Evaluating numerical methods by using asymptotic limit solutions

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This presentation covers the following areas

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Method

Governing equations

On all relevant scales, the atmosphere is governed by the compressible Navier-Stokes equations, the laws of thermodynamics, phase changes and source terms

The solutions of these equations are very complicated, reflecting the complex nature of observed flows

The accurate solution of these equations would require computers 10^{30} times faster than now available

Numerical models

Since solution cannot be accurate, how do we validate practical numerical models?

The flows that can be resolved, and are relevant to weather and climate prediction, may be accurately described by asymptotic limit equations:

e.g. rotation dominated flows are accurately described by the semi-geostrophic equations.

gravity waves on sharp interfaces can be described by shallow water equations with reduced gravity.

Method

Exploit mathematical results that the solution of the full equations converges to that of suitable limit equations at a rate $O(\varepsilon^k)$, where ε is a small parameter.

Test proposed numerical method for this rate of convergence to the limit solution, either by solving the limit equations directly or by calculating appropriate diagnostics.

Note that limit equations may be much harder to solve because of implicit coupling between variables and very high condition numbers.

Importance

Such tests are important given that existing numerical methods are being challenged on scalability grounds.

In particular, are semi-implicit methods better at converging to limit solutions; and are semi-Lagrangian methods better at approximating limit solutions governed by Lagrangian dynamics?

Shallow water example

Shallow water test

If the depth of the water is h, with mean value h_0, the linear gravity wave speed is $\sqrt{gh_0}$.

Define ε as the Froude number $U/\sqrt{gh_0}$.

Then the solution of the shallow water equations converges to that of the 2d incompressible Euler equation at a rate $O(\varepsilon^2)$.

Implementation

Use shallow water 'New Dynamics' code, semi-implicit, semi-Lagrangian, C grid. Prognostic variables (h,u,v).

Enforce incompressibility by altering Helmholz equation.

Choose initial data so that the discrete vorticity is the same for both shallow water and 2d incompressible, and the divergence is zero for both.

Calculate rms errors in h and vector velocity after a fixed time for different choices of h_0.

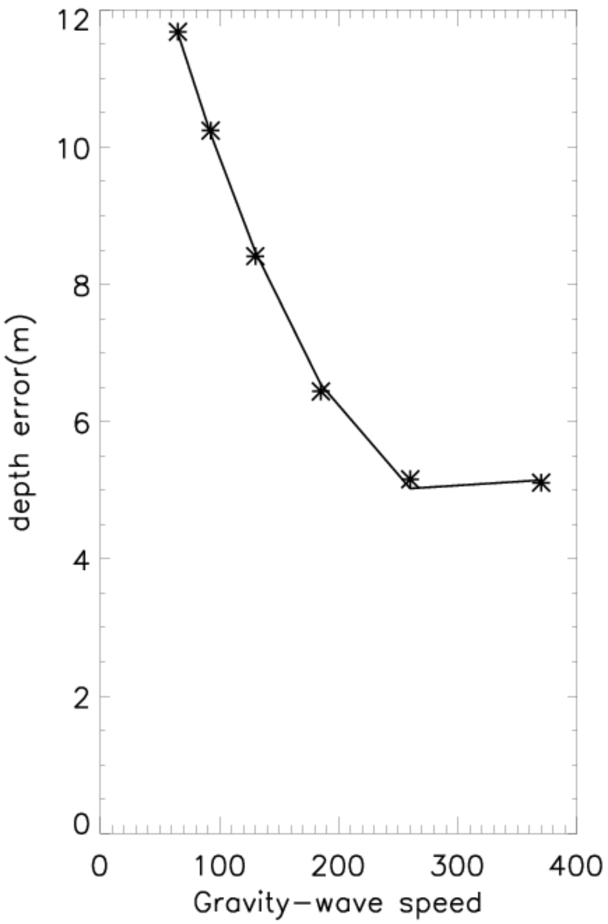
Demonstration

Data with typical max wind speed 15ms^{-1}.

