Met Office

Design of numerical methods to respect asymptotic limit solutions

Mike Cullen (Met Office), Bob Beare (Exeter), Abeed Visram and Colin Cotter (Imperial)

10 October 2012

Contents

This presentation covers the following areas

- Background
- The ‘anelastic’ limit of the 3d Euler equations
- Validation of frontal solutions
- Asymptotic limits in the boundary layer

Background

Governing equations

On all relevant scales, the atmosphere is governed by the compressible Navier-Stokes equations, the laws of thermodynamics, phase changes and source terms

The solutions of these equations are very complicated, reflecting the complex nature of observed flows

The accurate solution of these equations would require computers 10^{30} times faster than now available

Asymptotic limits

Important and computable solutions can be described by asymptotic limits of the governing equations.

These can be used to validate numerical methods.

Obvious examples are ‘sound-proof’ solutions, and ‘balanced’ solutions dominated by the Earth’s rotation.

Major development area is to obtain such solutions with physical forcing.

Method

Choose appropriate small parameters.

Examples are Rossby number, Froude number, Mach number, Reynolds number, aspect ratio, ratio of horizontal scale to earth's radius.

Choose appropriate limit, which may require several parameters to tend to zero in a prescribed ratio.

Correct asymptotic behaviour requires the smallest parameter to be taken to zero before the others.

Increased modelling capability means a wider selection of regimes is now relevant.

Method II

Derive limit equations by scale analysis.

Prove that these equations can be solved.

Typically this requires limit equations to have conservation properties, this may require selective inclusion of higher order terms.

Prove that there is a solution of the Navier-Stokes or Euler equations converging to the limit solution at the expected rate.

Needed to ensure that the scale analysis stays valid for large times.

Application

- Test the convergence of a numerical method for the Navier-Stokes equations to a numerical solution of the limit equations at the expected rate.
- This can be done by either modifying the algorithm to solve the limit equations or by computing appropriate diagnostics.
- Initialisation very important.

When can this be done?

Only complete theory I know is Bourgeois and Beale (1994) who proved existence of smooth solutions to QG for arbitrarily long time, and convergence of Euler to QG at the expected O(Ro) rate.

Smooth 2d Euler solutions known to exist for arbitrarily long time. Convergence proved, but no rate proved.

Existence of weak SG solutions proved for constant rotation. Convergence not yet proved for optimal rate (only *Ro* not Ro^2) and only for smooth solutions.

Importance of convergence proof

Typically higher order approximate models (e.g. nonlinear balance) appear not to be solvable for large times, so higher order convergence of Euler to nonlinear balance cannot be expected.

Such models can be regularised, but the error estimate now has to include the regularisation. This will result in much lower order accuracy.

Example-the 'anelastic' limit of the 3d Euler equations

Equations

3d compressible Euler equations.

$$\frac{D\mathbf{u}}{Dt}+\frac{1}{\rho}\nabla p+\mathbf{g}=0$$

$$\frac{D\theta}{Dt}=0$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{u}\right)=0$$

$$\left(\frac{p}{p_0}\right)^{1-\kappa}=\frac{\rho\theta}{\rho_0\theta_0}$$

Sound proof equations I

- Driven by the experience that sound waves ‘don’t matter’.
- On large scales eliminate vertically propagating sound waves by hydrostatic approximation. This is a limit as $NT^{-1} \rightarrow 0$, where N is the buoyancy frequency.
- On small scales, eliminate sound waves with anelastic approximation. This is a limit as the Mach number tends to zero.
- Not possible to combine these limits in a single reduced equation set in a satisfactory way.

Sound proof equations II

- Instead, solve fully compressible equations using semi-implicit methods to avoid timestep restriction.
- Need to show that method respects both the hydrostatic and anelastic limits.
- Discuss the anelastic case.

Anelastic limit

Consider the limit as the ratio ε of the speed of sound $\sqrt{\gamma p/\rho}$ to the velocity tends to zero. This gives, using standard scale analysis:

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \mathbf{g} = 0$$

$$\frac{D\theta}{Dt} = 0$$

$$\nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\left(\frac{p}{p_0}\right)^{1-\kappa} = \frac{\rho\theta}{\rho_0\theta_0}$$

Small scale limit

These equations do not conserve energy, and are unlikely to be solvable. On a small horizontal scale, ρ can be replaced by a reference profile $\rho_0(z)$. On a small horizontal and small vertical scale, ρ can be replaced by a constant ρ_0.

The resulting systems conserve energy. They are probably solvable if viscosity is included. The most accurate conservative form is given by Durran (1989)

Implications for numerical models

Consider the discretisation of the continuity equation. Assume a time-weighted semi-implicit scheme

$$\rho^{t+\delta t} = \rho_d{}^{t} + \delta t\left(\alpha\rho^{t}(\nabla\cdot\mathbf{u})^{t+\delta t} + (1-\alpha)\rho^{t}(\nabla\cdot\mathbf{u})^{t}\right)$$

For α~1, but second-order accurate in time discretisation of other terms, as ε→0 the solution will tend to that of the incompressible system where the continuity equation is replaced by

$$\nabla\cdot\mathbf{u} = 0$$

This is only accurate on small horizontal and vertical scales. An improved version (original version of Endgame inner loop) is

$$\rho = \rho_{ref}(z) + \rho'$$

$$\rho'^{t+\delta t} = \rho_d'^{t} + \alpha(\nabla \cdot \rho_{ref}\mathbf{u})^{t+\delta t} + (1-\alpha)\rho^{t}(\nabla \cdot \rho_{ref}\mathbf{u})^{t}$$

$$+\alpha(\rho'\nabla \cdot u)^{t+\delta t} + (1-\alpha)(\rho'\nabla \cdot u)^{t}$$

If $\rho' \ll \rho_{ref}$, this converges as $\alpha \sim 1$ to

$$\nabla \cdot \rho_{ref}\mathbf{u} = 0$$

This is accurate on small horizontal and large vertical scales. The expected solvability of the limit anelastic equations means that this solution can be used as a test of a model.

