Linear analyses of Runge-Kutta IMEX schemes for use in global atmospheric models

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with thanks to T. Allen, A. Staniforth, D. Durran

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- Motivation:
  "**Gung-Ho**": operational weather & climate forecasting model capable of exploiting $O(10^5)+$ processors
  For a semi-implicit model, from E. Mueller's talk:
  $\Rightarrow$ very fast solution of very large decomposed elliptic problems
  What do we do if the solvers can't deliver within the forecast slot?
- Framework for numerical analyses of **R**unge-**K**utta **IM**plicit-**EX**plicit methods
- Summary of stability properties for several RK IMEX schemes
- Results from a numerical experiment: non-linear test case (Durran and Blossey, 2012)
- Comments

Consider linearized "compressible Boussinesq system" (Durran and Blossey, 2012)

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) u + \frac{\partial P}{\partial x} &= 0 \\
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w + \frac{\partial P}{\partial z} &= b \\
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) b + N^2 w &= 0 \\
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) P + c_s^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) &= 0
\end{aligned}
$$

where U is constant mean windspeed, c_s is speed of sound,

$$b = -g\frac{\rho - \overline{\rho}(z)}{\rho_0},\ P = \frac{p - \overline{p}(z)}{\rho_0},\ N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}$$

Yields dispersion relations:

$$(\omega - k_x U)^2 = \frac{c_s^2}{2}\left\{k_x^2 + k_z^2 + \frac{N^2}{c_s^2} \pm \left[\left(k_x^2 + k_z^2 + \frac{N^2}{c_s^2}\right)^2 - \frac{4N^2 k_x^2}{c_s^2}\right]^{1/2}\right\}$$

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For a **H**orizontally-**E**xplicit **V**ertically-**I**mplicit (HEVI) time-integration scheme,

consider largest horizontal and vertical Courant numbers:

$$C_x = c_s k_x \Delta t \qquad C_z = c_s k_z \Delta t$$

Horizontal component defines a non-dimensional time-step that resolves c_s over Δx:

$$\Delta t^* = \max\{C_x\} = c_s \max\{k_x\} \Delta t \in [0, 2.5]$$

Then, the associated resolution of the vertical component can be described by

$$\frac{C_z}{\Delta t^*} = \frac{k_z}{\max\{k_x\}}$$

For global weather/climate, what is $\dfrac{C_z}{\Delta t^*} = \dfrac{k_z}{\max\{k_x\}}$?

Horizontal mesh-spacings vary across applications:

$$\Delta x = \begin{cases} 10^3\,\mathrm{m}, \ \mathrm{for\,Gung-Ho\,weather} \\ 10^4\,\mathrm{m}, \ \mathrm{for\,UM\,weather} \\ 10^5\,\mathrm{m}, \ \mathrm{for\,global\,climate,} \end{cases}$$

Vertical mesh-spacings vary within a model:

$$\Delta z = \Delta z_B \ldots \Delta z_T = 10 \ldots 10^3\,\mathrm{m}$$

Then,

$$\max\left\{\frac{C_z}{\Delta t^*}\right\} = \frac{\max\{k_z\}}{\max\{k_x\}} = \frac{\Delta x}{\Delta z_B} = \begin{cases} 10^2 \\ 10^3 \\ 10^4 \end{cases}, \quad \min\left\{\frac{C_z}{\Delta t^*}\right\} = \frac{\Delta x}{z_T} = \begin{cases} 10^{-2} \\ 10^{-1} \\ 10^0 \end{cases}$$

$\Rightarrow$ For our linear analyses, we will consider schemes' performance over

$$\Delta t^* \in [0, 2.5], \qquad \frac{k_z}{\max\{k_x\}} \in \left[10^{-2}, 10^4\right]$$

For the linear stability analysis, we (further) simplify the system to

1. an ODE with fast and slow wave contributions:

$$F_t = -iK_H F - iK_V F,$$

where $K_H = K_H(k_x)$ and $K_V = K_V(k_z)$;

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2. a system of horizontally & vertically propagating acoustic waves:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} &= 0 \\ \frac{\partial w}{\partial t} + \frac{\partial P}{\partial z} &= 0 \\ \frac{\partial P}{\partial t} + c_s^2 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) &= 0 \end{aligned}$$

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\end{aligned}$$

"UFPreF" (**U**-**F**orward **P**-**F**orward)

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\end{aligned}$$

"UFPreB" (**U**-**F**orward **P**-**B**ackward)

