An Orthogonal Curvilinear Terrain-Following Coordinate for Atmospheric Models

Yiyuan Li ¹, Bin Wang ¹, Donghai Wang ²

¹. Institute of Atmospheric Physics, Chinese Academy of Sciences
². Chinese Academy of Meteorological Sciences

Isaac Newton Institute for Mathematical Sciences
2012.09.26
Outline

1. Motivation
2. An Orthogonal Sigma Coordinate
3. The Idealised Experiments
4. Conclusion
The surface of the earth is complex – which is the lower boundary for a numerical model.

The choice of vertical coordinate system is the most important aspect of designing a model (Griffies et al., 2000; Steppeler et al., 2003; Ji et al., 2005; Staniforth and Wood, 2008) – As it can simplify the lower boundary!
Most atmospheric and oceanic models preferred the terrain-following coordinate (σ-coordinate)

- Pressure-based σ-coordinate (Phillips, 1957)
  \[ \sigma = \frac{p}{p_s} \]

- Height-based σ-coordinate (Gal-Chen and Somerville, 1975)
  \[ \sigma = H \frac{z-h}{H-h} \]

Both which are non-orthogonal coordinate

red, blue and green grid surfaces: x-, y-, and σ-coordinate surfaces; blue lines: coordinate lines; black arrow lines: basis vectors (Li et al., 2011)
1. Motivation – 2 Common Computational Errors in This $\sigma$-coordinate

1. The errors of advection

$$u \frac{\partial F}{\partial x} = u \frac{F_2 - F_1}{\Delta x}$$

2. The errors of pressure gradient force (PGF errors)

- Same order of magnitude
- Opposite in sign

1. Motivation — the Existing Methods to Overcome These Errors

◆ To alleviate the PGF errors to an acceptable level, upon the **two-term** computational form (Zeng, 1963; Corby et al., 1972; Gary, 1973; Qian and Zhong, 1986; Blumberg and Mellor, 1987; Yu, 1989; Sikirić et al., 2009; Berntsen, 2011)

  - Adjusting parameters
  - Designing special computational schemes

◆ To bypass the PGF errors, via keeping **one-term** computational form of PGF in the σ-coordinate (Li et al., 2012)

  - Using the covariant scalar equations of the σ-coordinate

\[
\frac{\partial v_1}{\partial t} + v^1 \frac{\partial v_1}{\partial x} + v^2 \frac{\partial v_1}{\partial y} + v^3 \frac{\partial v_1}{\partial \sigma} - \frac{1}{2} v^m v^n \frac{\partial g_{nm}}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - 2\sqrt{g} \left( \Omega^2 v^3 - \Omega^3 v^2 \right) + g_1
\]

All based on the **non-orthogonal** sigma coordinate
1. Motivation – the Existing Methods to Overcome These Errors

✈ Via decreasing the slope of sigma levels to handle the “advection errors” (Arakawa and Lamb, 1977; Simmons and Burridge, 1981; Simmons and Strüfing, 1983; Schär et al., 2003)

– Hybrid coordinate, the new smooth level vertical coordinate, ...

These two significant errors are solved separately.
Wouldn't it be great if BOTH can be killed by ONE stone?

Momentum equation in the $z$-coordinate

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - (2\Omega \cos \phi w - 2\Omega \sin \phi v)$$

- One-term computational form of PGF
- None curvilinear terms

✓ Design an orthogonal terrain-following coordinate!
Outline

1. Motivation

2. An Orthogonal Sigma Coordinate

3. The Idealised Experiments

4. Conclusion
The orthogonal sigma coordinate (OS-coordinate)

**Boundary condition**
(to turn terrain into a sigma level)

- **Definition of coordinate**
  - Coordinate transformation

- **Basis vectors**
  - Terrain-following
  - orthogonal
2. An Orthogonal Sigma Coordinate – Basis Vectors

◆ 3D-rotation of the basis vectors of z-coordinate
  - Twice of the 2D-rotation
  - To make the “vertical direction of z-coordinate” in line with the “normal vector of the terrain”

Coordinate after the first rotation $[o; i_1, j_1, k_1]$
Coordinate after the second rotation $[o; i_2, j_2, k_2]$

dark red arrow: the normal vector of the terrain; black arrows: the basis vectors of z-coordinate
2. An Orthogonal Sigma Coordinate – Basis Vectors

◆ The basis vectors after 3D-rotation

\[ i_o = i \cos \lambda - j \sin \theta \sin \lambda - k \cos \theta \sin \lambda \]
\[ j_o = j \cos \theta - k \sin \theta \]
\[ k_o = i \sin \lambda + j \sin \theta \cos \lambda + k \cos \theta \cos \lambda \]