The use of ρ_{ref} means that the solution will not be accurate on large horizontal scales where $\rho' \sim \rho_{ref}$. Therefore use (current version of EndGame inner loop)

$$\rho = \rho_d{}^t + \rho'$$

$$\rho'^{t+\delta t} = \rho_d'^{\,t} + \alpha(\nabla \cdot \rho^t{}_d \mathbf{u}^{t+\delta t}) + (1-\alpha)\rho^t(\nabla \cdot \rho^t{}_d \mathbf{u}^t)$$

$$+\alpha(\rho'\nabla \cdot \mathbf{u})^{t+\delta t} + (1-\alpha)(\rho'\nabla \cdot \mathbf{u})^t$$

This will have the correct small-scale asymptotic limit as then $\rho^t_d \sim \rho_{ref}$.

On large scales we would like to get the hydrostatic limit as $NT^{-1} \rightarrow 0$. This requires a proper cancellation between

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \text{ and } g$$

after removal of a reference state with uniform θ

Schemes involving ρ_{ref} will not give the right limit because $\rho'_d \sim \rho_{ref}$.

Met Office

Summary

No set of solvable limit equations describes ‘sound-proof’ behaviour on all spatial scales.

Operational models have to work on all scales. Validation against separate limit solutions for different scales is a good way of ensuring satisfactory behaviour.

Even a decentered semi-implicit scheme has a much lower condition number than the Poisson equation built into an anelastic model.

A semi-implicit Euler solver, which doesn’t treat acoustic waves correctly, can be viewed as a physically correct regularisation of an anelastic model-balance is not imposed on inappropriate scales.

Met Office

Validation of frontal solutions

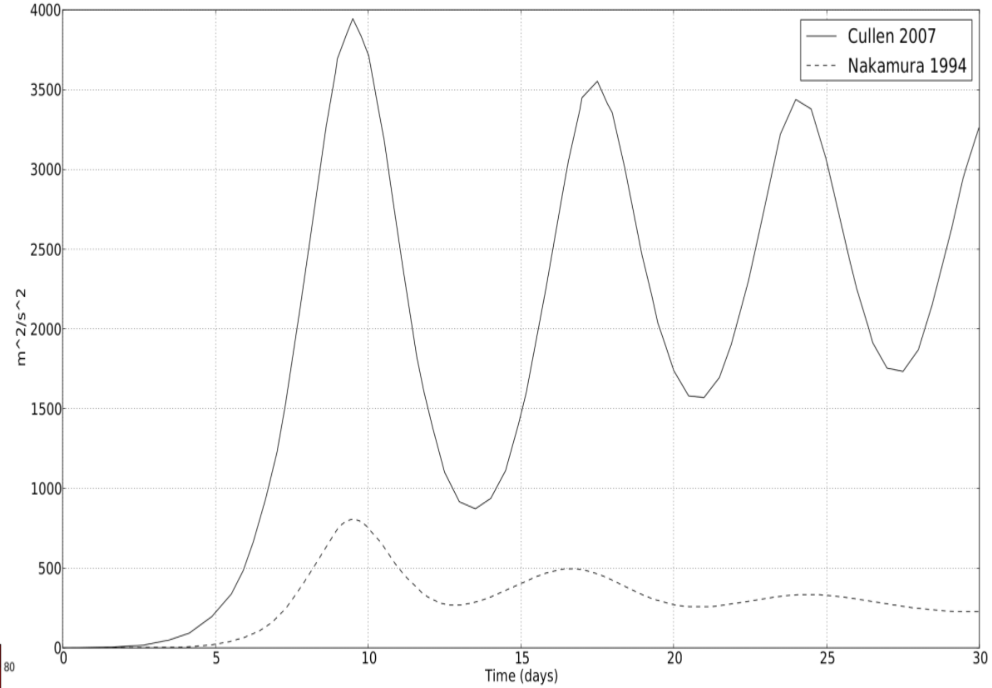
Eady problem

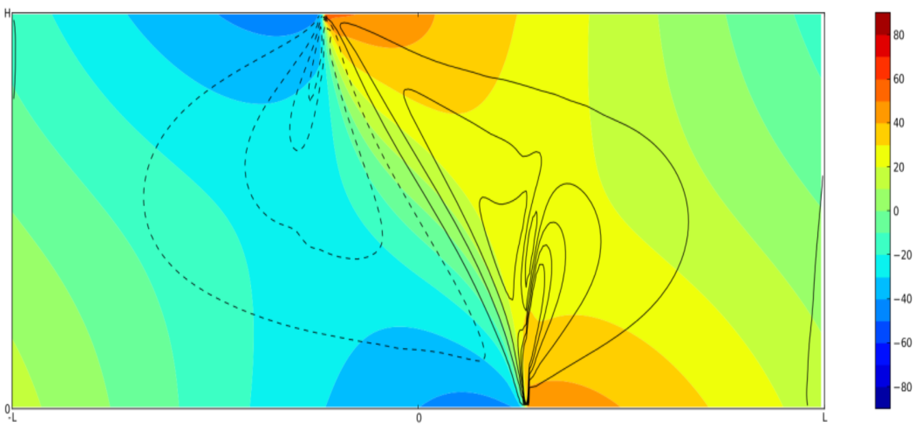
Consider incompressible Boussinesq equations in a vertical cross-section (x,z) with periodic boundary conditions in x and rigid upper and lower boundaries.

Solutions of these equations are a special case of solutions of 3d equations.

Eady model of fronts

- Lifecycles in SG model suggest predictability that is not captured in full equations
- Slice showing pertinent features of Eady model – sharp gradient in velocity, large scale balance and localised unbalanced motion of gravity waves





Incompressible equations

Met Office

$$\frac{Du}{Dt} + \frac{\partial p}{\partial x} - fv = 0$$

$$\frac{Dv}{Dt} + fu = C\left(z - H/2\right)$$

$$\frac{Dw}{Dt} + \frac{\partial p}{\partial x} - \frac{g\theta}{\theta_0} = 0$$

$$\frac{D\theta}{Dt} = Cv$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

C represents a basic state potential temperature gradient in *y*.