The ν-stage RK IMEX scheme that steps system

$$\mathbf{y}_t = \mathbf{s}\left(\mathbf{y}, t\right) + \mathbf{f}\left(\mathbf{y}, t\right),$$

from time $t = n\Delta t$ to $t = (n+1)\Delta t$ is given by:

$$\begin{aligned}
\mathbf{y}^{(j)} &= \mathbf{y}^{n} + \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk}\mathbf{s}\left(\mathbf{y}^{(k)}, t + \tilde{c}_k \Delta t\right) + \Delta t \sum_{l=1}^{j} a_{jl}\mathbf{f}\left(\mathbf{y}^{(l)}, t + c_l \Delta t\right),\ j = 1, \nu, \\
\mathbf{y}^{n+1} &= \mathbf{y}^{n} + \Delta t \sum_{j=1}^{\nu} \tilde{\omega}_j \mathbf{s}\left(\mathbf{y}^{(j)}, t + \tilde{c}_j \Delta t\right) + \Delta t \sum_{j=1}^{\nu} \omega_j \mathbf{f}\left(\mathbf{y}^{(j)}, t + c_j \Delta t\right)
\end{aligned}$$

or by the double Butcher tableau:

$$\begin{array}{c|ccc}
\tilde{c}_1 & \tilde{a}_{11} & \cdots & \tilde{a}_{1\nu} \\
\vdots & \vdots & & \vdots \\
\tilde{c}_\nu & \tilde{a}_{\nu 1} & \cdots & \tilde{a}_{\nu\nu} \\
\hline
 & \tilde{\omega}_1 & \cdots & \tilde{\omega}_\nu
\end{array}
\qquad
\begin{array}{c|ccc}
c_1 & a_{11} & \cdots & a_{1\nu} \\
\vdots & \vdots & & \vdots \\
c_\nu & a_{\nu 1} & \cdots & a_{\nu\nu} \\
\hline
 & \omega_1 & \cdots & \omega_\nu
\end{array}$$

Trap2$(2+\delta_f, 2+\delta_s, 2)$

based on ENDGame formulation (T. Allen)

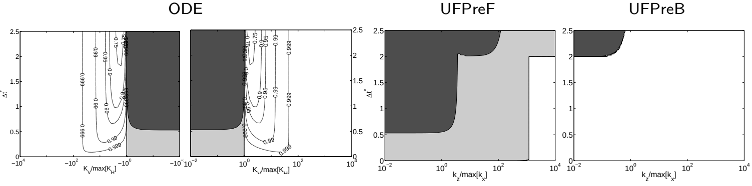
$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\delta_s & \delta_s & 0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\hline
 & \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{array}
\qquad
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\delta_f & \delta_f\left(\frac{1-\alpha}{2}\right) & \delta_f\left(\frac{1+\alpha}{2}\right) & 0 & 0 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\hline
 & \frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{array}
$$

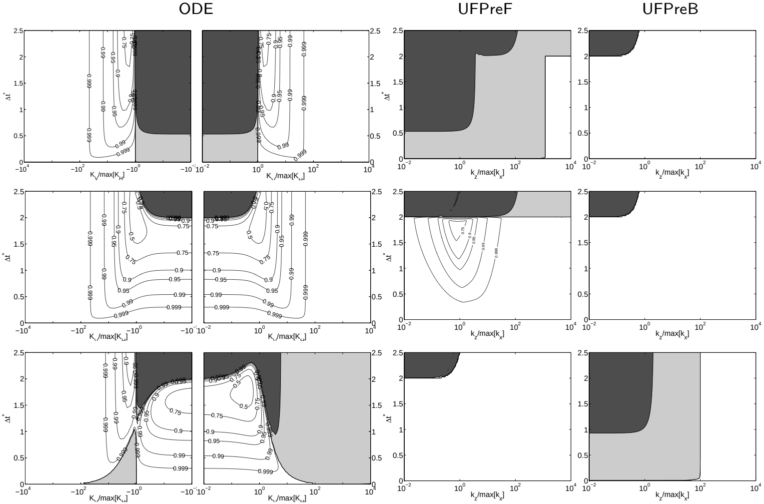
where $\delta_s, \delta_f = 0, 1$

2(2, 2, 2)

$\delta_s = 0$

$\delta_f = 0$





2(2, 2, 2)
$\delta_s = 0$
$\delta_f = 0$

2(2, 3, 2)
$\delta_s = 1$
$\delta_f = 0$

2(3, 3, 2)
$\delta_s = 1$
$\delta_f = 1$
$\alpha = 0$

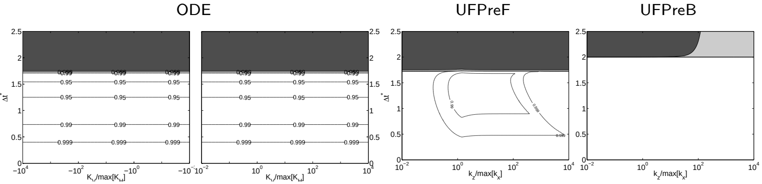
UJ3$(1, 3, 2)$

"Strang carryover" scheme (Ullrich and Jablonowski, 2012)

