t \langle G(t - s, x - \cdot)(A + B)(X(s, \cdot)), e_j \rangle_{\mathcal{H}} W_j(ds) \]
\[ + \langle G(t - \cdot, x - \ast)D(X(\cdot, \ast)), h \rangle_{\mathcal{H}_t} \]
\[ + \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)b(X(s, y))dsdy, \]

\[ X_n(t, x) = \sum_{j \in \mathbb{N}} \int_0^t \langle G(t - s, x - \cdot)(A)(X(s, \cdot)), e_j \rangle_{\mathcal{H}} W_j(ds) \]
\[ + \langle G(t - \cdot, x - \ast)B(X_n(\cdot, \ast)), w^n \rangle_{\mathcal{H}_t} \]
\[ + \langle G(t - \cdot, x - \ast)D(X_n(\cdot, \ast)), h \rangle_{\mathcal{H}_t} \]
\[ + \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)b(X_n(s, y))dsdy, \]
With an appropriate choice of the coefficients $A, B, D, b$:

1. $A = D = 0, B := \sigma$;
2. $A = -B = D = \sigma$,

the two convergences follow from the next

**Theorem**

The coefficients are Lipschitz. Suppose also that

$$\theta_1 \in \left[0, \frac{6 - \beta}{4}\right].$$

Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Then for any $\rho \in \left]0, \frac{2 - \beta}{2}\right[$, $\lambda > 0$,

$$\lim_{n \to \infty} \mathbb{P} (\|X_n - X\|_{\rho, t_0, K} > \lambda) = 0.$$
**Local $L^p(\Omega)$ convergence**

Prove that for a sequence $L_n(T) \uparrow \Omega$,

$$\lim_{n \to \infty} \mathbb{E} \left( \| X_n - X \|_{\rho, t_0, K}^p 1_{L_n(T)} \right) = 0.$$ 

(Similar idea as in Millet–S.-S (2000) for 2-d wave equation).

**Choice of the localization**

$$L_n(t) = \left\{ \sup_{1 \leq j \leq n} \sup_{0 \leq i \leq [2^n T^{-1}]^+} 2^{n \theta_1} |W_j(\Delta_i)| \leq \alpha 2^{n \theta_2} n^{1/2} \right\}$$

**Property**

$$\| w^n 1_{L_n(t')} 1_{[t, t']} \|_{\mathcal{H}_T} \leq C n^{2^{\theta_2}} |t' - t|^{1/2}.$$ 

**Lemma** For $\alpha > (2 \ln 2)^{1/2}$ and $\theta_2 - \theta_1 + \frac{1}{2} \geq 0$,

$$\lim_{n \to \infty} \mathbb{P}(L_n(T)^c) = 0.$$
Ingredients
For any $\theta_1 \in ]0, \infty[, \theta_2 \in ]0, \frac{4-\beta}{4} [,$

- **Local $L^p(\Omega)$ estimates of increments**
  \[ \sup_{n \geq 1} \left\| \left[ X_n(t, x) - X_n(\bar{t}, \bar{x}) \right] 1_{L_n(t)} \right\|_p \leq C \left( |\bar{t} - t| + |\bar{x} - x| \right)^\rho, \]
  \[ \rho \in ]0, \frac{2-\beta}{2} [. \]

- **Pointwise convergence**
  \[ \lim_{n \to \infty} \| (X_n(t, x) - X(t, x)) 1_{L_n(t)} \|_p = 0, \quad p \in [1, \infty). \]

To obtain the convergence in probability, $\theta_2 - \theta_1 + \frac{1}{2} \geq 0$, thus

\[ \theta_1 \in ]0, \frac{6 - \beta}{4} [. \]
Some Comments on The Proof
Increments in space

Notation

\[ \varphi_{n,p}(t, x, \bar{x}) = \mathbb{E}\left( \left| X_n(t, x) - X_n(t, \bar{x}) \right|^p 1_{L_n(t)} \right), \]

\( t \in [t_0, T], x, \bar{x} \in K, p \in [1, \infty[. \)

Proposition (a simplified version)

\[ \varphi_{n,p}(t, x, \bar{x}) \leq C \left[ f_n + |x - \bar{x}|^{\alpha_2 p / 2} + \int_0^t ds(\varphi_{n,p}(s, x, \bar{x})) \right. \]
\[ \left. + |x - \bar{x}|^{\alpha_1 p / 2} \int_0^t ds \left[ \varphi_{n,p}(s, x, \bar{x}) \right]^{1/2} \right], \]

with \( \lim_{n \to \infty} f_n = 0, \alpha_1 \in [0, (2 - \beta) \wedge 1)[, \alpha_2 \in ]0, (2 - \beta)[. \)
Lemma (Gronwall’s type) $u, b$ and $k$ are nonnegative continuous functions in $J = [\alpha, \beta]$; $\bar{p} \geq 0$, $\bar{p} \neq 1$, $a > 0$. Suppose that