Limit equations

The small parameter ε is the Rossby number U/fL.

The SG limit is obtained by neglecting (D*u*/D*t*,D*w*/D*t*), giving

$$\nabla p = \left(fv, \frac{g\theta}{\theta_0} \right)$$

$$\frac{Dv}{Dt} + fu = C\left(z - H/2\right)$$

$$\frac{D\theta}{Dt} = Cv$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Limit equations

Existence of a solution to the SG equations in this case has been proved.

Scale analysis shows that SG is accurate to $O(\varepsilon^2)$ in this geometry.

Equations produce identical balanced (SG) solution if rescaled $L \rightarrow \beta L$, $U \rightarrow \beta U$, $f \rightarrow \beta f$ (f is Coriolis parameter)

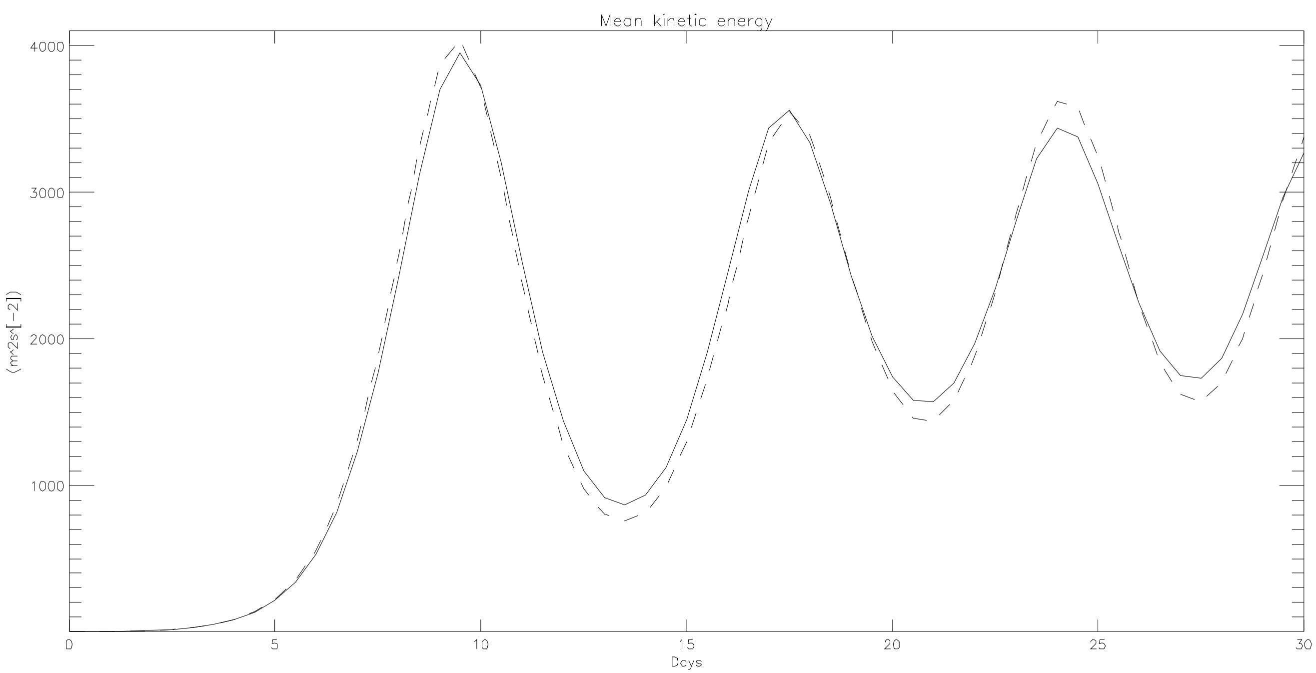
This rescaling replaces Ro by βRo.

The SG solution becomes discontinuous in finite time. The solution of the incompressible equations may not.

The challenge

Long time integrations of the SG Eady problem using special methods show very high predictability. Can this be reproduced by a standard scheme suitable for the UM?

Currently being investigated by Abeed Visram and Colin Cotter (Imperial).