0	0					
0	0	0				
1	0	1	0			
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0		
1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	0	
1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	0	0
	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	0	0

0	0					
$\frac{1}{2}$	$\frac{1}{2}$	0				
$\frac{1}{2}$	$\frac{1}{2}$	0	0			
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0		
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	
1	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$
	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$

3(1, 3, 2)

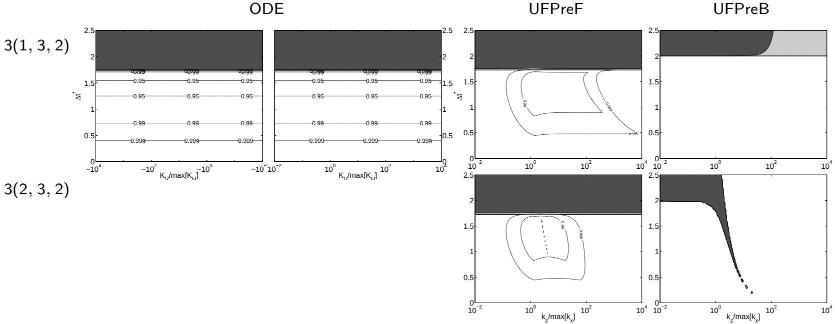


$UJ3(1,3,2)$

$$
\begin{array}{c|cccccc}
0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
1 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 \\
1 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccccc}
0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
\hline
 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{array}
$$

$UJ^*3(2,3,2)$

$$
\begin{array}{c|cccccc}
0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
1 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 \\
1 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccccc}
0 & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\
\hline
 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4}
\end{array}
$$



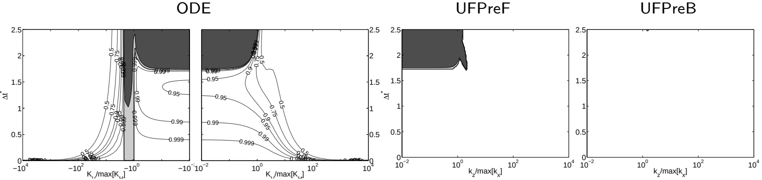
SSP3(3, 3, 2)

from Pareschi and Russo (2005)

$$
\begin{array}{c|ccc}
0 & 0 & & \\
1 & 1 & 0 & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|ccc}
\gamma & \gamma & & \\
1-\gamma & 1-2\gamma & \gamma & \\
\frac{1}{2} & \frac{1}{2}-\gamma & 0 & \gamma \\
\hline
 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

$$\gamma = 1 - 1/\sqrt{2}$$

3(3, 3, 2)

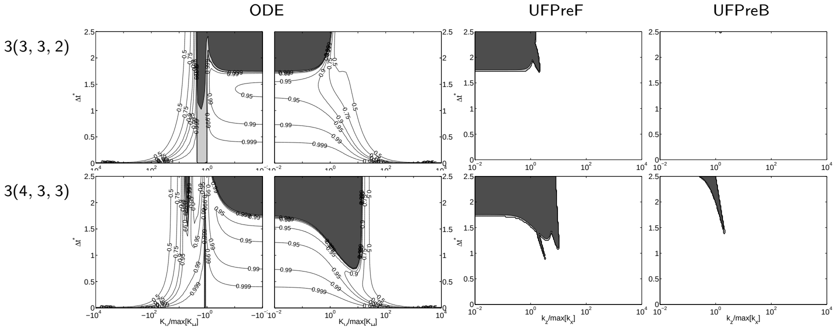


$$
\begin{array}{c|ccc}
0 & 0 & & \\
1 & 1 & 0 & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|ccc}
\gamma & \gamma & & \\
1-\gamma & 1-2\gamma & \gamma & \\
\frac{1}{2} & \frac{1}{2}-\gamma & 0 & \gamma \\
\hline
 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

$$\gamma = 1 - 1/\sqrt{2}$$

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
0 & 0 & 0 & & \\
1 & 0 & 1 & 0 & \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|cccc}
\alpha & \alpha & & & \\
0 & -\alpha & \alpha & & \\
1 & 0 & 1-\alpha & \alpha & \\
\frac{1}{2} & \beta & \eta & \delta & \alpha \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