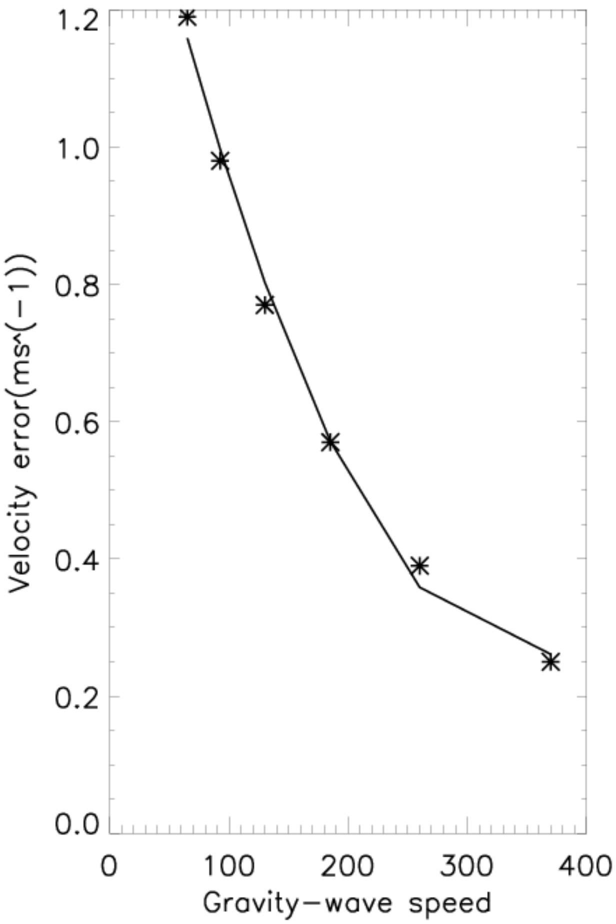
Mean depth chosen to give gravity wave speed 65ms^{-1} to 360ms^{-1}.

Gives Fr~0.05-0.3.

Rms errors



Depth



Vector velocity

Comments

Convergence in winds at expected rate, but to non-zero residual.

Appears to reflect poor discretisation of implied vorticity equation-not surprising since scheme works on velocities.

Poor convergence in *h*. Depth is only a diagnostic quantity in the incompressible equations, so large-scale errors not well controlled.

Eady wave example

Eady problem

Consider incompressible Boussinesq equations in a vertical cross-section (x,z) with periodic boundary conditions in x and rigid upper and lower boundaries.

Solutions of these equations are a special case of solutions of 3d equations.

Incompressible equations

$$\frac{Du}{Dt} + \frac{\partial p}{\partial x} - fv = 0$$

$$\frac{Dv}{Dt} + fu = C\left(z - H/2\right)$$

$$\frac{Dw}{Dt} + \frac{\partial p}{\partial x} - \frac{g\theta}{\theta_0} = 0$$

$$\frac{D\theta}{Dt} = Cv$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

C represents a basic state potential temperature gradient in *y*.

Limit equations

The small parameter ε is the Rossby number U/fL.

The SG limit is obtained by neglecting $(Du/Dt, Dw/Dt)$.