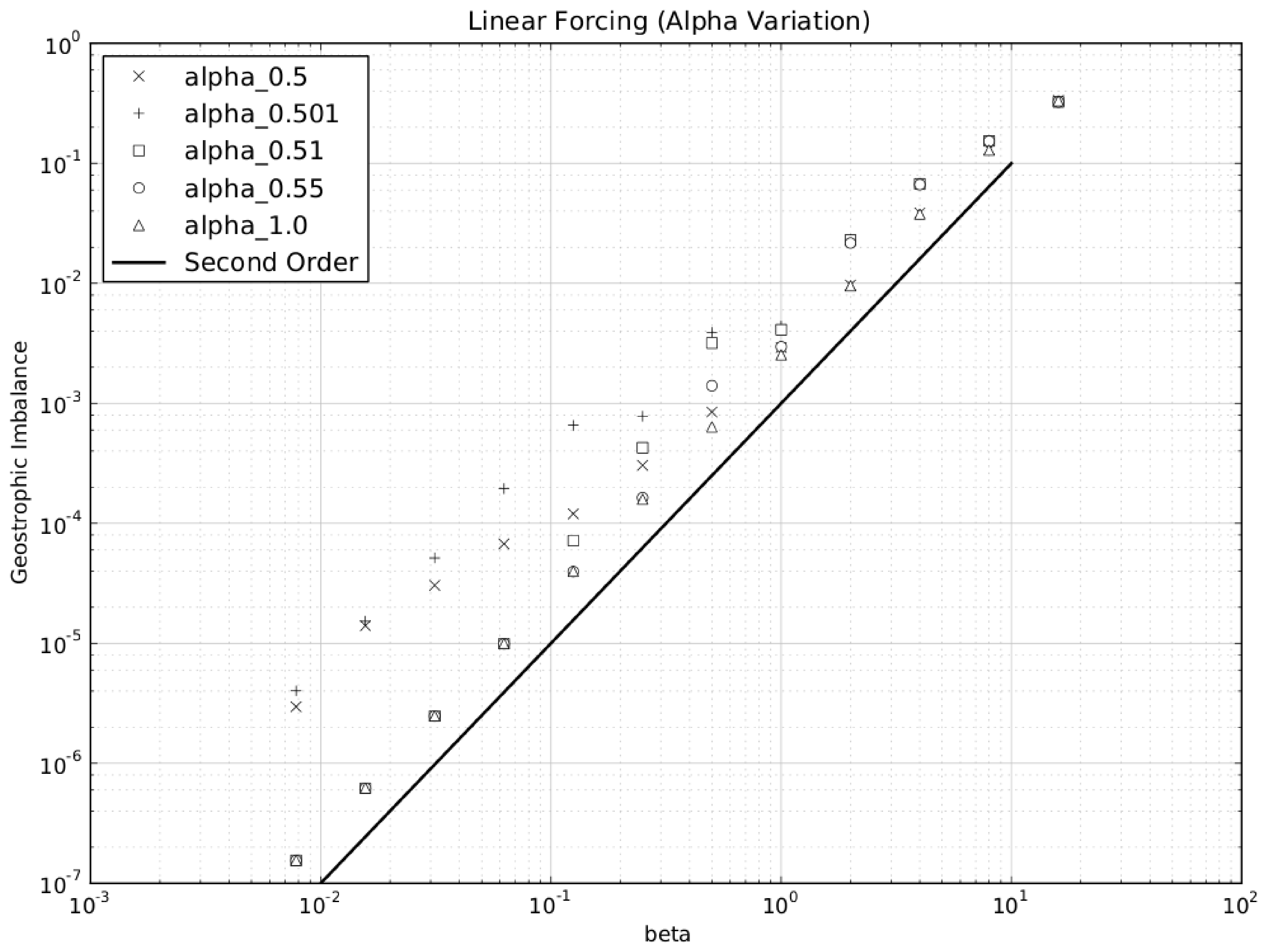


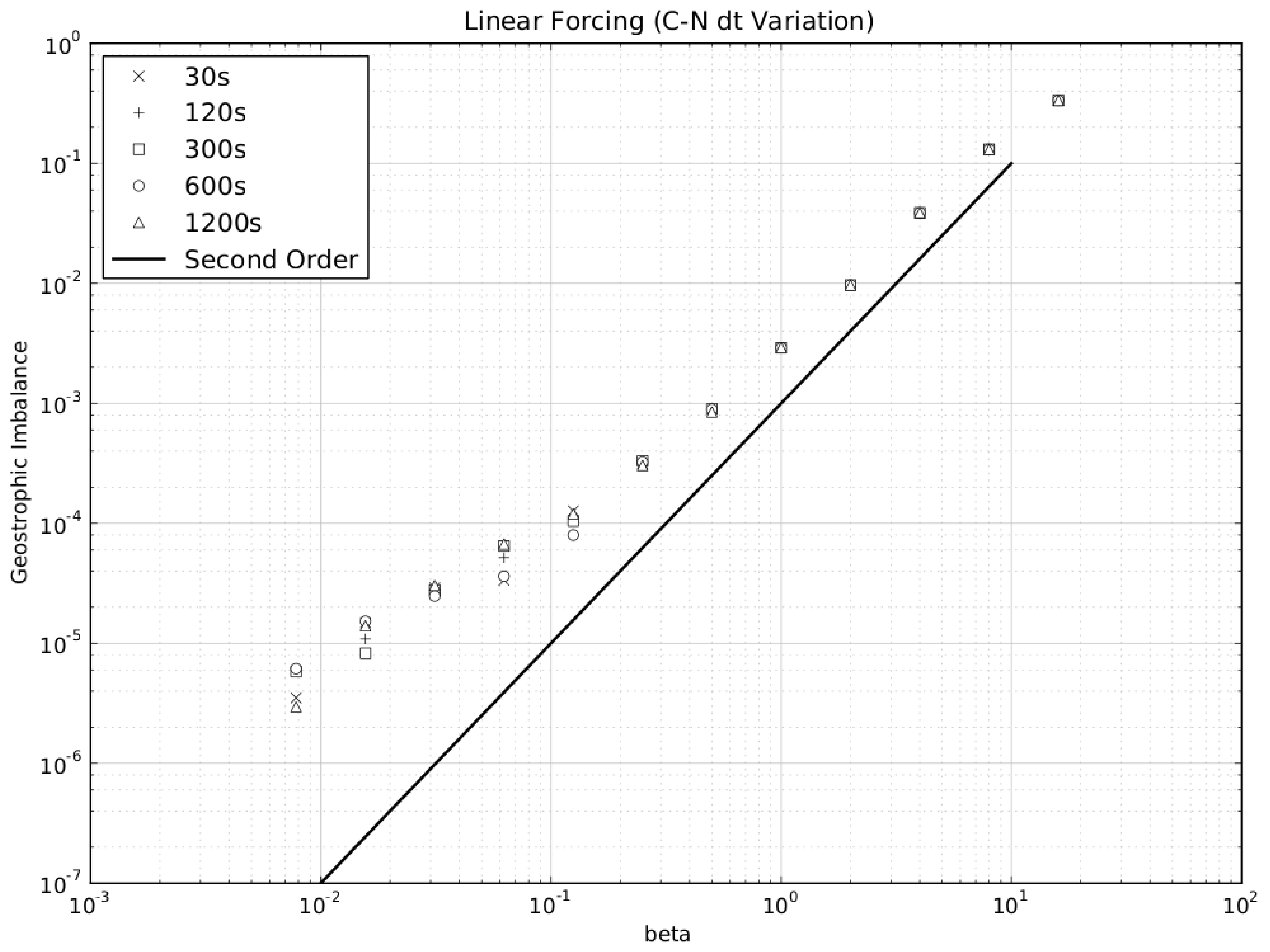
Linear computations

Initially demonstrate correct limit using linearised equations-so discontinuities do not occur.

