



Time-Parallel Algorithms for Weather Prediction and Climate Simulation

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Time Parallelism

- Small visionary paper by Nievergelt (1964)
- Need to parallelize seemingly essentially serial algorithms as **solving evolution equations** to obtain **real time** solutions on highly parallel computers
- **Crucial point**: perform redundant operations as long as this allows to reduce **real time**
- Beneficiary: **Numerical Weather Prediction**
- “that more general and improved versions of these methods will be of great importance when computers capable of executing many computations in parallel become available”



Why Time-Parallelism in NWP?

- Computer parallelism is expected to increase more rapidly than model resolution
- If there is an optimal number of grid points per processor (core) and the number of horizontal points is fixed then there is a limit on the number of processors that a model can use efficiently
- If the vertical can't be parallelized (because of physics parameterization, etc) then the time direction should be parallelized!
- A research problem – not for tomorrow



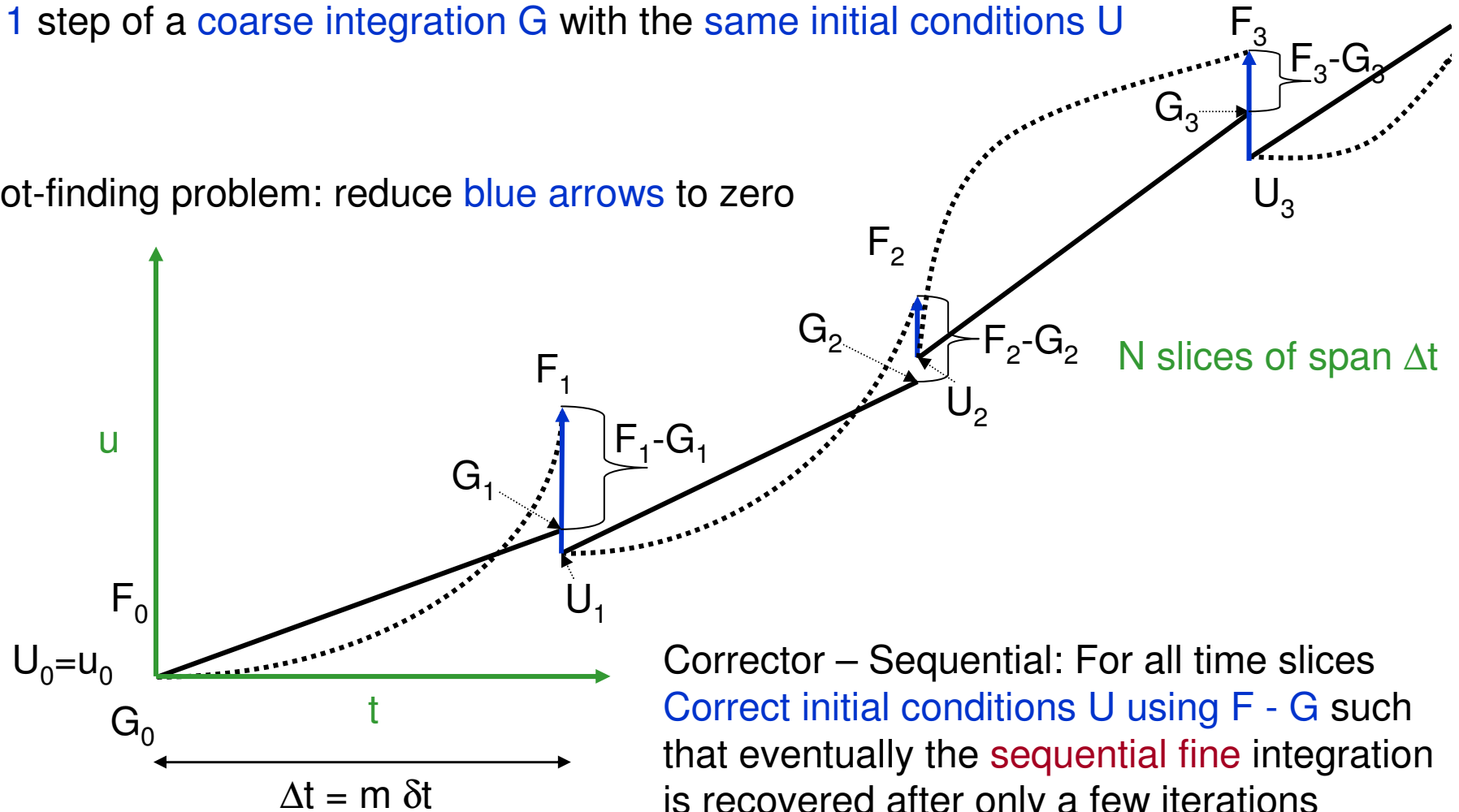
Iterative Scheme

- Suppose we have the discrete sequential solution of an ODE/PDE at every timestep δt for a time span T
- If we were to sample it at every $\Delta t = m \delta t$ ($T = N \Delta t$)
- Then for every time slice we could obtain in parallel the solution with timestep δt starting from our sampled values
- That solution would be obtained much faster in wall clock time
- An idea for the parallel solution of our ODE/PDE is to start from a set of guessed samples which will give us an approximation of the discrete solution we are interested in
- Somehow correct our guesses and obtain a better solution
- Iterate until convergence to the discrete solution
- If the convergence is fast enough we will need only few iterations and the wall clock time will be reduced (at the expense of spending more CPU time)

Parallel Iterative Scheme for $u' = f(u)$

Predictor – Parallel: For all the time slices Δt compare m steps of a **fine integration F** to 1 step of a **coarse integration G** with the **same initial conditions U**

Root-finding problem: reduce **blue arrows** to zero





MATLAB Code (adapted from MJ Gander)

```
% initial coarse integration
U = mdl( u_0, T, N, 'coarse' );

% iteration loop
for k = 1 : K

    % parallel loop - predictor
    for n = 1 : N
        Go = mdl( U( :, n ), coarse_step, 1, 'coarse' );
        Fo = mdl( U( :, n ), coarse_step, m, 'fine' );
        G( :, n ) = Go( :, end );
        F( :, n ) = Fo( :, end );
    end % end parallel loop

    % sequential loop - corrector
    if k < K
        for n = 1 : N
            Gn = mdl( U( :, n ), coarse_step, 1, 'coarse' );
            U( :, n + 1 ) = Gn( :, end ) - G( :, n ) + F( :, n );
        end % sequential loop
    end

end % iteration loop
```



The “Parareal” Algorithm ...

- If convergence is reached with few iterations ($\ll N$): real time gain at the expense of CPU time
- At convergence $U = F_{\text{serial}}$ at the common times
 - No need to iterate further than the accuracy of F
- Acceleration = sequential time/parallel time

$$\frac{N}{K(1 + \frac{N}{m})}$$

- Efficiency = Acceleration/number of processors

$$\frac{1}{K(1 + \frac{N}{m})}$$

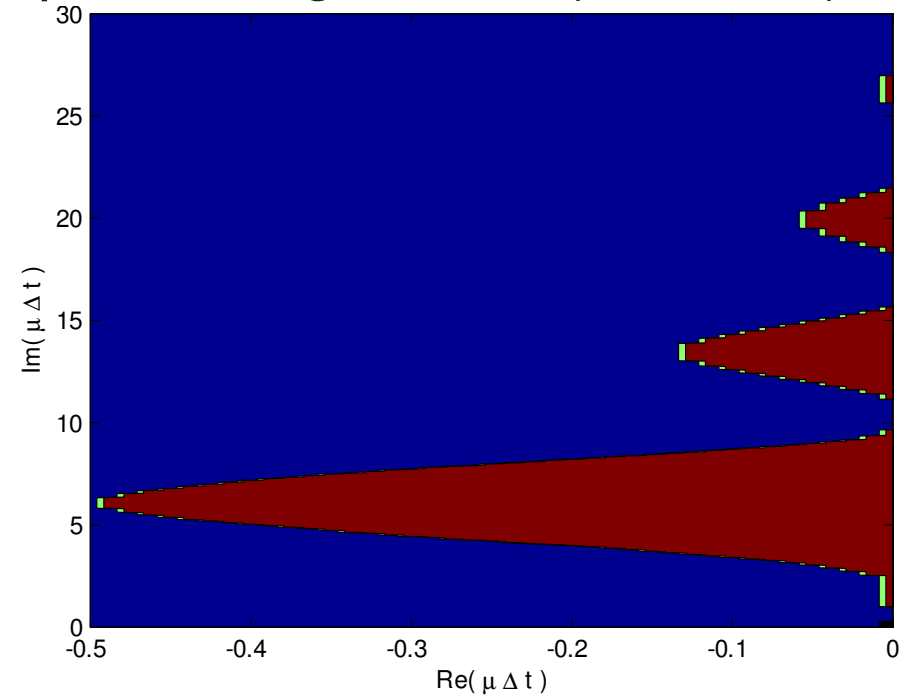
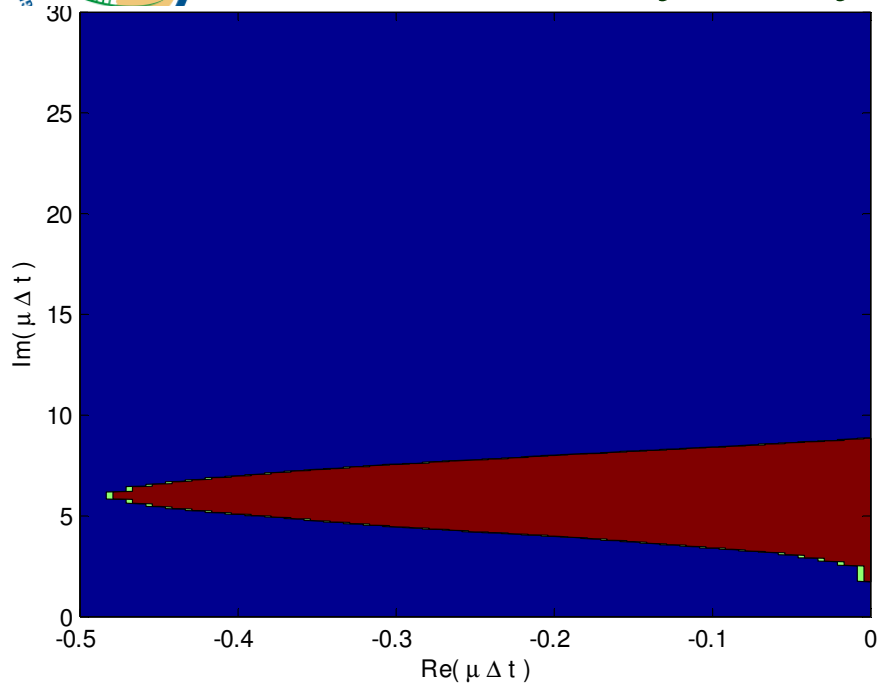
of processors = # of coarse steps = N