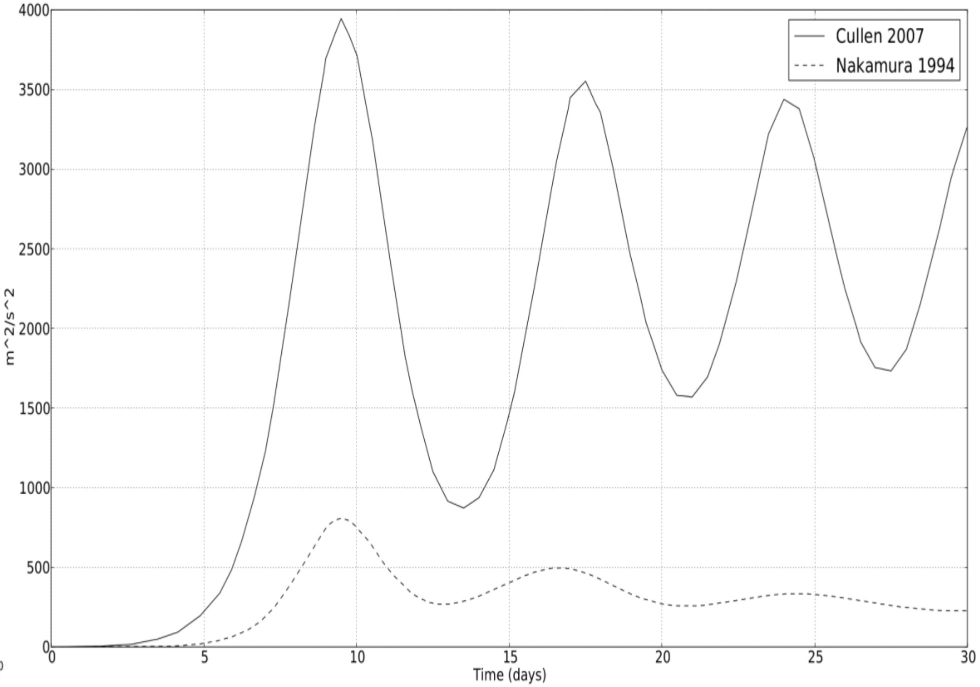
Existence of a solution to the SG equations in this case has been proved.

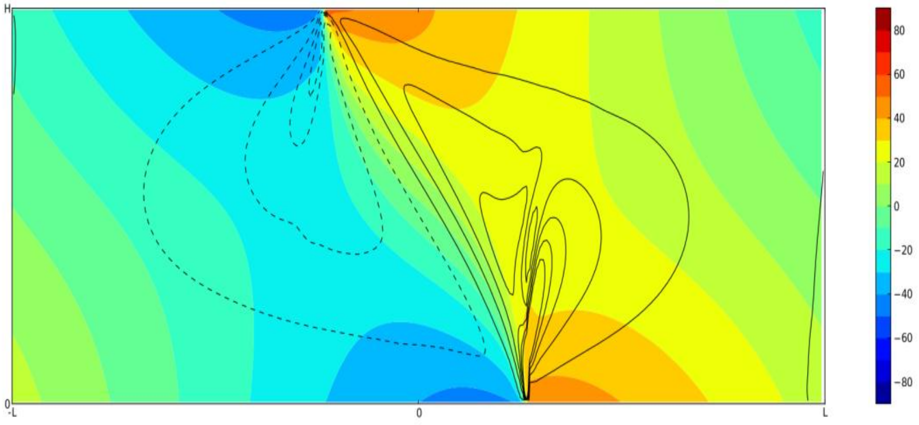
Scale analysis shows that SG is accurate to $O(\varepsilon^2)$ in this geometry.

The SG solution becomes discontinuous in finite time. The solution of the incompressible equations may not.

Eady model of fronts

- Lifecycles in SG model suggest predictability that is not captured in full equations
- Slice showing pertinent features of Eady model – sharp gradient in velocity, large scale balance and localised unbalanced motion of gravity waves