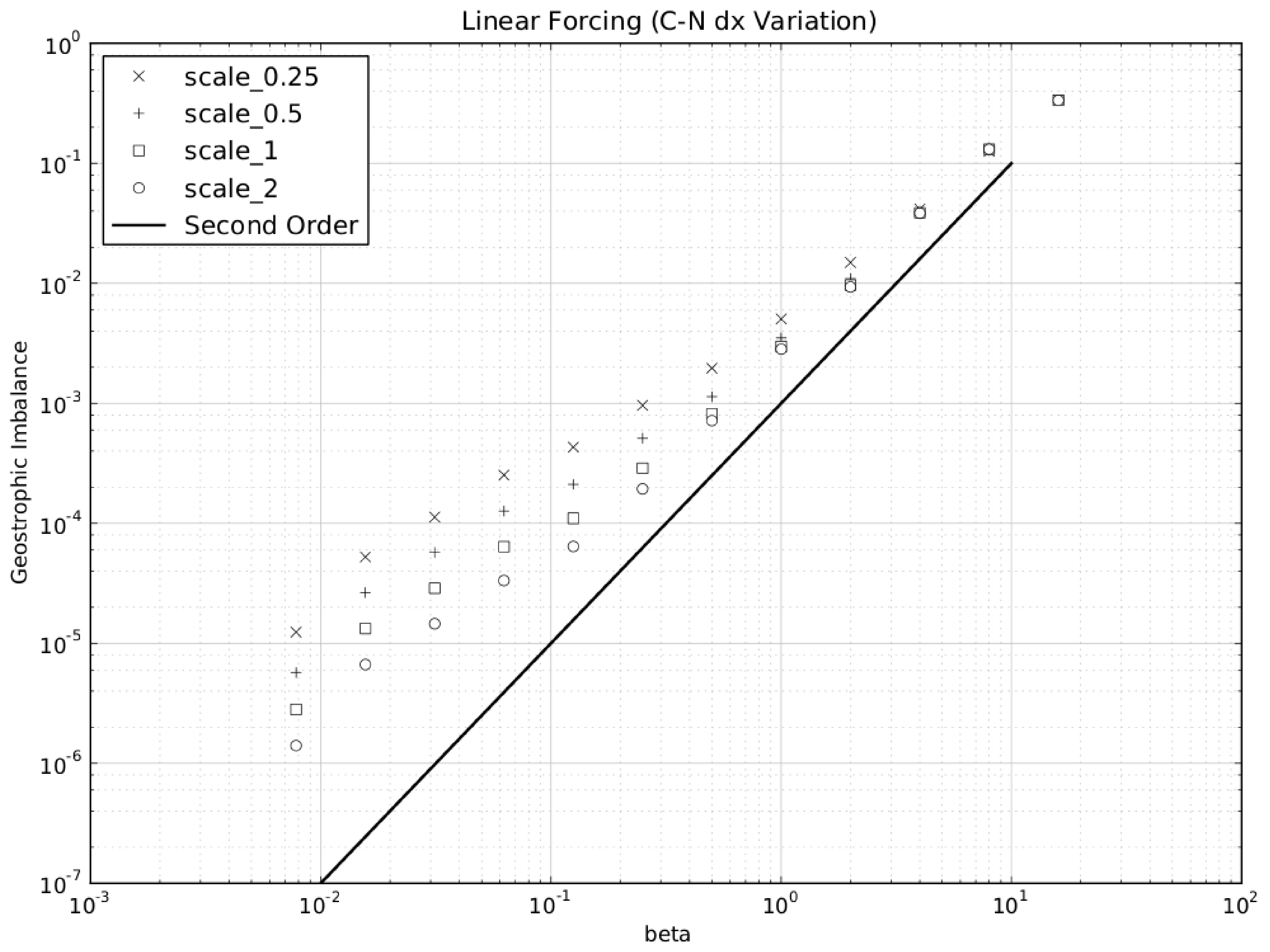
3 experiments:

- Decentered time integration, improving convergence to limit solution as β varies.
- Centred time integration with different timesteps
- Centred time integration with different resolutions.


Linear Forcing (Alpha Variation)
alpha_0.5
alpha_0.501
alpha_0.51
alpha_0.55
alpha_1.0
Second Order
Geostrophic Imbalance
beta
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-3
10^-2
10^-1
10^0
10^1
10^2



Linear Forcing (C-N dt Variation)
30s
120s
300s
600s
1200s
Second Order
Geostrophic Imbalance
beta
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}


Linear Forcing (C-N dx Variation)
scale_0.25
scale_0.5
scale_1
scale_2
Second Order
Geostrophic Imbalance
beta
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-3
10^-2
10^-1
10^0
10^1
10^2

Boundary layer limits

Inclusion of ‘physics’

Most work on asymptotic limits has been for adiabatic dynamics.

Dynamics/physics interaction has been much less studied in numerical model development.

Asymptotic limits studied very little in this context.

Effects of boundary layer and moisture on atmospheric flow very large.

Boundary layer

Met Office

Work with
Bob Beare.

Equations

(2d slice)

$$\frac{Du}{Dt} - fv + \frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(K_m \frac{\partial \mathbf{u}}{\partial z}\right)$$

$$\frac{Dv}{Dt} + fu = \frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z}\right)$$

$$\frac{\partial p}{\partial z} = b$$

$$\frac{Db}{Dt} + fu = \frac{\partial}{\partial z}\left(K_h \frac{\partial b}{\partial z}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = v = 0 \text{ at } z = 0$$

Small parameters

Rossby number $Ro=1/fT$

T is advective time scale of geostrophic wind above boundary layer

Ekman number $Ek=H^2/K_mT$

K_m depends strongly on dU/dz. In the surface layer this will translate to a dependence on U.