Analysis

- Stability (Staff & Rønquist, Bal)
 - Considering the model equation $\phi' = \mu \phi$
 - Can be unstable if the imaginary part of μ is much larger in modulus than the modulus of the real part of μ (linear stability)
 - There exists a work around for these important cases ([Krylov Space Enhanced Parareal](#), modified Parallel Implicit Time Algorithm (PITA))
- Convergence (Maday, Gander & Vandewalle, Gander & Hairer &c)
 - Supralinear
 - Not a problem, if it converges, it converges fast

Linear Stability Analysis: Implicit Runge-Kutta (Radau 3)



$$z = \mu\Delta t = \text{Re}(z) + \text{Im}(z)i$$

Amplification factor for 1 coarse timestep (λ) and for m fine timesteps (Λ)

$$\Lambda \equiv R(z) = \frac{1 + z/3}{1 - 2z/3 + z^2/6}$$

$$\lambda \equiv R^m(z/m)$$

Amplification factor of "Parareal" at coarse timestep n of iteration k

$$H(n, k, \lambda, \Lambda) = \sum_{j=0}^k \binom{n}{j} (\lambda - \Lambda)^j \Lambda^{n-j}$$

Stability domain (in blue) for $\text{Re}(z) < 0$

$$|H(n, k, \lambda, \Lambda)| \leq 1, \text{ for all } n, k \leq N$$



Applications (partial)

- Finance (Bal, 2002)
- Molecular Dynamics (Baffico & al., 2002)
- Navier-Stokes Equations for diffusion dominated flows (Fischer & al., 2005)
- Reservoir simulation (Garrido & al., 2005)
- Inverse parabolic problems (Daoud, 2007)
- Non-linear Ordinary Differential Equations + Burgers Equation (Gander & Hairer, 2008)
- Korteweg-deVries-Burgers Equations (Kaber & Maday, 2007)
- Chemical Kinetics (Blouza & al. 2010)
- Chemical Reaction-Diffusion (Duarte, & al., 2011)
- Method of Characteristics (Gander, 2008)
- Krylov Space Enhanced Parareal (Gander & Petcu, 2008)



Linear Stability Analysis: Semi-Lagrangian

- System of ODEs dependent on scale (wave number) of perturbation
- For constant advection and diffusion in a periodic domain we can obtain the amplification factor
 - Various types of interpolation (linear, cubic)
 - Various discretizations for the diffusion operator
 - Various temporal weighting of advection and diffusion
- Preliminary experiments show that the criteria for convergence is that λ and Λ be close enough
- Is the long timestep advantage of the Semi-Lagrangian scheme maintained?
- Result so far don't include the work around for pure advection
- It works for well resolved waves

Semi-Lagrangian advection with cubic spline

$$\theta = \pi k \Delta x$$

Residual CFL number

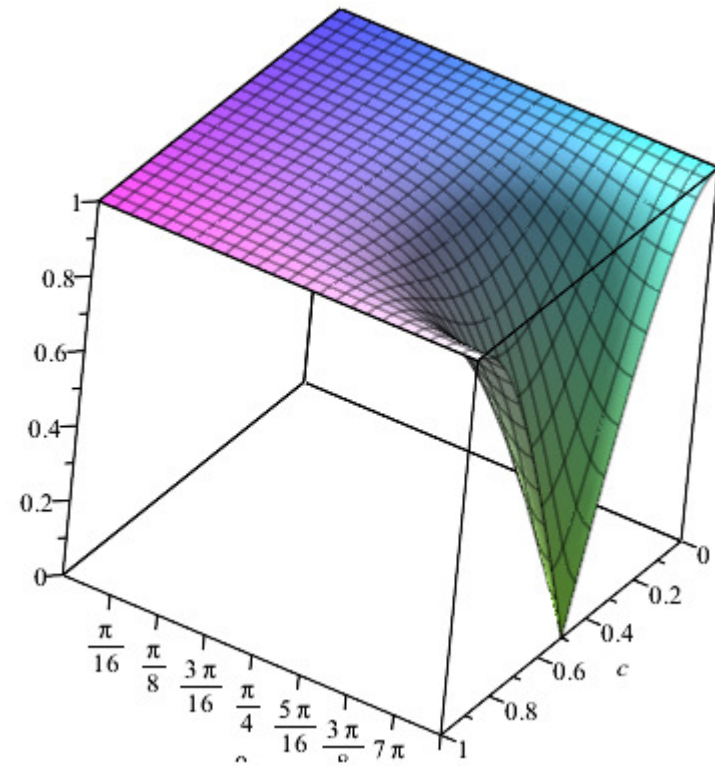
$$c = CFL - n_{CFL} \quad CFL = \frac{u_0 \Delta t}{\Delta x} \quad n_{CFL} = \text{trunc}(CFL)$$

Response of second derivative operator

$$\rho_{xx} = \frac{\sin(\theta)^2}{1 - \frac{2}{3} \sin(\theta)^2}$$

Response of cubic spline interpolation

$$\Lambda = e^{-(1+n_{CFL})2\theta I} \left[\begin{array}{l} c + (1-c)e^{2\theta I} + \dots \\ \frac{2c(1-c)\rho_{xx}}{3} (1 + c + (2-c)e^{2\theta I}) \end{array} \right]$$



$\text{Abs}(\Lambda), \theta = 0, \pi/2, c = 0, 1$



... and in Numerical Weather Prediction?

- Lorenz Equations (Gander & Hairer, 2008)
- An effort began at MeteoSwiss in collaboration with the German Weather Service (DWD)
 - Ruprecht & Krause (June 2011)
 - Schematic problems
 - They are looking at waves that are potentially problematic for “Parareal” but by applying the work around (Krylov Space Enhanced Parareal) and adding damping (divergence damping)



Project

- Stability & convergence
- Performance gain (predicted & measured)
- Choice of schemes on fine (less implicit?) & coarse grid
- Stabilization mechanism
- Ratio between fine & coarse grid timesteps
- Spatial resolution of the coarse grid model
- Relative performance of direct vs. iterative solvers
- Impact on the balance of the flow
- ...