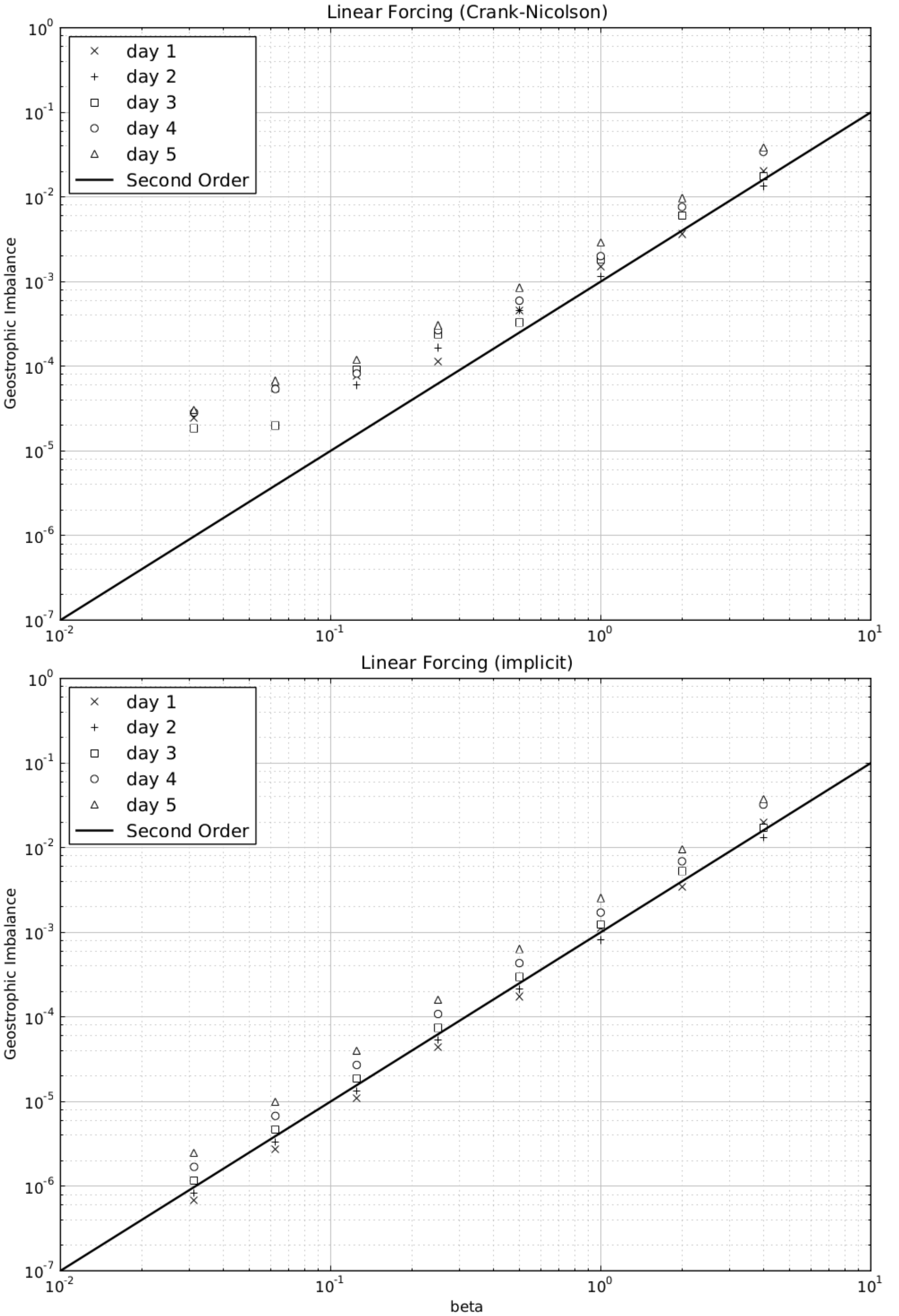


Linear computations

Initially demonstrate correct limit using linearised equations-so discontinuities do not occur.

Convergence of Linear Problem

- Non-linearity in advection isolated
- Convergence correctly at second order
- Correctly second order with full decentering ($\alpha=1$), only correct for $\beta>0.25$ for $\alpha=0.5$



Nonlinear problem

The theory shows that weak solutions of the Lagrangian SG equations exist.

Discontinuities can form. In general weak solutions of the Eulerian SG equations do not exist.

Suggests that numerical methods respecting this limit have to discretise the Lagrangian form of the equations.

Possible advantage for (conservative) semi-Lagrangian methods.