◆ To satisfy the vertical distribution of basis vectors of classic sigma coordinate, we add a rotation parameter \( b \), so we set

- \( b = 1 \) on the surface of terrain; \( b = 0 \) on the top of a model

Basis vectors
Of the OS-coordinate

\[ i_o = i \cos(b \cdot \lambda) - j \sin(b \cdot \theta) \sin(b \cdot \lambda) - k \cos(b \cdot \theta) \sin(b \cdot \lambda) \]
\[ j_o = j \cos(b \cdot \theta) - k \sin(b \cdot \theta) \]
\[ k_o = i \sin(b \cdot \lambda) + j \sin(b \cdot \theta) \cos(b \cdot \lambda) + k \cos(b \cdot \theta) \cos(b \cdot \lambda) \]

\( \lambda \) and \( \theta \) are angle of each rotation,

\[ \cos \theta = \frac{1}{\sqrt{\left(\frac{\partial h}{\partial y}\right)^2 + 1}}, \quad \sin \theta = \frac{-\frac{\partial h}{\partial y}}{\sqrt{\left(\frac{\partial h}{\partial y}\right)^2 + 1}}, \quad \cos \lambda = \frac{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1}}{\left(\frac{\partial h}{\partial y}\right)^2 + 1}, \quad \sin \lambda = \frac{-\frac{\partial h}{\partial x}}{\left(\frac{\partial h}{\partial y}\right)^2 + 1} \]
Basis vectors of the orthogonal sigma coordinate

- Near the terrain: terrain-following and orthogonal
- Away from the terrain: horizontal and vertical

color arrows: basis vectors of the OS-coordinate; black line: terrain
Three kinds of the rotation parameter $b$

- Obtaining the smoother sigma levels above steep terrain

**Linear b**
(OsBr1)

$$b = \frac{H_t - z}{H_t - h}$$

**Square b**
(OsBr2)

$$b = \left[ \frac{H_t - z}{H_t - h} \right]^2$$

**Exponent b**
( OsBr3)

$$b = \frac{e^{2(H_t-h)}}{e^{2(H_t-h)} - 1} \cdot e^{-(z-h)} + \frac{-1}{e^{2(H_t-h)} - 1} \cdot e^{(z-h)}$$

black lines: the sigma levels of CS-coordinate; color lines: the sigma levels of the OS-coordinate
2. An Orthogonal Sigma Coordinate – the Equations

- Using the basis vectors of OS-coordinate to expand the vector equation of the atmosphere
  - Computational form of PGF are one-term
  - None curvilinear terms

\[
\frac{\partial u_o}{\partial t} + u_o \frac{\partial u_o}{\partial x_o} + v_o \frac{\partial u_o}{\partial y_o} + w_o \frac{\partial u_o}{\partial \sigma} = -\frac{1}{\rho} \frac{\partial p}{\partial x_o} - 2(\Omega_{o2} w_o - \Omega_{o3} v_o) + g_{o1}
\]

\[
\frac{\partial v_o}{\partial t} + u_o \frac{\partial v_o}{\partial x_o} + v_o \frac{\partial v_o}{\partial y_o} + w_o \frac{\partial v_o}{\partial \sigma} = -\frac{1}{\rho} \frac{\partial p}{\partial y_o} + 2(\Omega_{o1} w_o - \Omega_{o3} u_o) + g_{o2}
\]

The Orthogonal sigma coordinate

The Classic sigma coordinate
The unified framework of z-coordinate, the classic and orthogonal sigma coordinate

- Difference in form ← setting corresponding parameters
- Difference in value ← defining the value of scalars

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial q^1} + v \frac{\partial u}{\partial q^2} + w \frac{\partial u}{\partial q^3} + \lambda_1 = -\frac{1}{\rho} \delta_{1i} \frac{\partial p}{\partial q^i} - 2\lambda_2 \left( \Omega_2 w - \Omega_3 v \right) + g_1
\]

<table>
<thead>
<tr>
<th></th>
<th>z-coordinate</th>
<th>The orthogonal sigma coordinate</th>
<th>The classic sigma coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_{11})</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\delta_{1i})</td>
<td>(1, 0, 0)</td>
<td>(1, 0, 0)</td>
<td>(1, 0, (\frac{H - z}{H - h} \cdot \frac{\partial h}{\partial x}))</td>
</tr>
<tr>
<td>(\lambda_2)</td>
<td>1</td>
<td>1</td>
<td>(H-h)/H</td>
</tr>
</tbody>
</table>

\(\alpha, \beta, \gamma = (a \ b \ c) \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}^{-1}\)

\(a, b, \text{ and } c\) represent the scalars in the z-coordinate

H is the top of a model, and h is the terrain height.
We use a reversed way of the classic sigma coordinate to design an orthogonal sigma coordinate.

