A multi-moment constrained finite volume model for non-hydrostatic atmospheric dynamics

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Local high-order spectral converging methods

Accurate(high-order), Conservative, Highly scalable on massively parallel system (peta, exa, ...)

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   - Nair et al., 2007, 2009
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2. Spectral element method
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Multi-moment constrained finite volume method (Li and Xiao, JCP, 2009)

For conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$  \hspace{1cm} (1)

Multi-moment: the line-integrated average value (LIA moment), the point value (PV moment) and the derivative value (DV moment):

$$\bar{q}^{(x)}(t) \equiv \frac{1}{\Delta x_i} \int_{\delta x} q(x, t) dx,$$  \hspace{1cm} (2)

$$q_{cp}(t) \equiv q(x_{cp}, t),$$  \hspace{1cm} (3)

$$\partial_x^k q_{cp}(t) \equiv \frac{\partial^k}{\partial x^k} q(x_{cp}, t); \text{ with } k = 1, 2, \cdots$$  \hspace{1cm} (4)

where $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ and $x_{cp}$ represents a constraint point within or at the two ends of line segment $x_1 x_L$ where constraints in terms of multi-moments are imposed. The constraint points can but don’t have to coincide with the solution points. The solution points (the unknowns, or the degrees of freedom (DOFs)) can be more flexibly chosen in an MCV scheme, not limited to the Gauss points or Gauss-Lobatto points.
Multi-moment constrained finite volume method

A Lagrange interpolation polynomial of degree \((L - 1)\) is constructed as

\[
\Psi (\phi : x) = \sum_{l=1}^{L} (B_l \phi_l), \tag{5}
\]

where the Lagrange basis function is

\[
B_l = \prod_{p=1, p \neq l}^{L} \frac{(x - x_p)}{(x_l - x_p)}. \tag{6}
\]

\(\phi\) represents the either the conservative state variables \(q\) or the flux component of \(f\) and \(g\) (shown later).
Splitting of reference state

\[
\rho(x, t) = \bar{\rho}(z) + \rho'(x, t) \tag{8}
\]
\[
p(x, t) = \bar{p}(z) + p'(x, t) \tag{9}
\]
\[
(\rho \theta)(x, t) = (\bar{\rho} \theta)(z) + (\rho \theta)'(x, t) \tag{10}
\]

where the reference pressure \(\bar{p}(z)\) and density \(\bar{\rho}(z)\) are in local hydrostatic balance.

The governing equations with the effects of topography in the Cartesian curvilinear coordinate via \(\zeta = \zeta(x, z)\).

\[
\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial \zeta} = \mathbf{s}(\mathbf{q}) \tag{11}
\]

where \(\mathbf{q} = (\sqrt{G} \rho', \sqrt{G} \rho u, \sqrt{G} \rho w, \sqrt{G}(\rho \theta)')^T\) are the conservative state variables. \(\mathbf{f} = (\sqrt{G} \rho u, \sqrt{G} \rho u^2 + \sqrt{G} p', \sqrt{G} \rho w u, \sqrt{G} \rho u \tilde{w}, \sqrt{G} \rho w \tilde{w} + p', \sqrt{G} \rho \theta \tilde{w})^T\) and \(\mathbf{g} = (\sqrt{G} \rho \tilde{w}, \sqrt{G} \rho u \tilde{w} + \sqrt{G} G^{13} p', \sqrt{G} \rho w \tilde{w} + p', \sqrt{G} \rho \theta \tilde{w})^T\) denote the vectors of the flux functions in \(x\) and \(\zeta\) directions, \(\mathbf{s}(\mathbf{q}) = (0, 0, -\sqrt{G} \rho' g, 0)^T\) are the source terms, \(\sqrt{G} = \frac{\partial \zeta}{\partial \zeta}\) is the Jacobian of the transformation and \(G^{13} = \frac{\partial \zeta}{\partial x}\) are the contravariant metric, and \(\tilde{w} = \frac{d\zeta}{dt}\) is the velocity component in the transformed coordinate.