Roadmap

- 1D (advection, rotation, Shallow-Water) & 2D (advection, Shallow-Water, vertical slice)
 - cover most problems met in NWP
 - Advection + (diffusion/reaction, sources)
 - Rotation
 - Mix of slow waves (Rossby) & fast waves (gravity, sound)
- Stopping criteria for iterations
- Williamson & al. (1992) cases, for horizontal
- Rising Bubble case (Robert, 1993) & mountain flow mountain (Pinty & al. 1995), for vertical
- If time & work force permit
 - 3D Primitive Equations
 - 3D Non-Hydrostatic Equations
 - 3D model Full Model (including physics parameterization)



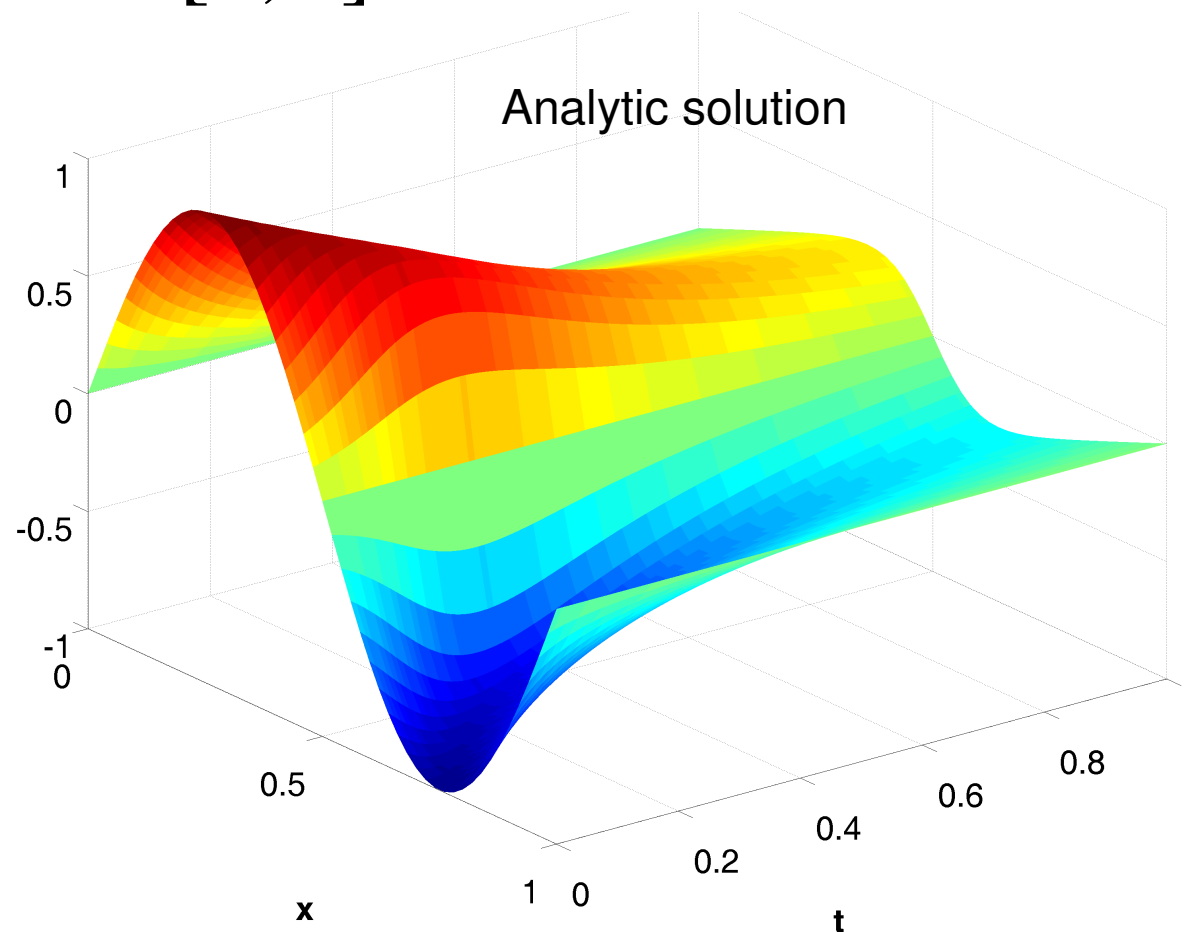
First Steps

- Reproduce Gander & Hairer 2008
 - Always 5 iterations ($K = 5$)
- Ordinary Differential Equations (ODEs)
 - Brusselator
 - Arenstorf Orbits
 - Lorenz Equations
 - 4th Order Runge-Kutta (“classic”)
- Partial Differential Equations (PDEs)
 - Burgers Equation
 - Finite-Difference model similar to paper
 - Semi-Lagrangian model similar to GEM model

Burgers Equation

$$u_t + uu_x = \nu u_{xx} \text{ in } [0,1], u(x,0) = \sin(2\pi x)$$

$$\nu = 1/50 \text{ for } [0,T] \text{ with } T=1$$





Burgers Eqn: Eulerian FE BTCS

$$T = 1$$

$$N = 10 \quad m = 10 \quad K = 5$$

$$\Delta t = T/N = 1/10$$

$$\delta t = \Delta t/m = 1/100$$

$$n_x = 51$$

$$\Delta x = 1/50$$

$$\text{Advection } \alpha = 1 \quad (\text{iter} = 3)$$

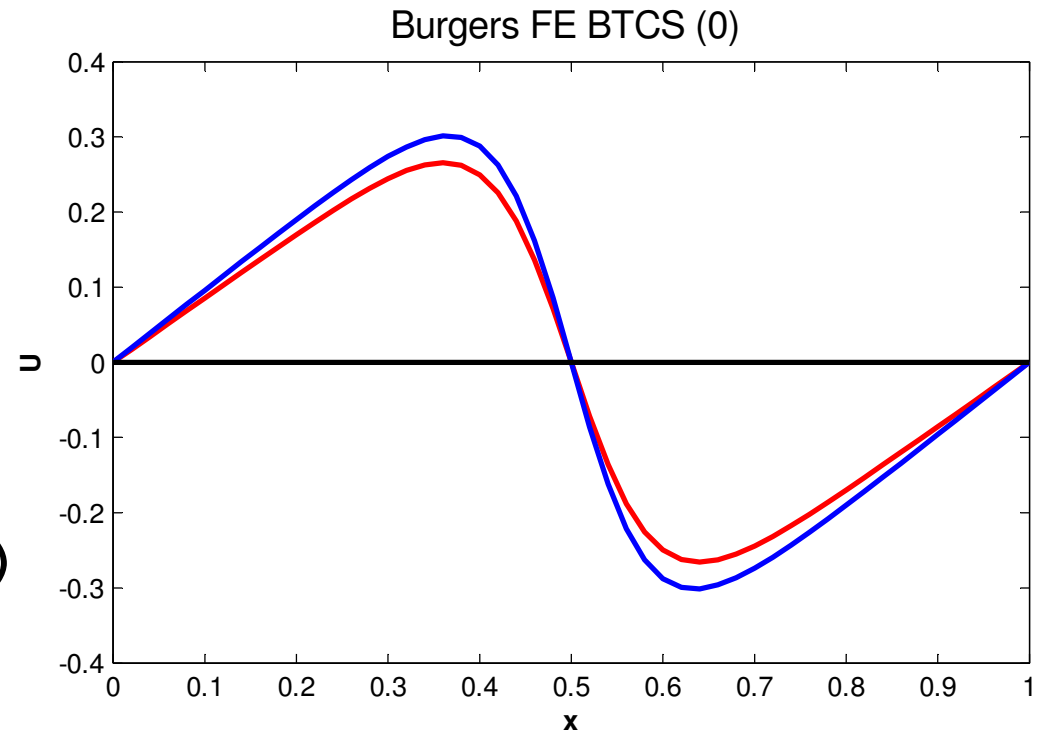
$$v = 1/50$$

$$\text{Diffusion } \gamma = 1$$

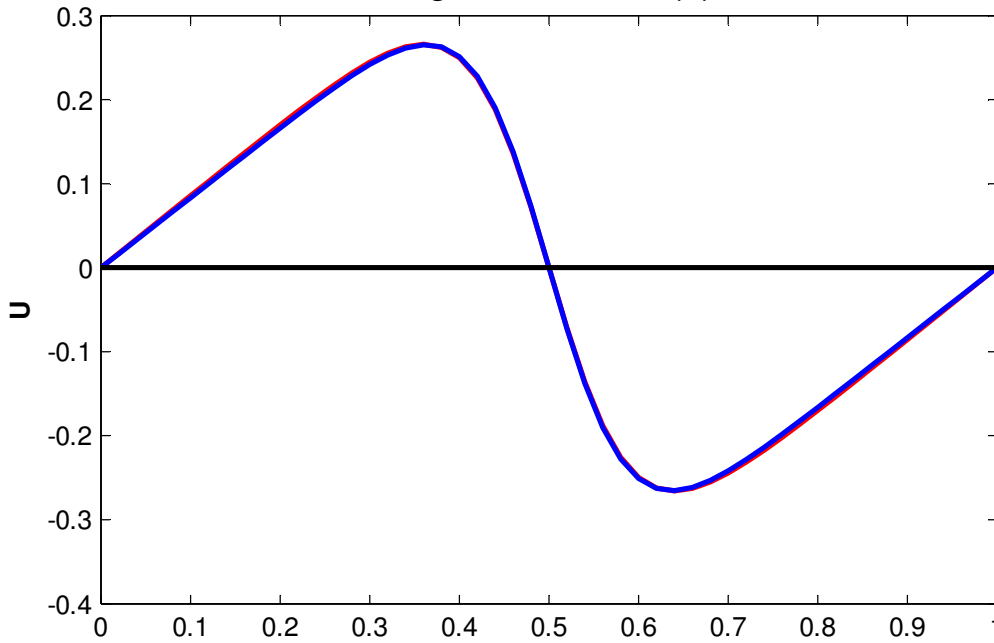
$$\text{Initial Condition} = \sin(2\pi x)$$