Limit for small Ro, Ek

Leading order approx is

$$-fv + \frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(K_m \frac{\partial \mathbf{u}}{\partial z} \right)$$

$$fu = \frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z} \right)$$

$$\frac{\partial p}{\partial z} = b$$

$$\frac{Db}{Dt} = \frac{\partial}{\partial z}\left(K_h \frac{\partial b}{\partial z} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = v = 0 \text{ at } z = 0$$

Met Office

Further approximations

This system is the 'planetary geotriptic' system, consistent with planetary geostrophic flow above the boundary layer.

To get to higher order, define,

$$-fv_e + \frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(K_m \frac{\partial u_e}{\partial z} \right)$$

$$fu_e = \frac{\partial}{\partial z}\left(K_m \frac{\partial v_e}{\partial z} \right)$$

$$u_e = v_e = 0 \text{ at z} = 0$$

Boundary layer

Met Office

Second order approx

$$\frac{Du_e}{Dt} - fv + \frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z} \right)$$

$$\frac{Dv_e}{Dt} + fu = \frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z} \right)$$

$$\frac{\partial p}{\partial z} = b$$

$$\frac{Db}{Dt} + fu = \frac{\partial}{\partial z}\left(K_h \frac{\partial b}{\partial z} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = v = 0 \text{ at } z = 0$$

Properties

Met Office

Rewrite momentum equations as

$$\frac{Du_e}{Dt} - f\left(v - v_e\right) = \frac{\partial}{\partial z}\left(K_m \frac{\partial\left(u - u_e\right)}{\partial z}\right)$$

$$\frac{Dv_e}{Dt} + f\left(u - u_e\right) = \frac{\partial}{\partial z}\left(K_m \frac{\partial\left(v - v_e\right)}{\partial z}\right)$$

While this would give second order accuracy, the equations do not have a negative definite energy integral, and the solutions will blow up.

Second order accuracy cannot be achieved sustainably (by this method).

Alternative

Instead, write momentum equations as

$$\frac{Du_e}{Dt} - f\left(v - v_e\right) = \frac{\partial}{\partial z}\left(K_m \frac{\partial\left(u_e - u\right)}{\partial z}\right)$$

$$\frac{Dv_e}{Dt} + f\left(u - u_e\right) = \frac{\partial}{\partial z}\left(K_m \frac{\partial\left(v_e - v\right)}{\partial z}\right)$$

This gives second order accuracy if Ro<<Ek, there is a negative definite energy integral. The equations are only first order accurate if Ro~Ek.

Met Office

Demonstration-low level jet

An ageotriptic wind is required to maintain the jet

15

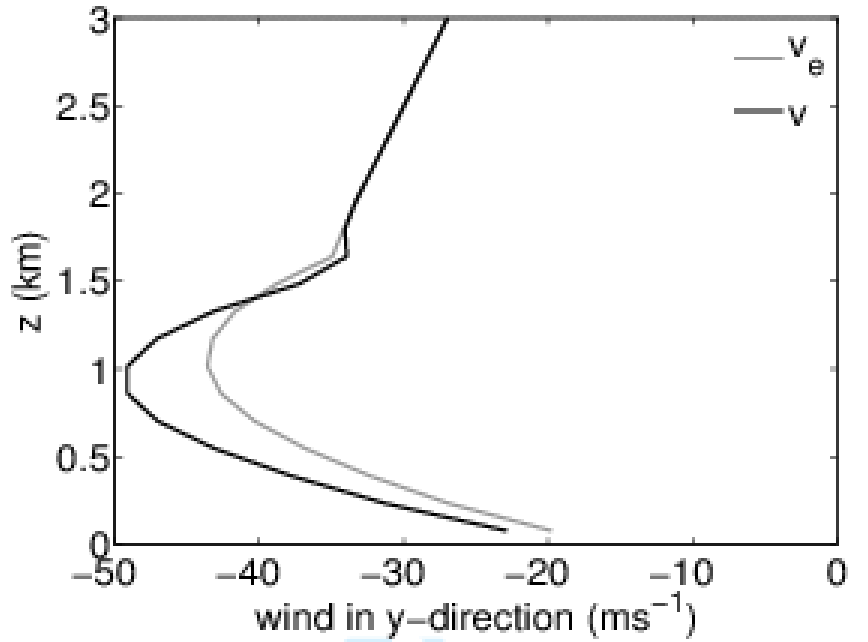


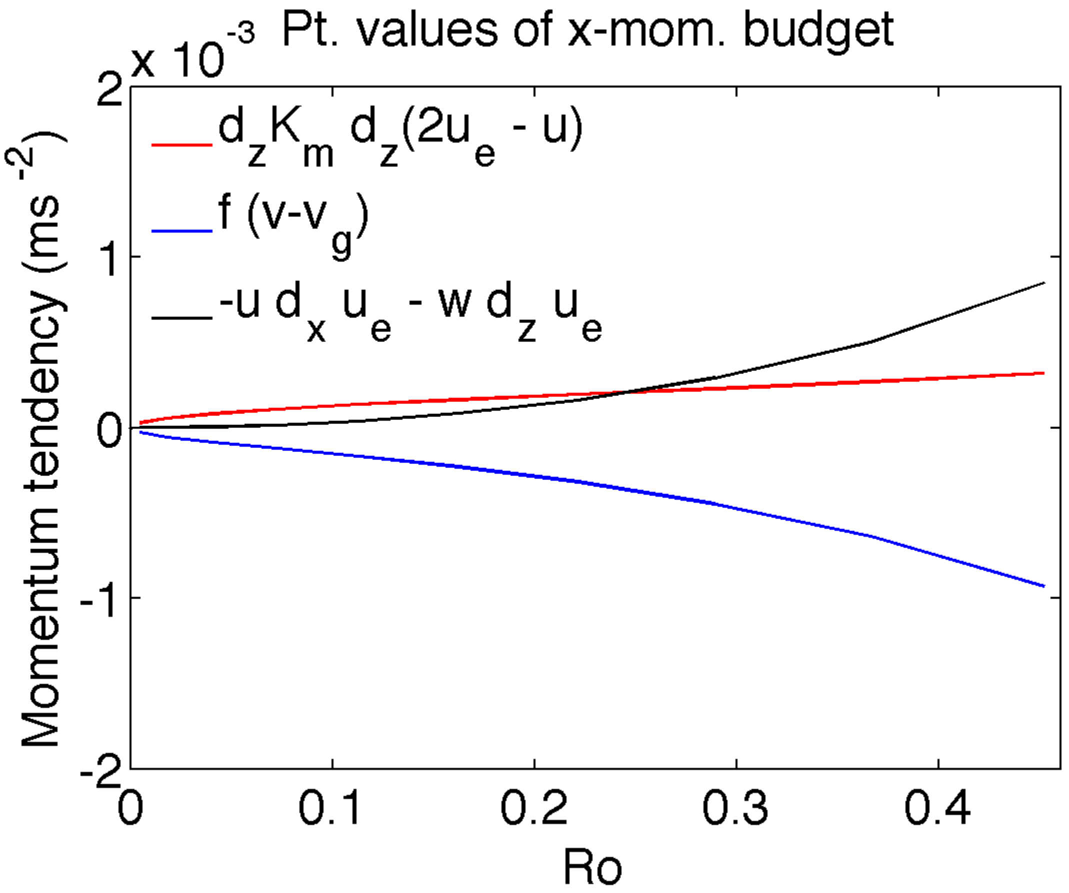
Figure 7. Vertical profiles of v_e (grey) and v (black) at the horizontal location of the jet maximum.