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)ds + \int_{\alpha}^{t} k(s)u^{\bar{p}}(s)ds, \quad t \in J.$$ 

Then

$$u(t) \leq \exp \left( \int_{\alpha}^{\beta} b(s)ds \right) \left[ a^{\bar{q}} + \bar{q} \int_{\alpha}^{\beta} k(s) \exp \left( -\bar{q} \int_{\alpha}^{s} b(\tau)d\tau \right) ds \right]^{\frac{1}{\bar{q}}},$$

for $t \in [\alpha, \beta_1)$, where $\bar{q} = 1 - \bar{p}$ and $\beta_1$ is choosen so that the expression beween [...] is positive in the subinterval $[\alpha, \beta_1)$ ($\beta_1 = \beta$ if $\bar{q} > 0$).

D. Bainov, P. Simenov: Integral Inequalities and Applications.
Where \((\cdot)^{\frac{1}{2}}\) and the increments come from?

\[
\mathbb{E} \left( |X_n(t, x) - X_n(t, \bar{x})|^p 1_{L_n(t)} \right) \leq C \sum_{i=1}^{4} R_n^i(t, x, \bar{x}),
\]

\[
R_n^1(t, x, \bar{x}) =
\mathbb{E} \left( \left| \sum_{j \in \mathbb{N}} \int_0^t \langle [G(t-s, x-\ast) - G(t-s, \bar{x}-\ast)]Z_n(s, \ast), e_k \rangle \mathcal{H} \right| W^j(ds) \right|^p
\]

\[
Z_n(s, y) = A(X_n(s, y)) 1_{L_n(s)}.
\]
Apply Burkholder’s inequality and Plancherel’s identity:

\[ R_n^1(t, x, \bar{x}) \leq C \mathbb{E} \left( \left\| \int_0^t ds \left\| \left[ G(t - s, x - \ast) - G(t - s, \bar{x} - \ast) \right] Z_n(s, \ast) \right\|^2 \mathcal{H} \right\|^{p/2} \right) \]

\[ \overset{(*)}{=} C \mathbb{E} \left( \int_0^t ds \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left[ G(t - s, x - du) - G(t - s, \bar{x} - du) \right] f(u - v) \times \left[ G(t - s, x - dv)) - G(t - s, \bar{x} - dv) \right] Z_n(s, u) Z_n(s, v) \right)^{p/2} \]

\[ = \int_0^t ds \int_{\mathbb{R}^3 \times \mathbb{R}^3} G(s) G(s) [f \Delta Z_n \Delta Z_n + Z_n \Delta Z_n \Delta f + Z_n Z_n \Delta^2 f], \]

\[ \overset{(*)}{=} f(x) = |x|^{-\beta}, \beta \in [0, 2[. \]
\[ C \varphi_{n,p}(t, x, \bar{x}) \leq f_n \quad \text{(correction stochastic-pathwise integrals)} \]
\[ + |x - \bar{x}|^{\alpha_2 p/2} (Z_n Z_n \Delta^2 f) \]
\[ + \int_0^t ds(\varphi_{n,p}(s, x, \bar{x})) \quad (f \Delta Z_n \Delta Z_n) \]
\[ + |x - \bar{x}|^{\alpha_1 p/2} \int_0^t ds \left[ \varphi_{n,p}(s, x, \bar{x}) \right]^{1/2} \quad (Z_n \Delta Z_n \Delta f) \]

**Stationarity!!**

For non vanishing initial conditions, use estimates with fractional Sobolev norms (Dalang and S.-S., 2009).
Comparison with $d = 2$

- Different approach to $G(\bar{t} - s, x - dy) - G(t - s, \bar{x} - dy)$ (method from Dalang–S.-S., 2009).
- The approximation of

$$\sum_{j \geq 1} \int \cdots W_j(ds) \quad \text{by} \quad \sum_{j \geq 1} \int \cdots W_j^n(s)ds$$

is more difficult.

- smoother approximations of the noise (parameter $\theta_1$),
- combination of the two processes: approximation and localization.
References


Many Thanks!