Red line: Fine serial solution at $T = 1$

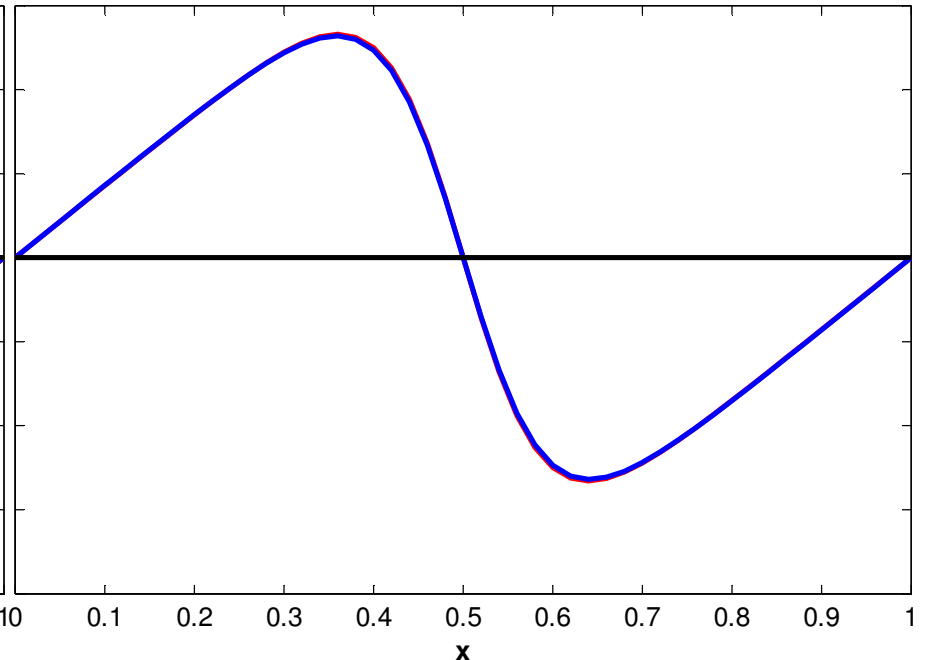
Blue line: Coarse solution at $T = 1$



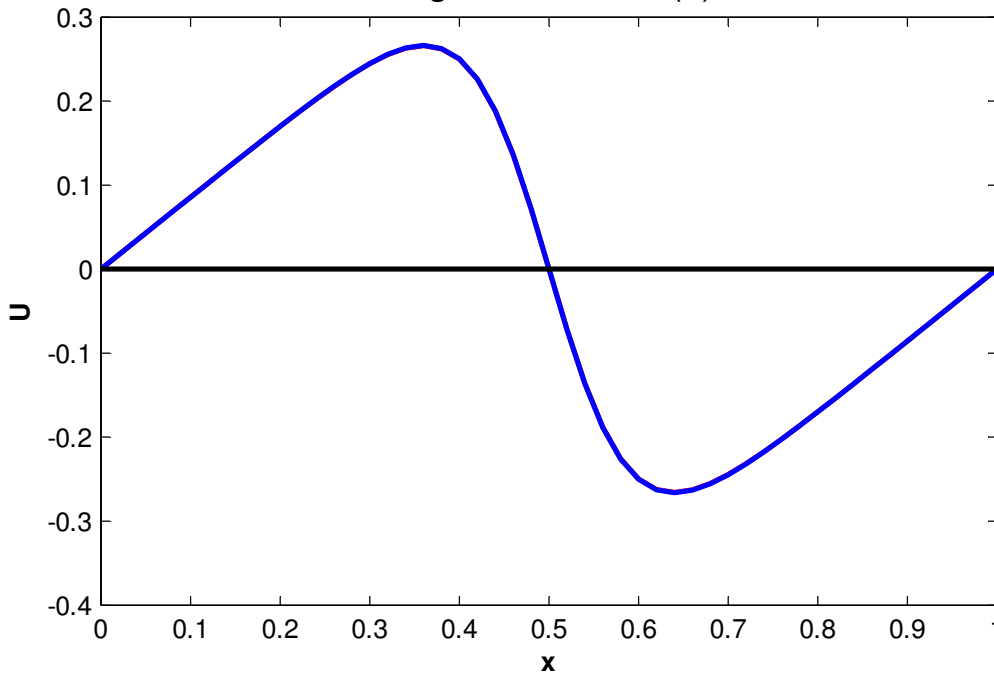
Burgers FE BTCS (1)



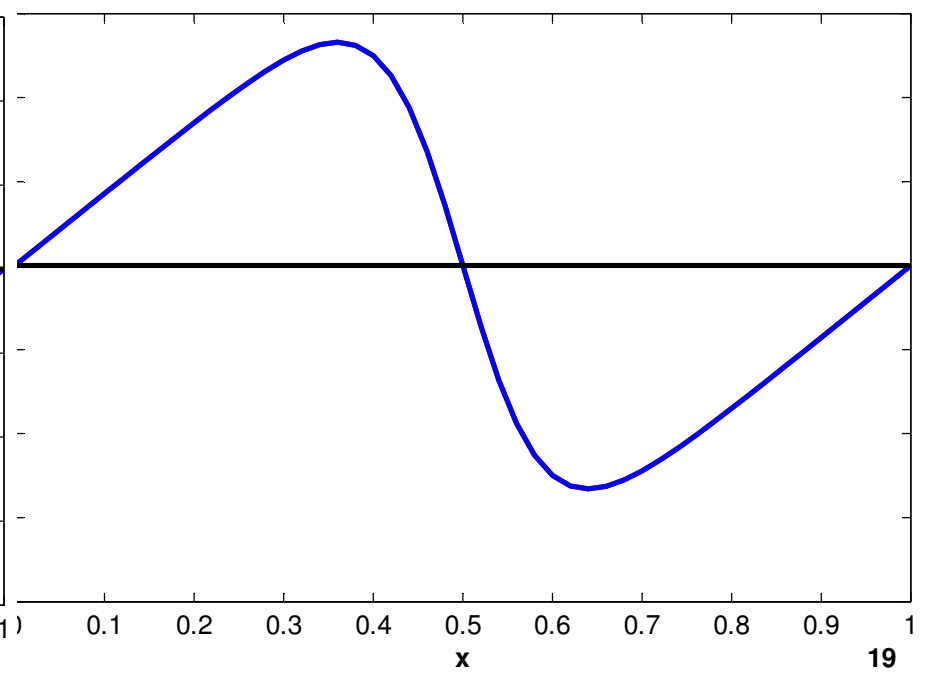
Burgers FE BTCS (2)



Burgers FE BTCS (3)



Burgers FE BTCS (4)



Convergence History - BTCS

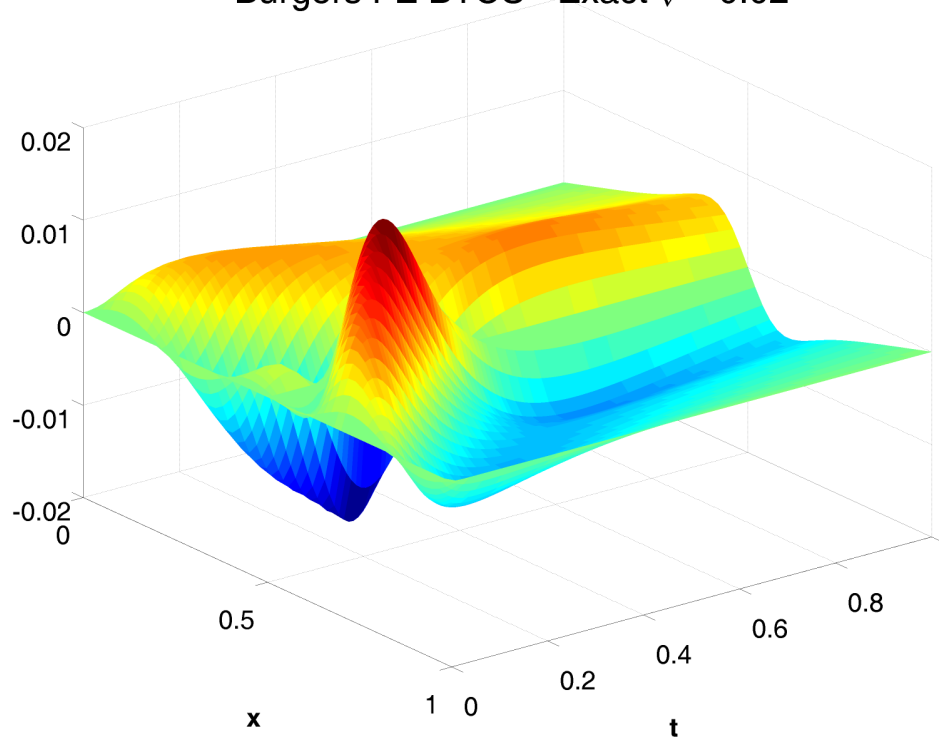
- Difference between the Fine serial solution and the exact solution