Nonlinear compressible test

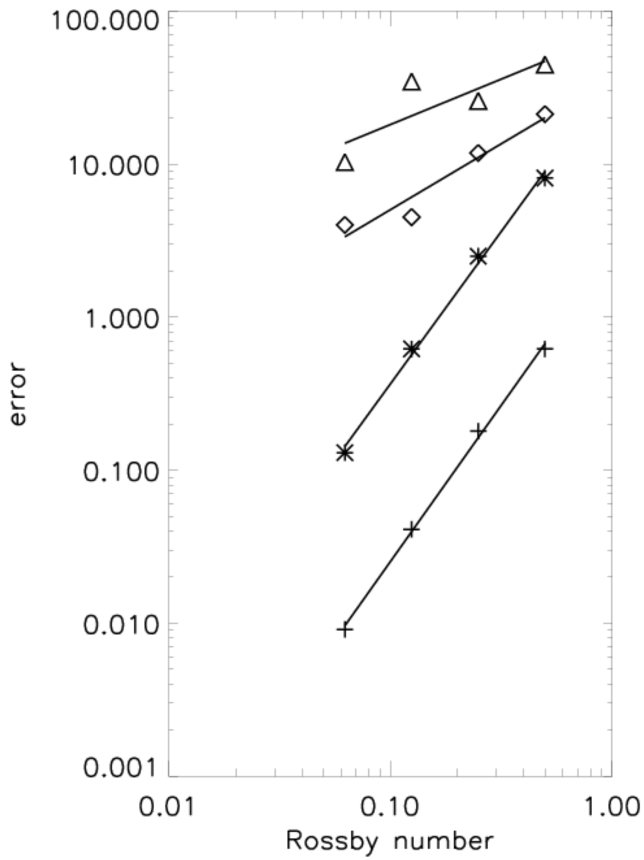
Use compressible equations; 2d solution is OK for up to 10 days but then not valid solution of 3d problem.

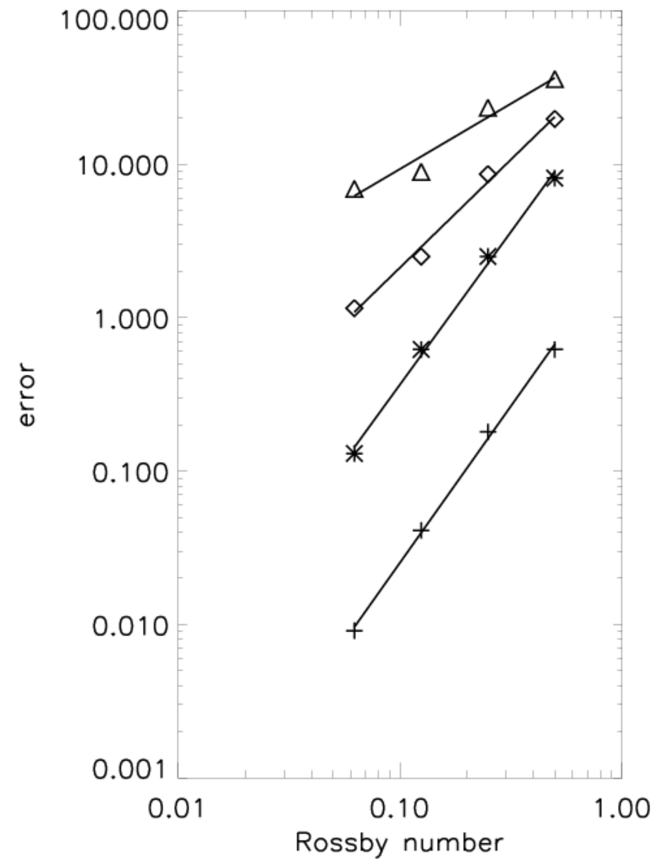
Use New Dynamics slice model with semi-Lagrangian advection-appropriate for discontinuous Lagrangian solutions.

Check limiting behaviour as $\varepsilon \to 0$ by using UM with $\varepsilon = 0.031$ as reference.

Check ratio of unbalanced along-front wind to total wind.