Diagnostic tests

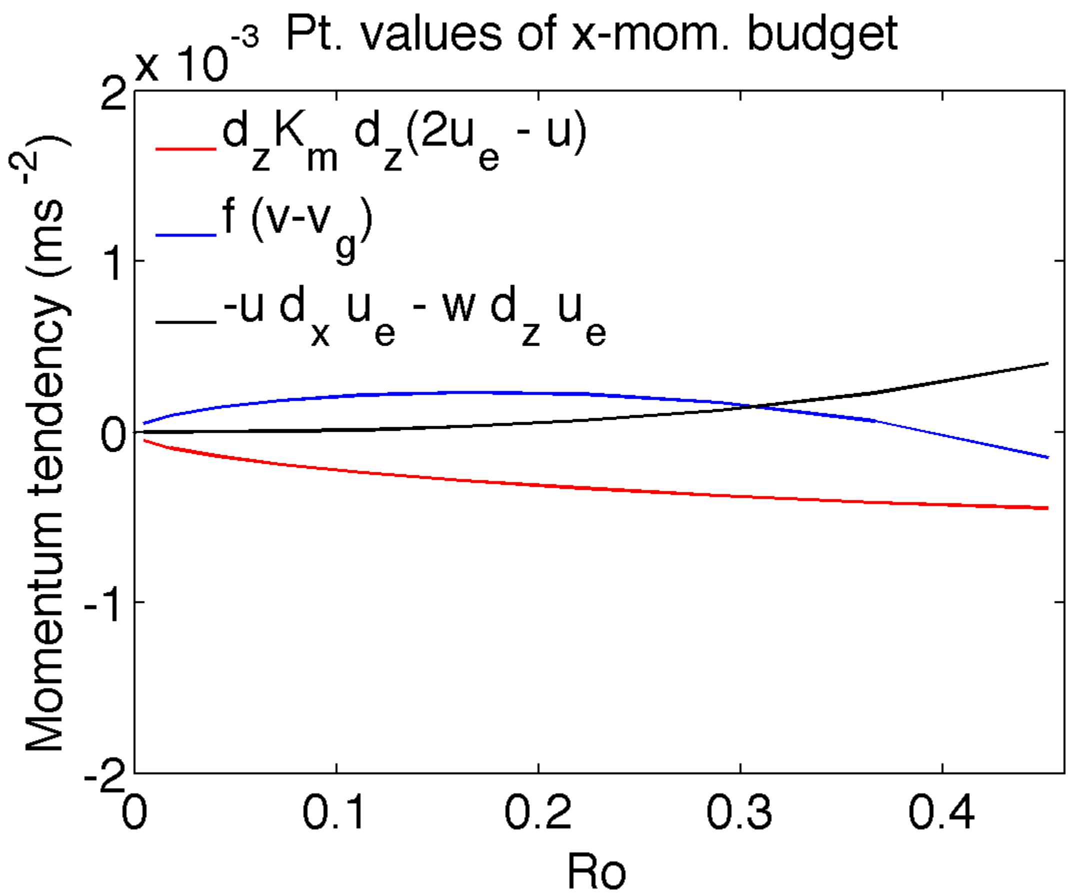
Integrate the full equations for a period, and calculate momentum budget. This allows scaling to be checked as *Ro*, *Ek* vary.

Results shown for varying *Ro*, achieved by changing horizontal scale with thermodynamic structure fixed.