The height-based terrain-following hybrid coordinate is used when including the effects of topography (Schär et al., 2002)

\[
z(\zeta) = \zeta + z_s(x) \frac{\sinh[(z_T - \zeta)/s]}{\sinh(z_T/s)} \tag{12}
\]
Constraint conditions

- The LIA moment over line segment $\overline{x_1x_L} \times \zeta_m$, which is cast in a finite volume formulation,
  \[
  \frac{d}{dt} \overline{q_m(x)}(t) = -\frac{1}{\Delta x_i} \left( \hat{f}_{Lm} - \hat{f}_{1m} \right)
  \]

- The PV-moment values at the two ends of segment $\overline{x_1x_L} \times \zeta_m$ are predicted by a collocation formulation of,
  \[
  \frac{d}{dt} q_{1m}(t) = -\partial_x \hat{f}_{1m} \quad \text{and} \quad \frac{d}{dt} q_{Lm}(t) = -\partial_x \hat{f}_{Lm},
  \]

\[
\begin{align*}
x_1 &= x_{i-\frac12} \\
x_2 &= \frac{x_{i-\frac12} + x_{i+\frac12}}{2} \\
x_3 &= x_{i+\frac12}
\end{align*}
\]

The 3 equi-distanced-point configuration along the line segment.

$\xi_{j+\frac12}$ $\xi_{j-\frac12}$

Constraint point

Solution point

$x_{i-\frac12}$ $x_{i+\frac12}$

$C_y$

$l$

$m$

$x_{i-\frac12}$ $x_{i+\frac12}$

$x_{i-\frac12}$ $x_{i+\frac12}$
The LIA moment over line segment \( \overline{x_1 x_L} \times \zeta_m \), which is cast in a finite volume formulation,

\[
\frac{d}{dt} [\bar{q}_m^{(x)}(t)] = -\frac{1}{\Delta x_i} \left( \hat{f}_{Lm} - \hat{f}_{1m} \right)
\]

The PV-moment values at the two ends of segment \( \overline{x_1 x_L} \times \zeta_m \) are predicted by a collocation formulation of,

\[
\frac{d}{dt} [q_{1m}(t)] = -\partial_x \hat{f}_{1m} \quad \text{and} \quad \frac{d}{dt} [q_{Lm}(t)] = -\partial_x \hat{f}_{Lm},
\]
3rd order MCV scheme

Constraint conditions

- The LIA moment over line segment $x_1x_L \times \zeta_m$, which is cast in a finite volume formulation, is given by:
  \[
  \frac{d}{dt} [\bar{\alpha}_m^{(x)}(t)] = -\frac{1}{\Delta x_i} \left( \hat{f}_{Lm} - \hat{f}_{1m} \right)
  \]

- The PV-moment values at the two ends of segment $x_1x_L \times \zeta_m$ are predicted by a collocation formulation of:
  \[
  \frac{d}{dt} [q_{1m}(t)] = -\partial_x \hat{f}_{1m} \quad \text{and} \quad \frac{d}{dt} [q_{Lm}(t)] = -\partial_x \hat{f}_{Lm}.
  \]

The 3 equi-distanced-point configuration along the line segment.

\[
\begin{align*}
  x_1 & = x_{i-1/2} & x_2 & = (x_{i-1/2} + x_{i+1/2})/2 & x_3 & = x_{i+1/2} \\
\end{align*}
\]
From the above constraints and the Lagrange interpolation (5),

\[
\begin{bmatrix}
\frac{d}{dt}(q_{1m}) \\
\frac{d}{dt}(q_{2m}) \\
\frac{d}{dt}(q_{3m})
\end{bmatrix} = \begin{bmatrix}
0 & 0 & -1 & 0 \\
\frac{3}{2\Delta x_i} & \frac{3}{2\Delta x_i} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
\hat{f}_{1m} \\
\hat{f}_{3m} \\
(\partial_x \hat{f})_{1m} \\
(\partial_x \hat{f})_{3m}
\end{bmatrix}.
\]  