Convergence of wind along front

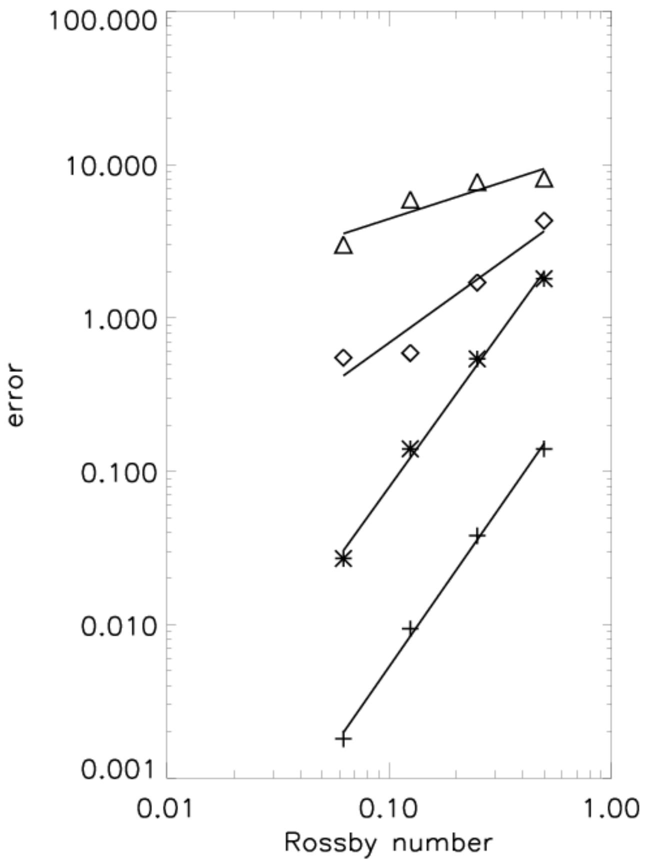




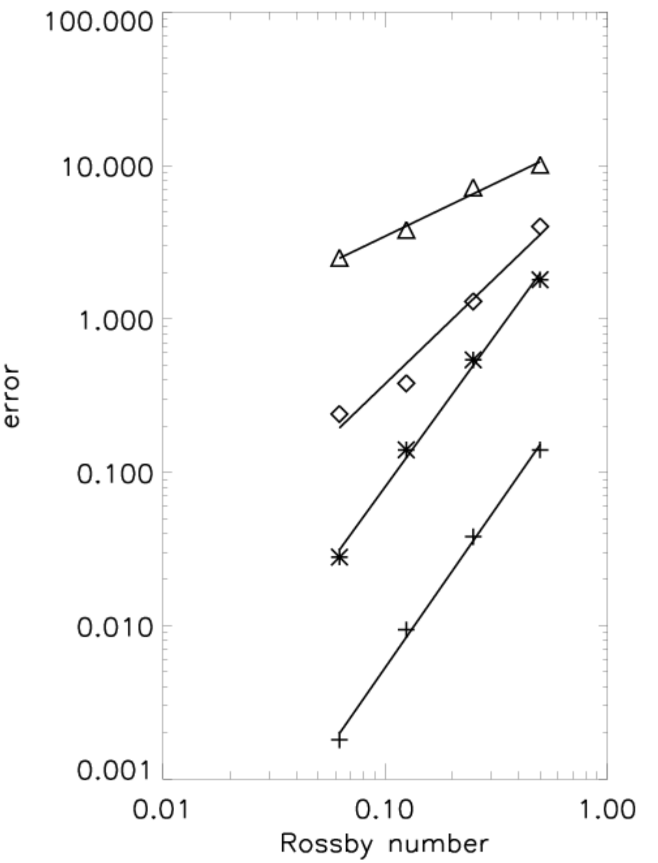
Standard days 3,6,7,8

Monotone days 3,6,7,9

Convergence test-potential temperature

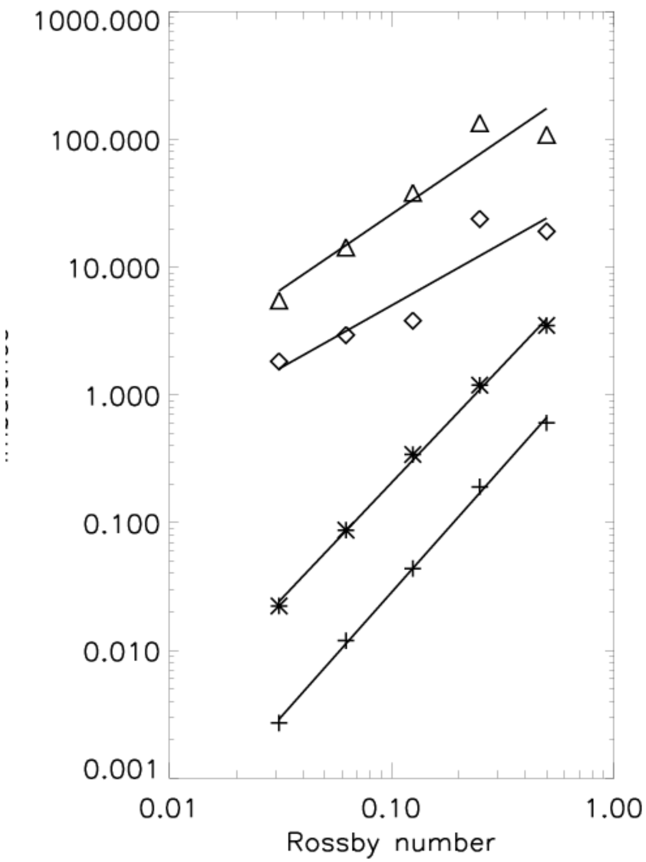


Standard days 3,6,7,8

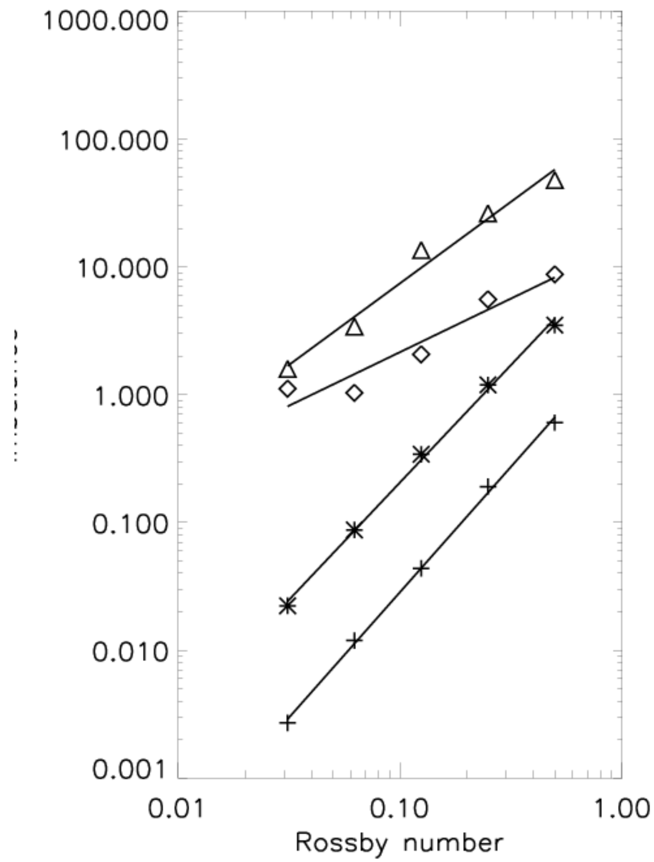


Monotone days 3,6,7,9

Convergence test-imbalance



Standard days 3,6,7,8



Monotone days 3,6,7,9

Comments

Shows second order convergence while SG limit smooth, then first order.

Better convergence if quasi-monotone advection used-but need to quantify whether too close to limit.

Boundary layer example

Boundary layer test

Results shown are from a 1d shallow water model with no rotation and friction.