X-momentum budget



X-momentum budget II



Comments

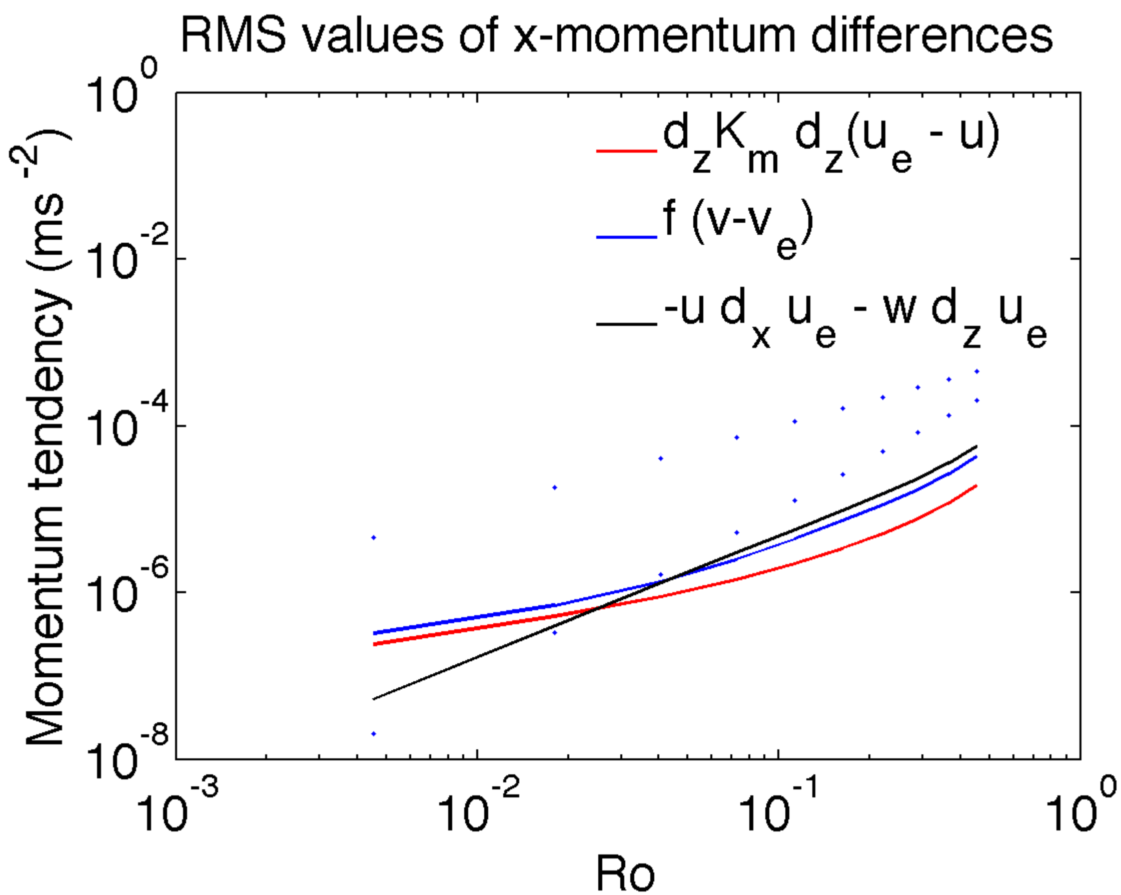
Two different points illustrated

For small Ro, good balance between rotation and friction.

For larger Ro, all three terms in SGT momentum equation contribute.

Met Office

Errors in SGT approximation



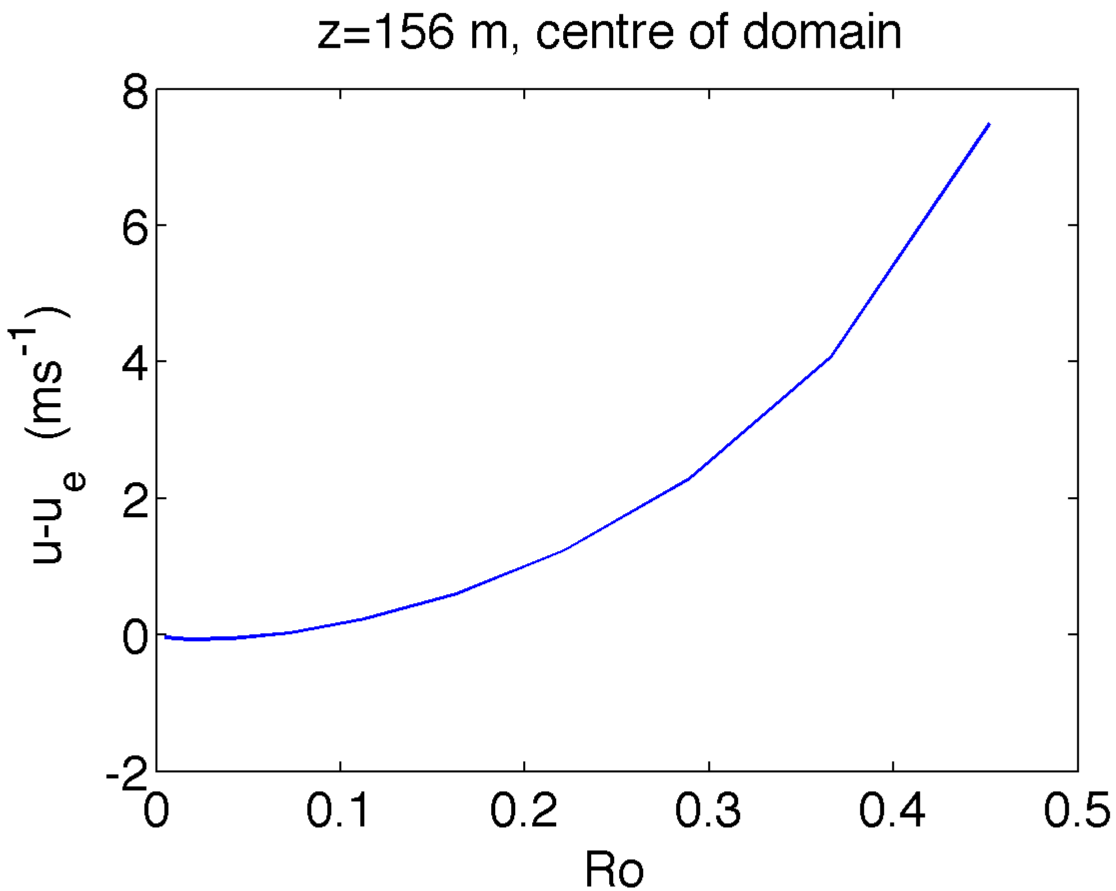
Comments

The leading order error in the SGT equations is $d_z(K_m d_z(u_e\text{-}u))$.

This converges to zero at second order rate for *Ro*>0.05, but then tails off.

Convergence not even first order for very small *Ro*, may indicate issue with model.

Convergence of u in surface layer



Comments

Changing β will reduce all velocities in proportion.

The second order reduction in u indicates that the frictional dominance is increasing, so that the boundary condition $u_e=0$ is obeyed more accurately.

Questions