(13)

Making use of the following notations,

\[
M^{(x)}_3 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
\frac{3}{2\Delta x_i} & \frac{3}{2\Delta x_i} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & -1
\end{bmatrix} \quad \text{and} \quad F_3 = \begin{bmatrix}
\hat{f}_{1m} \\
\hat{f}_{3m} \\
(\partial_x \hat{f})_{1m} \\
(\partial_x \hat{f})_{3m}
\end{bmatrix}
\]

(14)

and denoting the entries of matrix $M^{(x)}_3$ by $M^{(x)}_{3\alpha\beta}$ and components of $F_3$ by $F_{3\beta m}$, we re-write (13) into a component form,

\[
\frac{d}{dt}(q_{lm}) = \sum_{\beta=1}^{4} M^{(x)}_{3l\beta} F_{3\beta m}, \quad \text{for} \ l = 1, 2, 3.
\]

(15)
The semi discrete formulation for updating the conservative variables in the 2D system (11) is then obtained as

$$\frac{d}{dt}(q_{lm}) = \sum_{\beta=1}^{4} M^{(x)}_{3l\beta} F_{3\beta m} + \sum_{\beta=1}^{4} M^{(\zeta)}_{3m\beta} G_{3l\beta} + S(q_{lm}), \text{ for } l, m = 1, 2, 3. \quad (16)$$

where $M^{(\zeta)}_{3m\beta}$ and $G_{3l\beta}$ are the entries of the following matrix and vector generated from the spatial discretization in $\zeta$ direction,

$$M^{(\zeta)}_{3} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{3}{2\Delta \zeta_j} & \frac{3}{2\Delta \zeta_j} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad G_{3} = \begin{bmatrix} \hat{g}_{l1} \\ \hat{g}_{l3} \\ (\partial_\zeta \hat{g})_{l1} \\ (\partial_\zeta \hat{g})_{l3} \end{bmatrix}, \quad (17)$$

and $S(q_{lm})$ represents the point-wisely evaluated source term.
The LIA moment over line segment $\overline{x_1x_L} \times \zeta_m$, which is cast in a finite volume formulation, 
$$\frac{d}{dt} \left[ \overline{q_m}(x) \right](t) = -\frac{1}{\Delta x_i} \left( \hat{f}_{Lm} - \hat{f}_{1m} \right)$$

The PV-moment values at the two ends of segment $\overline{x_1x_L} \times \zeta_m$ are predicted by a collocation formulation of,
$$\frac{d}{dt}[q_{1m}(t)] = -\partial_x \hat{f}_{1m} \quad \text{and} \quad \frac{d}{dt}[q_{Lm}(t)] = -\partial_x \hat{f}_{Lm},$$

An extra constraint condition is imposed on the first-order DV moment at the segment center, 
$$\frac{d}{dt}[(q_x)_{cm}(t)] = -\partial_x^2 \hat{f}_{cm}.$$
Constraint conditions

- The LIA moment over line segment $\bar{x}_1 \bar{x}_L \times \zeta_m$, which is cast in a finite volume formulation, $\frac{d}{dt} \left[ \tilde{q}_{m}^{(x)} (t) \right] = - \frac{1}{\Delta x_i} \left( \hat{f}_{Lm} - \hat{f}_{1m} \right)$

- The PV-moment values at the two ends of segment $\bar{x}_1 \bar{x}_L \times \zeta_m$ are predicted by a collocation formulation of, $\frac{d}{dt} [q_{1m} (t)] = - \partial_x \hat{f}_{1m}$ and $\frac{d}{dt} [q_{Lm} (t)] = - \partial_x \hat{f}_{Lm}$.