- Evolution of the error norm with the iteration #

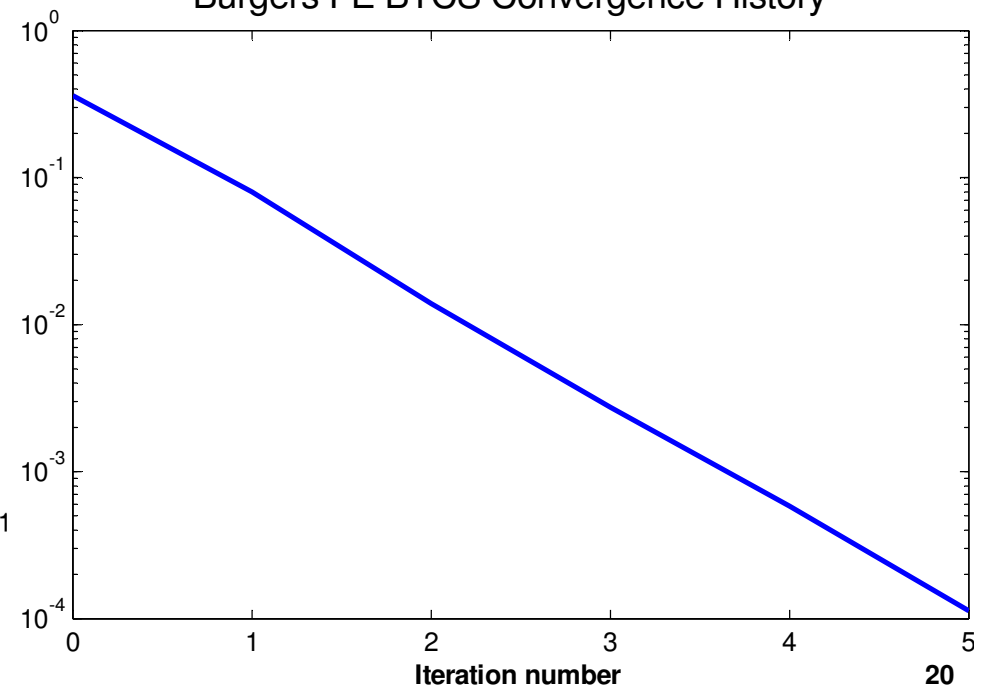
$$\max\left(\sqrt{\sum_{i=1}^{n_x} (U(i, n) - \text{Ref}(i, mn))^2}, n = 1, N\right)$$

- L_2 in space, L_{inf} in time

Burgers FE BTCS - Exact $\nu = 0.02$



Burgers FE BTCS Convergence History



Constant Advection-Diffusion SL

$c = 0.432$ $\nu = 5 \cdot 10^{-6}$ $n_x = 51$ $T = 30$ $N = 5$ $m = 20$ $K = 5$

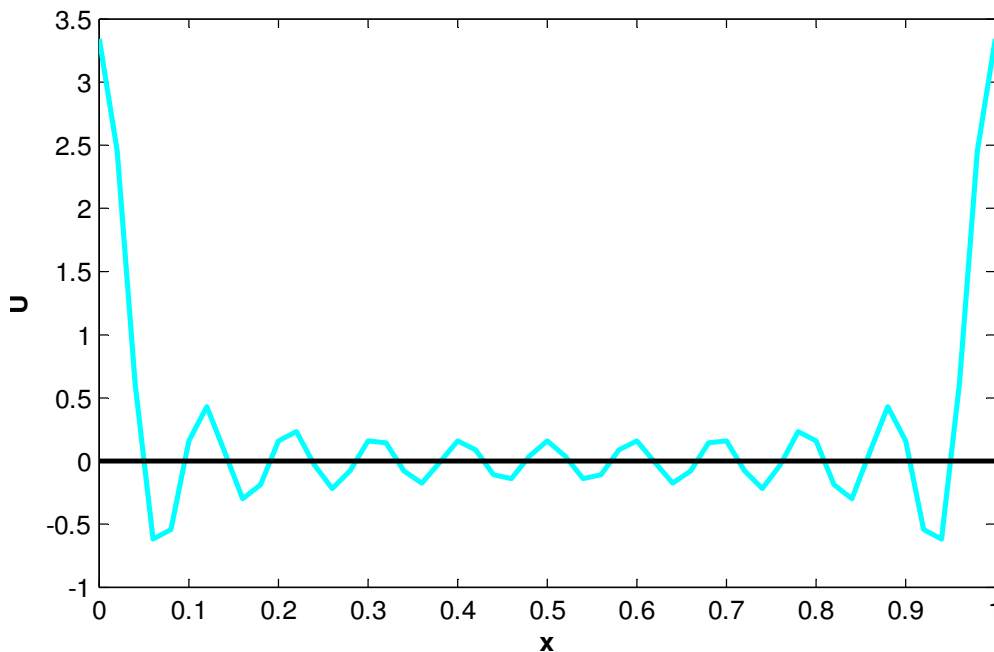
interpolation: cubic spline

diffusion: finite-element & implicit ($\gamma = 1$)

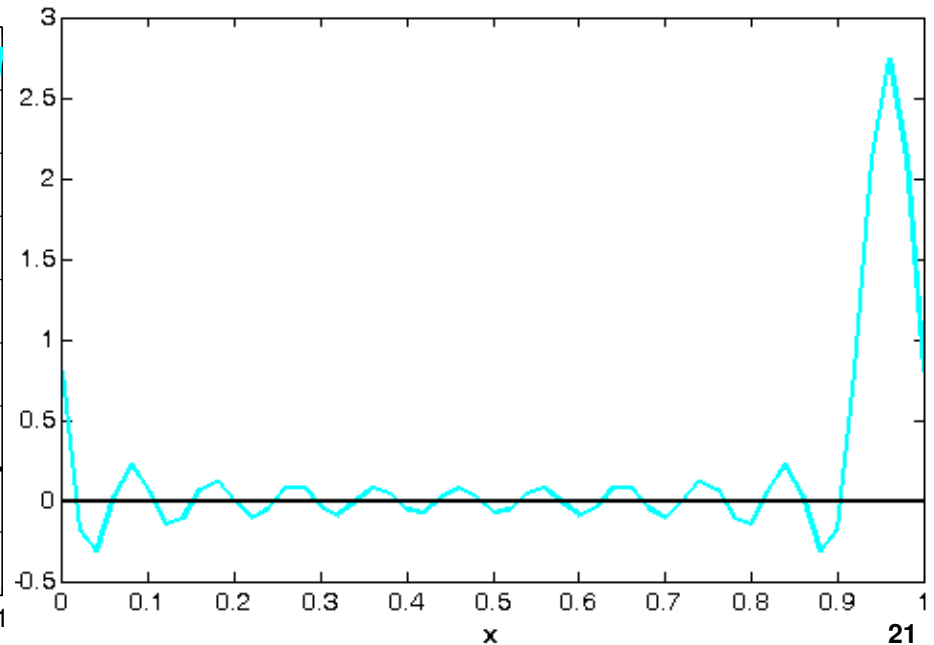
Exact solution

$$u_{exact}(x, t) = \left(1 + \sum_{l=1}^{10} 2 \exp(-\nu t (2\pi l)^2) \cos(2\pi l(x - ct)) \right) / \pi$$

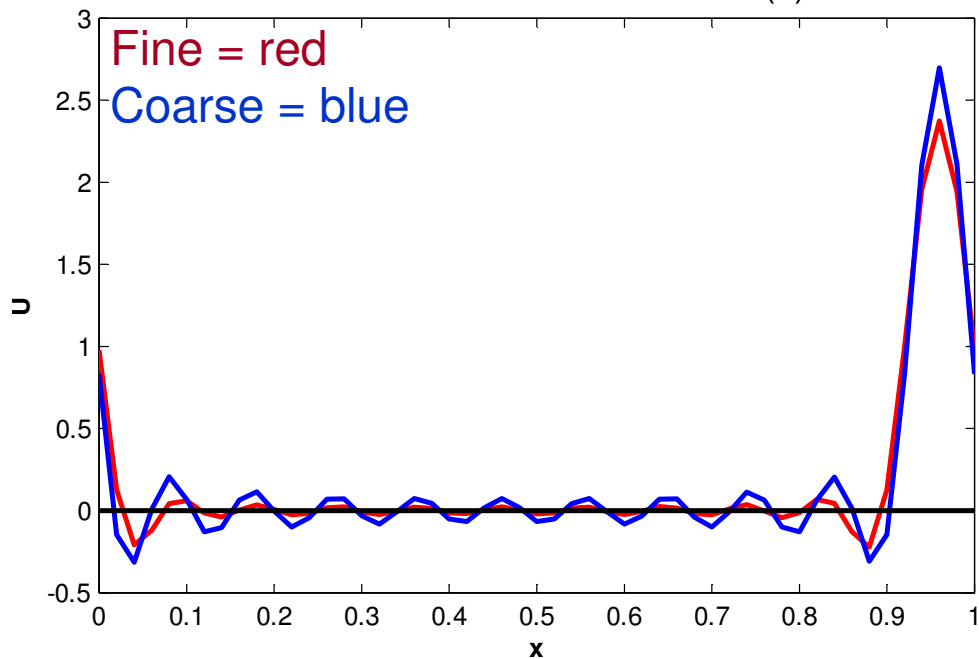
Constant Advection-Diffusion SL Initial Condition