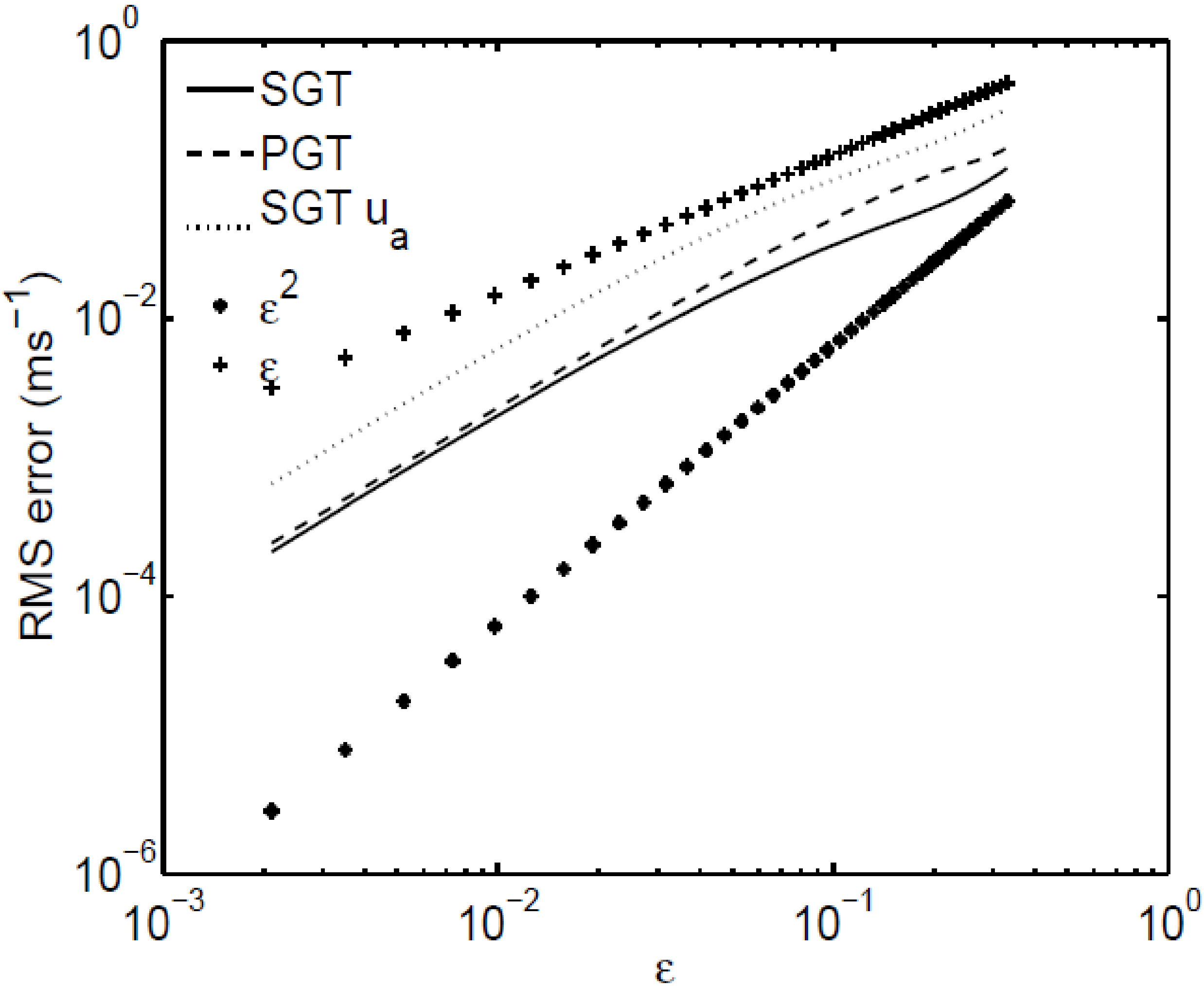
The small parameter ε is $U\tau/L$ where τ is the frictional timescale, U a velocity scale and L a length scale.

Simple Rayleigh friction used.

Limit equations replace the momentum by u_e, where the friction term u_e/τ is chosen to balance the pressure gradient.

Stable limit equations accurate to O(ε)

RMS error with respect to the SWE model with small parameter



$$\epsilon = \frac{\tau}{T} = \frac{U\tau}{L}$$

Comments

Results show better than expected convergence.

Numerics rather trivial in this case, so more a confirmation of the analytical theory.

Similar theory applies to 2d vertical slice model with a boundary layer.

This can be used to test convergence with a full boundary layer scheme and thus validate boundary layer coupling algorithms.

Questions and answers