- An extra constraint condition is imposed on the first-order DV moment at the segment center, $\frac{d}{dt} [(q_x)_{cm} (t)] = - \partial_x^2 \hat{f}_{cm}$.
where the second order derivative value of flux function $\partial_{x}^{2}f_{cm}$ at the segment center is computed from the 4th-order approximation,

$$
\left( \partial_{x}^{2} \hat{f} \right)_{cm} = \frac{-16f_{cm} + 8\hat{f}_{1m} + 8\hat{f}_{4m}}{\Delta x^2} + \frac{(\partial_{x}\hat{f})_{4m} - (\partial_{x}\hat{f})_{1m}}{\Delta x},
$$

(18)

and the line segment center value $f_{cm}$ is obtained from the Lagrange interpolation (5).
The 4th-order semi-discrete formulation

The 4th-order semi-discrete formulation for the 2D system (11) is obtained as

\[
\frac{d}{dt}(q_{lm}) = \sum_{\beta=1}^{5} M_{4l\beta}^{(x)} F_{4\beta m} + \sum_{\beta=1}^{5} M_{4m\beta}^{(\zeta)} G_{4l\beta} + S(q_{lm}), \quad \text{for } l, m = 1, 2, 3, 4. \tag{19}
\]

where

\[
M_{4}^{(x)} = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
\frac{4}{3\Delta x_i} & -\frac{4}{3\Delta x_i} & \frac{4}{27} & \frac{5}{27} & -\frac{4\Delta x_i}{27} \\
\frac{4}{3\Delta x_i} & -\frac{4}{3\Delta x_i} & \frac{4}{27} & \frac{5}{27} & -\frac{4\Delta x_i}{27} \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and

\[
F_{4}^{(x)} = \begin{bmatrix}
\hat{f}_{1m} \\
\hat{f}_{4m} \\
\left(\frac{\partial \hat{f}}{\partial \zeta}\right)_{1m} \\
\left(\frac{\partial \hat{f}}{\partial \zeta}\right)_{4m} \\
\left(\frac{\partial^{2} \hat{f}}{\partial \zeta^{2}}\right)_{cm} \\
\end{bmatrix}, \tag{20}
\]

\[
M_{4}^{(\zeta)} = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
\frac{4}{3\Delta \zeta_j} & -\frac{4}{3\Delta \zeta_j} & \frac{4}{27} & \frac{5}{27} & -\frac{4\Delta \zeta_j}{27} \\
\frac{4}{3\Delta \zeta_j} & -\frac{4}{3\Delta \zeta_j} & \frac{4}{27} & \frac{5}{27} & -\frac{4\Delta \zeta_j}{27} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and

\[
G_{4}^{(\zeta)} = \begin{bmatrix}
\hat{g}_{l1} \\
\hat{g}_{l4} \\
\left(\frac{\partial \hat{g}}{\partial \zeta}\right)_{l1} \\
\left(\frac{\partial \hat{g}}{\partial \zeta}\right)_{l4} \\
\left(\frac{\partial^{2} \hat{g}}{\partial \zeta^{2}}\right)_{cm} \\
\end{bmatrix}. \tag{21}
\]
Approximate Riemann solver

In the above discussions we are left with the need to find the numerical flux and its derivative at the cell boundaries.

**The numerical flux computation at the cell boundaries**

The numerical fluxes $\hat{f}_m$ and $\hat{f}_L$, as well as $\hat{g}_{IL}$ and $\hat{g}_{IL}$, are continuous at cell boundaries, and can be immediately calculated from the values of the state variables at the boundary solution points.

**The derivative Riemann problems** *(Toro et al. (2001) and Titarev and Toro (2002))*

For computational efficiency, a local Lax-Friedrich (LLF) approximate Riemann solver (Shu and Osher 1989) is adopted, which reads

$$
\partial_x \hat{f}_{bpm} = \frac{1}{2} \left( \partial_x f_{bpm}^- + \partial_x f_{bpm}^+ - |\lambda_{\text{max}}^{(x)}| \left( \partial_x q_{bpm}^+ - \partial_x q_{bpm}^- \right) \right)
$$