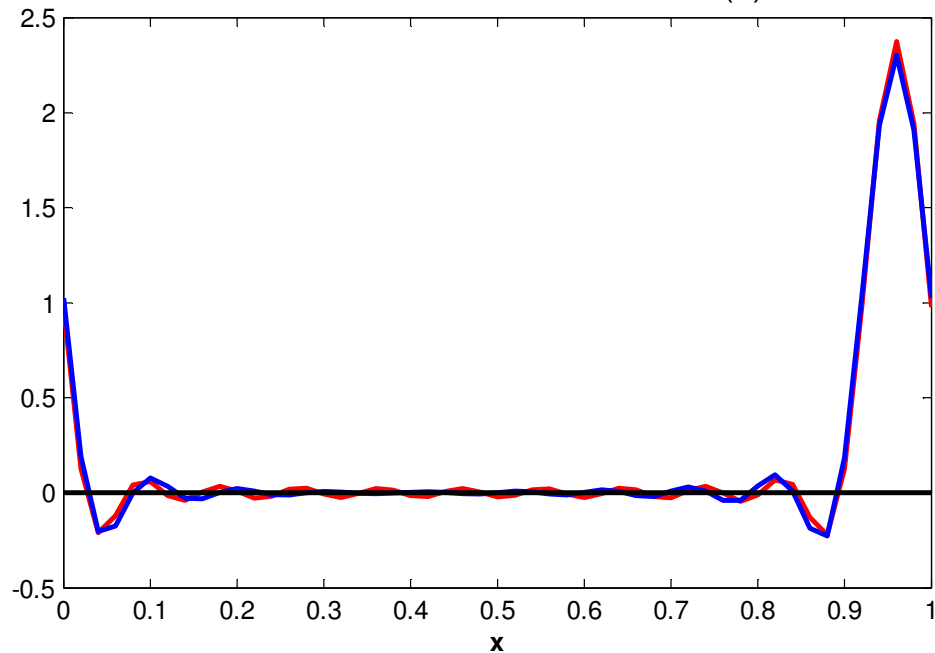
Exact Solution at T = 30



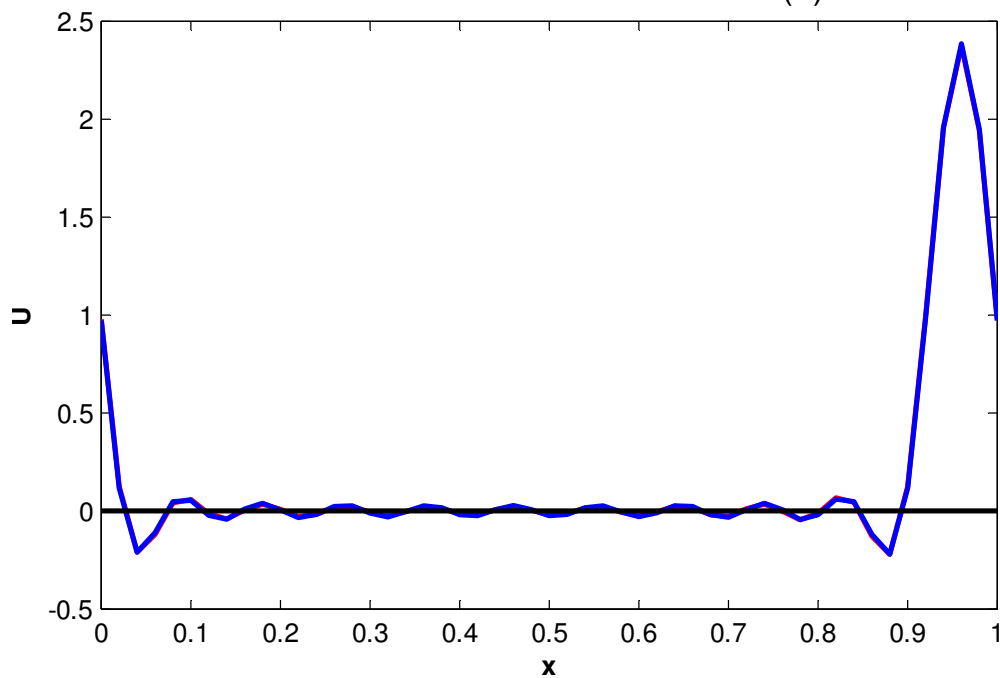
Constant Advection-Diffusion SL (0)



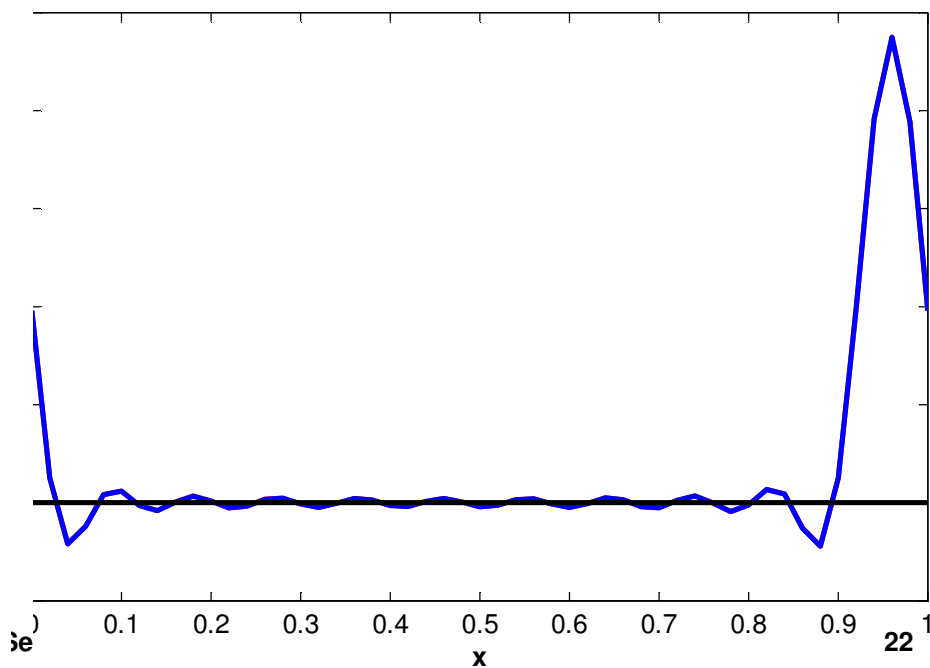
Constant Advection-Diffusion SL (1)



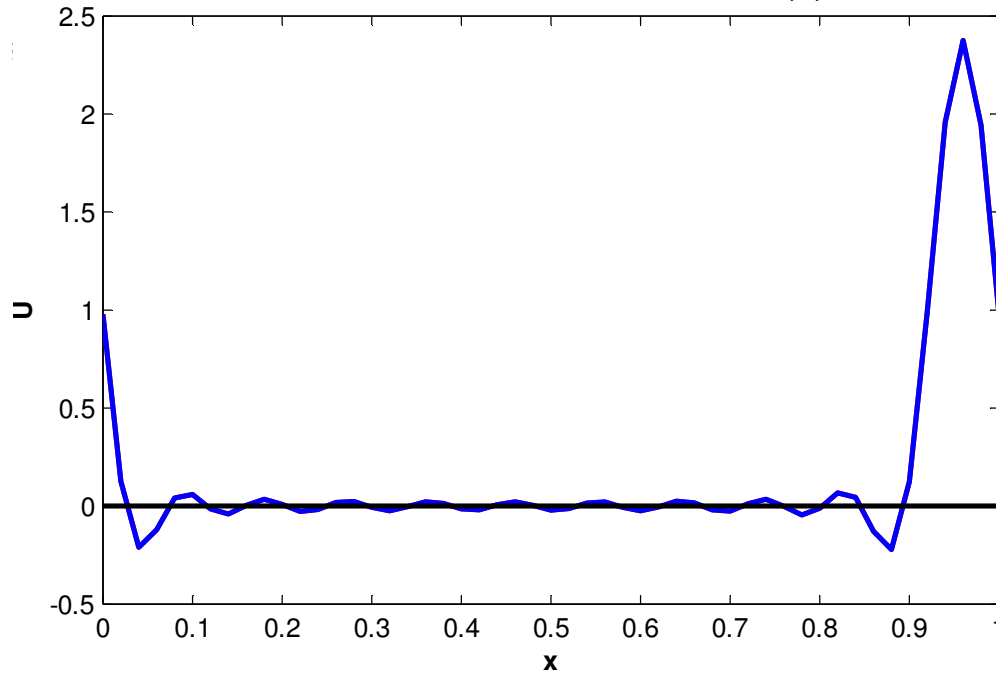
Constant Advection-Diffusion SL (2)



Constant Advection-Diffusion SL (3)