The basis vectors of the orthogonal sigma coordinate are terrain-following and orthogonal.

There are 3 advantages of the OS-coordinate:
- Only one term in the computational form of PGF
- Smoother sigma levels can be obtained through choosing an appropriate rotation parameter $b$ above steep terrain
- The scalar equations are as simple as those in the $z$-coordinate.
Outline

1. Motivation
2. An Orthogonal Sigma Coordinate
3. The Idealised Experiments
4. Conclusion
3. The Idealised Experiments – the Design

◆ “PGF errors”
  – Similar as the results in the experiments implemented in Li et al. (2012)
    http://www.newton.ac.uk/programmes/AMM/seminars/082411001.html

◆ “Advection errors”
  – Idealised advection experiments
  – Using a unified framework to implement the classic and orthogonal sigma coordinate
    • 4 groups in two coordinates
      – OS-coordinate with linear, square, and exponent rotation parameter b
      – CS-coordinate
    • 5 experiments with increasing horizontal resolution
      – dx = 0.5, 1.0, 2.0, 4.0, 8.0 m, ds = 0.5 m
3. The Idealised Experiments – the Advection Experiment

**Basic parameters** (based on the experiments designed by Schär et al., 2003)

- Two-dimension linear advection
- Bell-shaped mountain
- Leapfrog scheme in the horizontal discretization and forward scheme in the vertical
- Period boundary in horizontal and rigid-lid boundary in vertical

\[
\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + w \frac{\partial q}{\partial z} = 0
\]

Horizontal wind \( u \)

The analytical solution

Color contours: \( q \in [0, 1] \), intervals of the contour are 0.1, warm colors represent the larger value; black line: terrain
3. The Idealised Experiments – Comparison of the Pattern

◆ Results of the advection over the top of terrain and in the end
  – Deformation near the steep terrain are alleviated in OS-coordinate
  – RMSE are reduced in OS-coordinate, especially in OsBr3

\[\begin{array}{c|c|c}
\text{Cs} & \text{OsBr1} & \text{OsBr2} & \text{OsBr3} \\
\hline
\text{rmse:} & 0.0712 & 0.0693 & 0.0701 & 0.0680 \\
\text{rmse:} & 0.0584 & 0.0626 & 0.0374 & 0.0521 \\
\end{array}\]

\(dx = 8.0 \text{ m}\)
\(ds = 0.5 \text{ m}\)

color contours: \(q \in [0, 1]\), intervals of the contour are 0.1
3. The Idealised Experiments – Comparison of the Errors

◆ Variation of errors in both coordinates
  - Absolute errors in the OS-coordinate are always less than those in the CS-coordinate
  - Absolute errors are two orders of magnitude less than those in the CS-coordinate in OsBr3

- OsBr2
- OsBr3

Max = 0.904
Max = 0.687
Max = 0.007

dx = 1.0 m, ds = 0.5 m

shaded colors: the absolute errors
3. The Idealised Experiments – Spurious vertical velocity

◆ The spurious vertical velocity in both coordinates
  – Smaller in range and value of the OS-coordinate compared with those in CS-coordinate
  – Value in the OS-coordinate of OsBr3 is one order of magnitude less than those in the CS-coordinate

Truth: \( w = 0.0 \)

<table>
<thead>
<tr>
<th></th>
<th>Cs</th>
<th>OsBr1</th>
<th>OsBr2</th>
<th>OsBr3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max</td>
<td>0.186</td>
<td>0.163</td>
<td>0.144</td>
<td>0.016</td>
</tr>
</tbody>
</table>

\( dx = 2.0 \text{ m} \)
\( ds = 0.5 \text{ m} \)

shaded colors: the vertical velocity in both coordinates
RMSE in the experiments of increasing $dx$

- The RMSE in the OS-coordinate are always less than those in the CS-coordinate
- In high horizontal resolution, the RMSE in OsBr3 is one order of magnitude less than those in the CS-coordinate
Outline

1. Motivation
2. An Orthogonal Sigma Coordinate
3. The Idealised Experiments
4. Conclusion
4. Conclusion

◆ Designing an orthogonal sigma coordinate to overcome two computational errors of the classic sigma coordinate
  – To handle the terrain
  – To bypass the PGF errors
  – To alleviate the advection errors
The idealised advection experiments:

- The advection errors in the orthogonal sigma coordinate have been **reduced consistently by at least one order of magnitude** than those in the classic sigma coordinate

- The advection errors in the orthogonal sigma coordinate **decrease more significantly** in high horizontal resolution

- The advection errors can be **reduced much more via choosing the rotation parameter b**

Thank you!
Mountains