(22)

where $|\lambda_{\text{max}}^{(x)}|$ is the local maximum of the eigen values of the Jacobian matrix $A = \frac{\partial f}{\partial q}$, which is directly evaluated from the point values at $x_{bpm}$.
We give the formula for $(\partial_x^2 \phi)_{lm}$ in $x$ direction, for example,

$$(\partial_x^2 \phi)_{lm} = \frac{4\phi_{1m} - 8\phi_{2m} + 4\phi_{3m}}{\Delta x^2}, \quad l = 1, 2, 3,$$

(23)

for MCV3 and

$$(\partial_x^2 \phi)_{1m} = \frac{18\phi_{1m} - 45\phi_{2m} + 36\phi_{3m} - 9\phi_{4m}}{\Delta x^2},$$

(24)

$$(\partial_x^2 \phi)_{2m} = \frac{9\phi_{1m} - 18\phi_{2m} + 9\phi_{3m}}{\Delta x^2},$$

(25)

$$(\partial_x^2 \phi)_{3m} = \frac{9\phi_{2m} - 18\phi_{3m} + 9\phi_{4m}}{\Delta x^2},$$

(26)

$$(\partial_x^2 \phi)_{4m} = \frac{-9\phi_{1m} + 36\phi_{2m} - 45\phi_{3m} + 18\phi_{4m}}{\Delta x^2},$$

(27)

for MCV4. The same applies to $\zeta$ direction.
No-flux boundary conditions

\[ \mathbf{u} \cdot \mathbf{n} = 0, \quad (28) \]

where \( \mathbf{n} \) is the normal direction of the boundary and \( \mathbf{u} = (u, w)^T \) is the wind field. It can be also interpreted as the slip boundary condition, and is used in all test cases.

\[ (\rho')_{i0lm} = (\rho')_{1l(M+1-m)}, \quad (29) \]
\[ (\rho u)_{i0lm} = (\rho u)_{1l(M+1-m)}, \quad (30) \]
\[ (\rho w)_{i0lm} = -(\rho w)_{1l(M+1-m)}, \quad (31) \]
\[ (\tilde{w})_{i0lm} = -(\tilde{w})_{1l(M+1-m)}, \quad (32) \]
\[ [(\rho \theta)']_{i0lm} = [(\rho \theta)']_{1l(M+1-m)}. \quad (33) \]
Non-reflecting boundary conditions

\[
\frac{\partial \mathbf{q}}{\partial t} = [\text{governing equation terms}] - \tau (\mathbf{q} - \mathbf{q}_b), \tag{34}
\]

where \(\tau\) is the relaxation coefficients and \(\mathbf{q}_b\) represent some specified background fields such as the mean velocity or the reference state of the thermal-dynamic variables.

Following the existing works (Giraldo and Restelli 2008, JCP; Ullrich and Jablonowski 2012, MWR), the relaxation coefficients is defined as

\[
\tau(s) = \begin{cases} 
0, & \text{if } s < s_0 - s_T, \\
\tau_0 \left( \frac{s-(s_0-s_T)}{s_T} \right)^4, & \text{otherwise},
\end{cases} \tag{35}
\]

Note:

The no-flux condition is used along the bottom boundary. According to different test cases, periodical, no-flux and non-reflecting conditions are used respectively for the lateral boundaries. We use either no-flux or non-reflecting condition for the top boundary.
Denote the right hand side of (16) or (19) by $\mathcal{L}(q_{lm})$, we summarize the semi-discrete formulations as
\[ \frac{dq_{lm}}{dt} = \mathcal{L}(q_{lm}). \] (36)
Given the values $q_{lm}^n$ at step $n$, we use the following multi-step updating to obtain the values $q_{lm}^{n+1}$ at step $n + 1$,
\[ q_{lm}^{(1)} = q_{lm}^n + \Delta t \mathcal{L}(q_{lm}^n) \]
\[ q_{lm}^{(2)} = \frac{3}{4} q_{lm}^n + \frac{1}{4} q_{lm}^{(1)} + \frac{1}{4} \Delta t \mathcal{L}(q_{lm}^{(1)}) \] (37)
\[ q_{lm}^{n+1} = \frac{1}{3} q_{lm}^n + \frac{2}{3} q_{lm}^{(2)} + \frac{2}{3} \Delta t \mathcal{L}(q_{lm}^{(2)}). \]
A rising convective thermal bubble