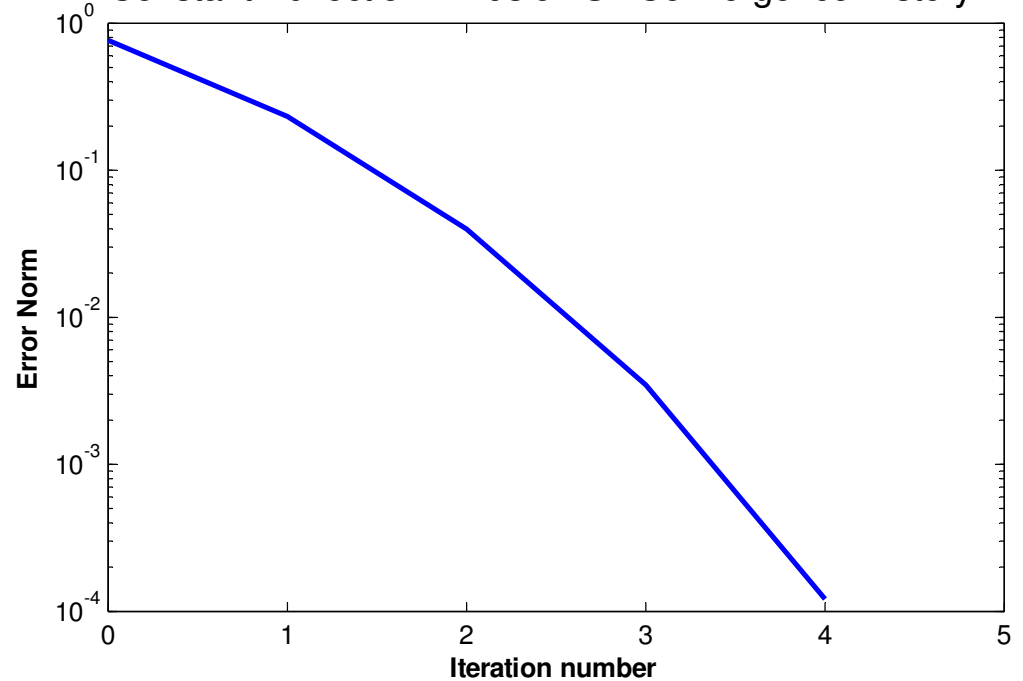
Constant Advection-Diffusion SL (4)



The **coarse** solution has converged towards the **fine** solution

Error Norm at the end of iteration 5 is zero because there are only 5 **coarse** steps ($K = N = 5$)

Constant Advection-Diffusion SL Convergence History





Burgers Eqn: Semi-Lagrangian

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$$n_x = 51$$

$$\Delta x = 1/50$$

Advection: cubic spline

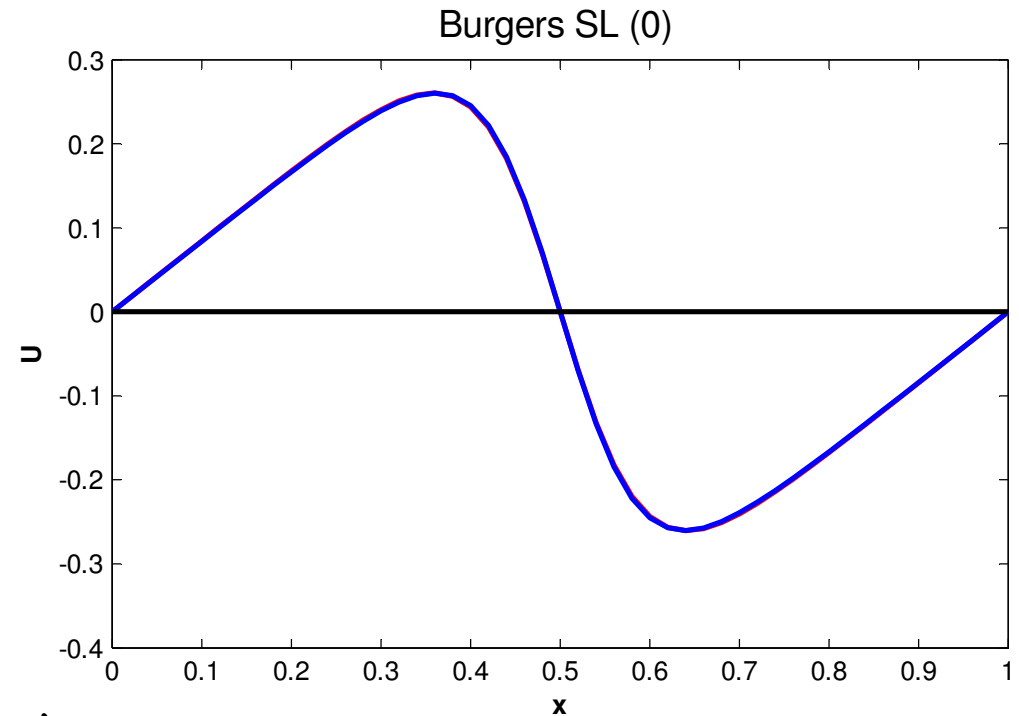
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$$\text{Diffusion } \gamma = 1$$

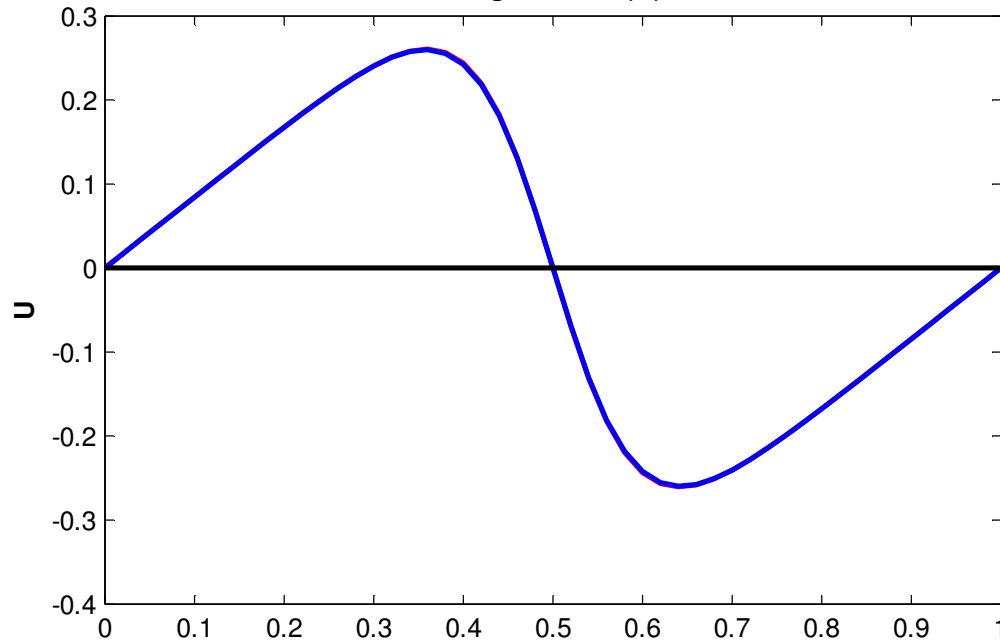
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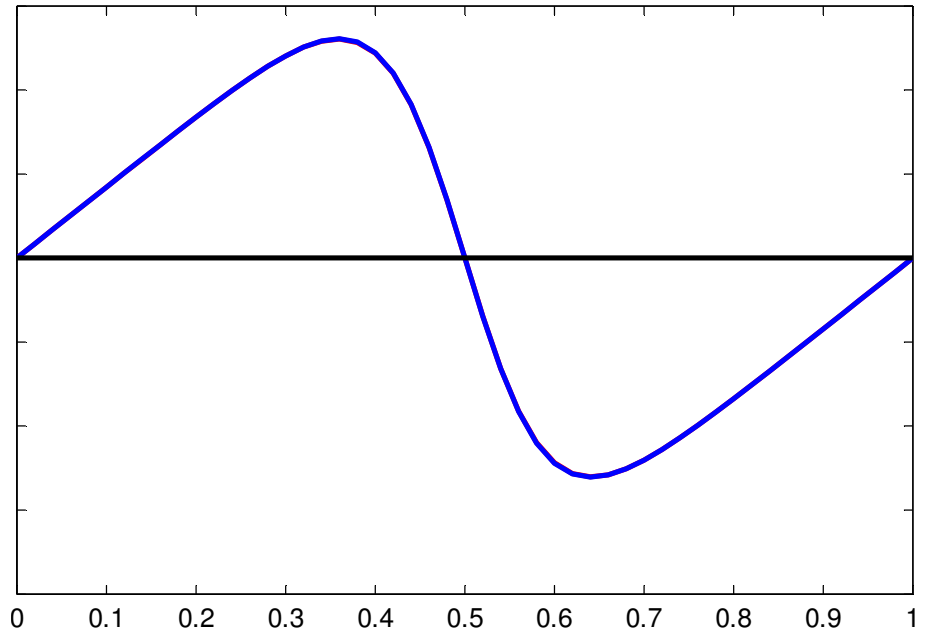
Blue line: Coarse solution at $T = 1$