**MCV3 results with filter** \( \Delta x = 125 \, \text{m}, \Delta t = 0.075 \, \text{s} \) and \( \mu = 10 \, \text{m}^2/\text{s} \)

---

(a) Potential temperature perturbation

(b) Horizontal wind

(c) Vertical wind
A rising convective thermal bubble

**MCV4 results with filter** $\Delta x = 200$ m, $\Delta t = 0.05$ s and $\mu = 10m^2/s$
Introduction

Multi-moment constrained formulations

Numerical results

Summary and future work

A rising convective thermal bubble

Max $\theta'$ and vertical velocity of MCV3 and MCV4 schemes without filter

It agrees well with other existing studies (e.g. Ahmad and Lindeman 2007; Norman et al. 2011).
Density current after 900 seconds, dissipation coefficient $\mu = 75 \text{ m}^2/\text{s}$ and $\Delta t = 0.015\text{s}$ for MCV3 and $\Delta t = 0.0125\text{s}$ for MCV4

Density current benchmark (Straka et al. 1993)

Using the initial potential temperature perturbation

(a) MCV3 $\Delta x = \Delta z = 25\text{ m}$

(b) MCV4 $\Delta x = \Delta z = 37.5\text{ m}$
Density current

MCV3 (left) and MCV4 (right), using the same DOF resolution
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Density current

(a) 12.5m DOF resolution for MCV3 and MCV4 schemes

(b) MCV3 for different grid spacing

(c) MCV4 for different grid spacing

Table:

<table>
<thead>
<tr>
<th>Schemes</th>
<th>Front location (in m)</th>
<th>$\rho_{\text{max}}'$</th>
<th>$\rho_{\text{min}}'$</th>
<th>$\theta_{\text{max}}'$</th>
<th>$\theta_{\text{min}}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCV3</td>
<td>14,883</td>
<td>0.036</td>
<td>-0.0019</td>
<td>$8.52 \times 10^{-13}$</td>
<td>-9.06</td>
</tr>
<tr>
<td>MCV4</td>
<td>14,896</td>
<td>0.036</td>
<td>-0.0019</td>
<td>$1.19 \times 10^{-7}$</td>
<td>-9.06</td>
</tr>
<tr>
<td>SE3</td>
<td>14,789</td>
<td>-</td>
<td>-</td>
<td>$-3.31 \times 10^{-3}$</td>
<td>-9.08</td>
</tr>
<tr>
<td>DG3</td>
<td>14,789</td>
<td>-</td>
<td>-</td>
<td>$-3.31 \times 10^{-3}$</td>
<td>-9.08</td>
</tr>
</tbody>
</table>

SE3 and DG3's data from Giraldo and Restelli, JCP, 2008

Table: comparison of the numerical results between the 3rd and 4th MCV schemes for the density current test. The total number of DOFs is equivalent to that of a conventional finite volume or finite difference scheme with a 12.5m grid spacing. Front location is defined by the location of $-1$ K contour value of potential temperature perturbation.