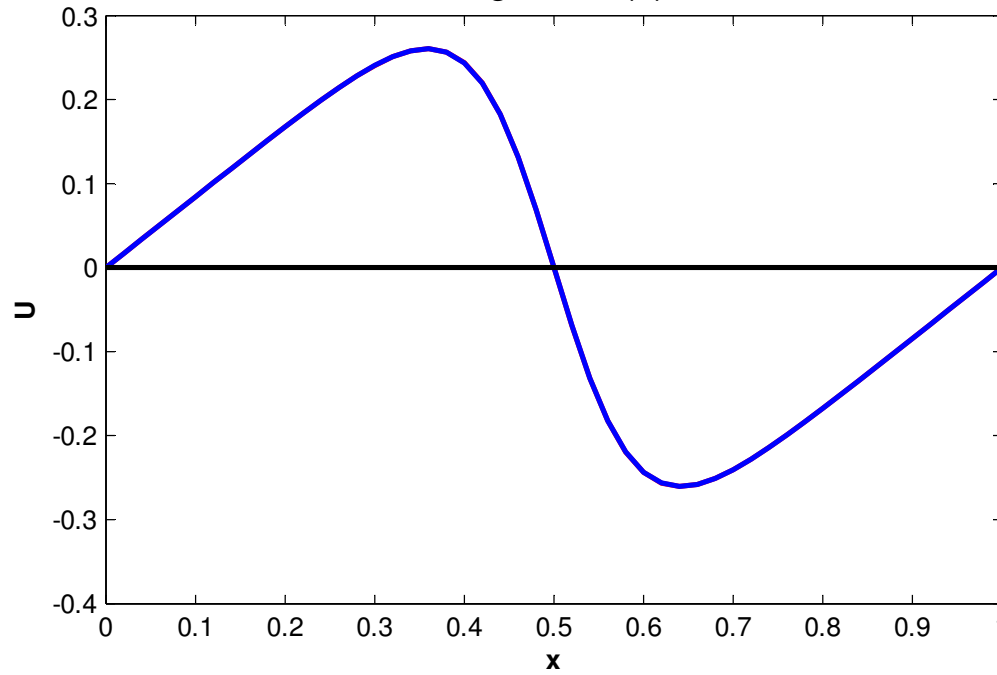
Burgers SL (1)



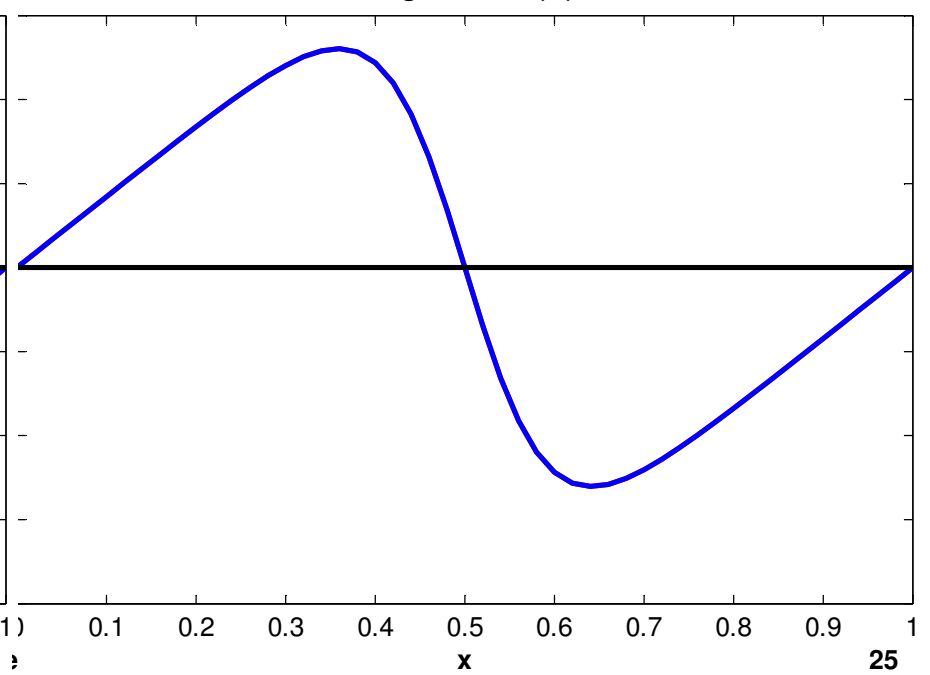
Burgers SL (2)



Burgers SL (3)



Burgers SL (4)



Convergence History - SL

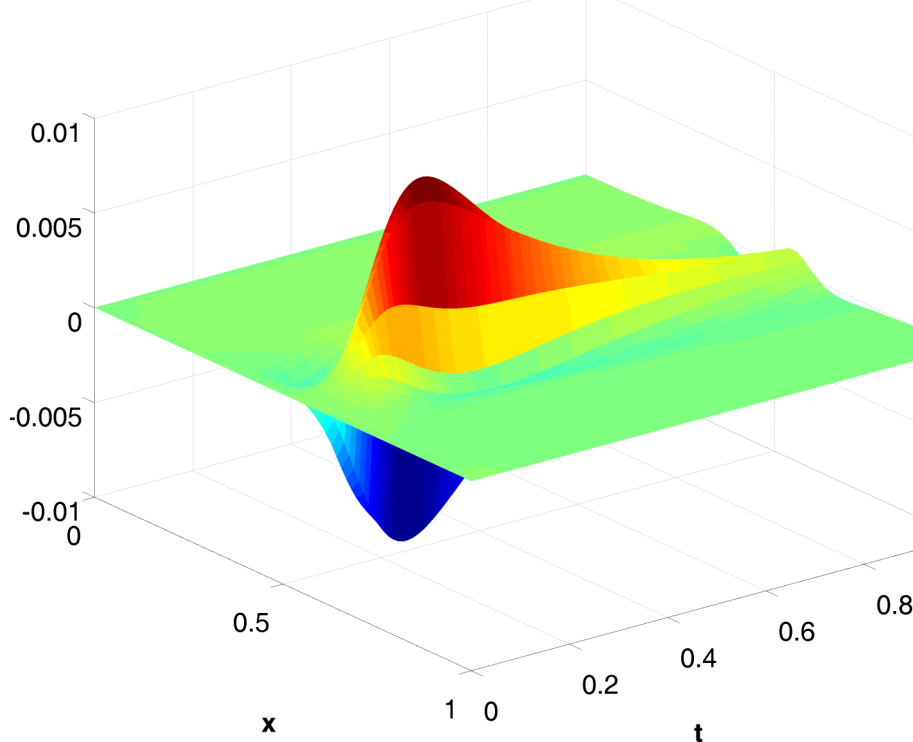
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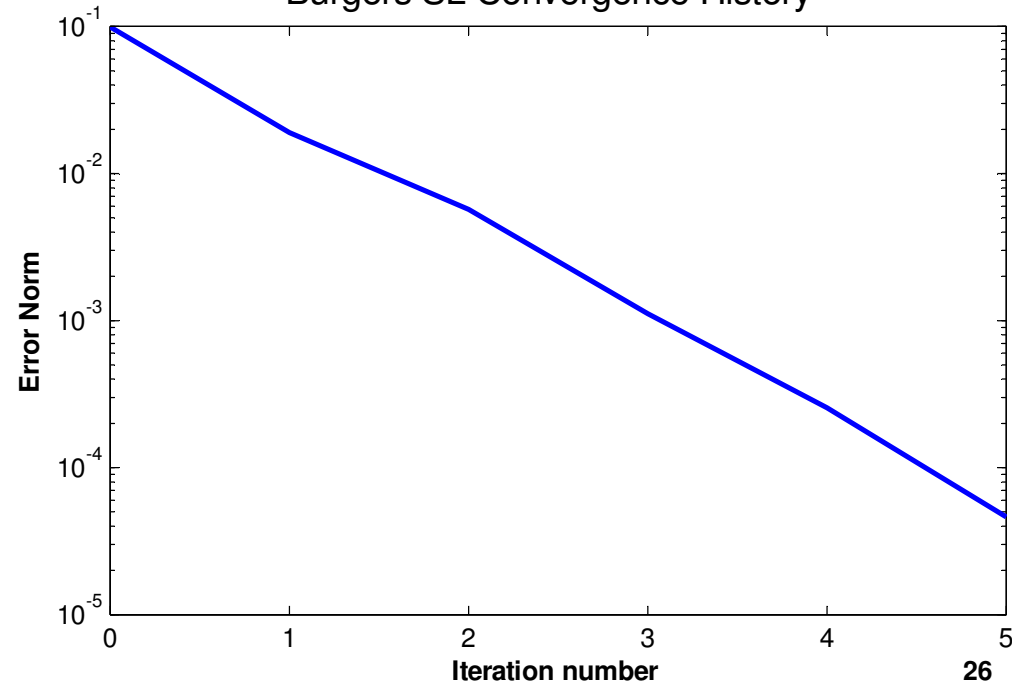
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- L_2 in space, L_{inf} in time

Burgers SL - Exact $\nu = 0.02$



Burgers SL Convergence History





Done (or almost)

- Finish advection-diffusion(-reaction) in 1-D
 - Reduction in spatial resolution of coarse integration
- Study solution for pure advection
 - Krylov Space Enhanced Parareal, modified PITA
 - Test with pure rotation ($\mu = i \omega$)



Future

- Study solution for pure advection
 - modified PITA
- Begin Shallow-Water Equations in 1-D
 - Linear & non-linear
 - 2-D horizontal
- Begin Vertical Structure Equation in 1-D
 - Hydrostatic
 - Non-hydrostatic
 - 2-D vertical slice



Thank You!

Questions?

Merci!