It agree well with other high order numerical methods such as spectral element and discontinuous Galerkin (Giraldo and Restelli (2008))
Internal gravity waves

\[ \Delta t = 0.1 \text{s for MCV3 and } \Delta t = 0.15 \text{s for MCV4} \]

\[
\theta(x, z) = \bar{\theta}(z) + \Delta \theta \frac{\sin(\pi z/H)}{1 + (x - x_0)^2/a^2}
\]

where \( H = 10 \text{ km}, \Delta \theta = 0.01 \text{ K}, x_0 = 100 \text{ km}, a = 5 \text{ km} \) and \( \bar{\theta}(z) = \theta_0 \exp\left(\frac{N_0^2}{g} z\right) \) with the constant temperature \( \theta_0 = 300 \text{ K} \).
It is quite similar with those in Skamarock and Klemp (1994) and Giraldo and Restelli (2008), as well as other existing studies (e.g. Ahmad and Lindeman 2007; Norman et al. 2011).
The computational domain is $[-25000, 25000] \text{m} \times [0, 21000] \text{m}$ with the grid spacings of $\Delta x = 250 \text{m}$ in $x$ direction and $\Delta \zeta = 210 \text{m}$ in $\zeta$ direction.

$$z_s(x) = h_0 \exp \left[ - \left( \frac{x}{a_0} \right)^2 \right] \cos\left( \frac{\pi x}{\lambda_0} \right) \quad (38)$$
Schär mountain

**MCV3 scheme** blue solid lines for numerical solution and red dashed lines for analytic solution, $\Delta t = 0.12s$

(a) Horizontal wind

(b) Vertical wind

Numerical results using the MCV3 scheme after 10 hours.
Numerical results using the MCV4 scheme after 10 hours.
Table : Root-mean-square errors of the schär mountain for different physical fields after 10 hours for 250 m (in $x$) and 210 m (in $\zeta$) resolution using the MCV3 and MCV4 schemes.

<table>
<thead>
<tr>
<th>Schemes</th>
<th>MCV3</th>
<th>MCV4</th>
<th>SE3</th>
<th>DG3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>$5.25 \times 10^{-6}$</td>
<td>$5.27 \times 10^{-6}$</td>
<td>$8.27 \times 10^{-6}$</td>
<td>$7.36 \times 10^{-6}$</td>
</tr>
<tr>
<td>$u$</td>
<td>$9.21 \times 10^{-2}$</td>
<td>$9.24 \times 10^{-2}$</td>
<td>$2.26 \times 10^{-1}$</td>
<td>$1.94 \times 10^{-1}$</td>
</tr>
<tr>
<td>$w$</td>
<td>$2.76 \times 10^{-2}$</td>
<td>$2.93 \times 10^{-2}$</td>
<td>$7.66 \times 10^{-2}$</td>
<td>$7.51 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$4.19 \times 10^{-2}$</td>
<td>$4.39 \times 10^{-2}$</td>
<td>$6.78 \times 10^{-2}$</td>
<td>$5.84 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

SE3 and DG3’s data from Giraldo and Restelli, JCP, 2008

Acknowledgment: we gratefully thank professor F. X. Giraldo for providing the semi-analytic solution of the schär mountain test for comparison.
Summary

- We have presented a 2D non-hydrostatic compressible atmospheric core by using the 3rd and 4th order multi-moment constrained finite volume (MCV) schemes.
- Different from conventional finite volume method, the predicted variables (unknowns) in an MCV scheme are the values at the solution points collocated within each mesh cell.
- Rigorously the numerical conservation.
- There is no any numerical quadrature explicitly involved.
Two important features make MCV method particularly attractive as an accurate and practical numerical framework for atmospheric and oceanic modelling:

1) The predicted variables are the nodal values at the solution points that can be flexibly located within a mesh cell (equidistant solution points are used in the present model). It is computationally efficient and provides great convenience in dealing with complex geometry and source terms.

2) High order and physically consistent formulations can be built by choosing proper constraints in view of not only numerical accuracy and efficiency but also underlying physics.
Future work

Further researches will be continued to develop 3D dynamical cores using the same methodology for global atmosphere based on the popular spherical grids, such as structured cubed grid, Yin-Yang grid, hexagonal geodesic grid, triangular geodesic grid.
Some results of shallow water on the popular grids using multi-moment constrained finite volume method
Some References using multi-moment constrained finite volume method

- C. G. Chen and F. Xiao, 2012: A high-order conservative multi-moment collocation shallow water model on cubed sphere, prepared for Journal of Computational Physics