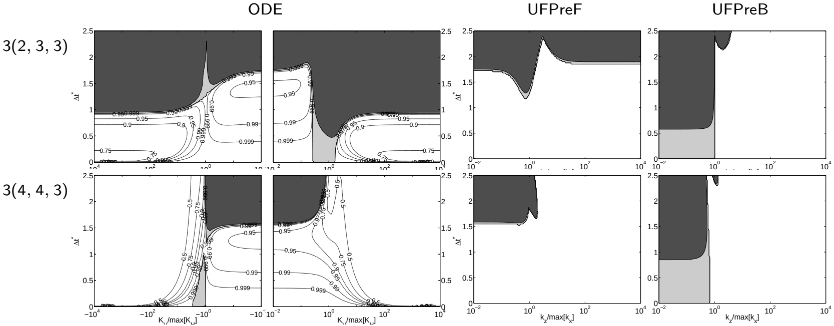
$\alpha = 0.24169426078821,\ \beta = 0.06042356519705$
$\eta = 0.12915286960590,\ \delta = \frac{1}{2} - \beta - \eta - \alpha$



$$
\begin{array}{c|ccc}
0 & 0 & & \\
\gamma & \gamma & 0 & \\
1-\gamma & \gamma-1 & 2(1-\gamma) & 0 \\
\hline
 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
\quad
\begin{array}{c|ccc}
0 & 0 & & \\
\gamma & 0 & \gamma & \\
1-\gamma & 0 & 1-2\gamma & \gamma \\
\hline
 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

$$\gamma = \left(3+\sqrt{3}\right)/6$$

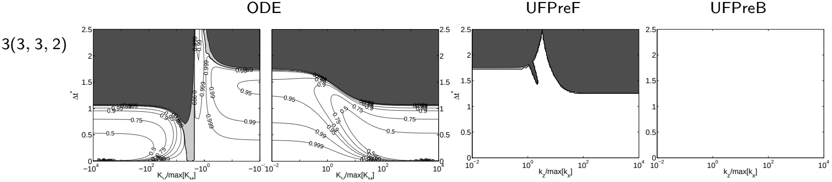
$$
\begin{array}{c|ccccc}
0 & 0 & & & & \\
\frac{1}{2} & \frac{1}{2} & 0 & & & \\
\frac{2}{3} & \frac{11}{18} & \frac{1}{18} & 0 & & \\
\frac{1}{2} & \frac{5}{6} & -\frac{5}{6} & \frac{1}{2} & 0 & \\
1 & \frac{1}{4} & \frac{7}{4} & \frac{3}{4} & -\frac{7}{4} & 0 \\
\hline
 & \frac{1}{4} & \frac{7}{4} & \frac{3}{4} & -\frac{7}{4} & 0
\end{array}
\quad
\begin{array}{c|ccccc}
0 & 0 & & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & & \\
\frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{2} & & \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\
1 & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\
\hline
 & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
$$



ARK2(2, 3, 2)
from Giraldo et al. (2012)

$$
\begin{array}{c|ccc}
0 & 0 & & \\
2-\sqrt{2} & 2-\sqrt{2} & 0 & \\
1 & 1-a_{32} & a_{32} & 0 \\
\hline
 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}}
\end{array}
\qquad
\begin{array}{c|ccc}
0 & 0 & & \\
2-\sqrt{2} & 1-\frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} & \\
1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \\
\hline
 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}}
\end{array}
$$

$a_{32} = 1/4$ $\qquad$ $\gamma = 1 - 1/\sqrt{2}$



Non-linear near-hydrostatic test case (Durran and Blossey, 2012)

- Equations: non-linear Boussinesq vertical-slice
- Localized forcing ($\nabla \times \psi$) generating gravity waves in stratified shear flow

Non-linear near-hydrostatic test case (Durran and Blossey, 2012)

- Equations: non-linear Boussinesq vertical-slice
- Localized forcing $(\nabla \times \psi)$ generating gravity waves in stratified shear flow
- Set-up:

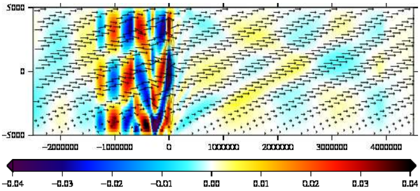
$$u_0(z) = 5 + z + 0.4(5 - z)(5 + z)\,\mathrm{m\,s^{-1}}$$

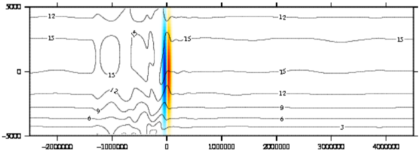
$$\psi(x, z, t) = \psi_0 \left(\frac{\pi x}{L_x}\right) \sin(\omega t) \exp\left[-\left(\frac{\pi x}{L_x}\right)^2 - \left(\frac{\pi z}{L_z}\right)^2\right] \mathrm{m^2\,s^{-1}}$$

$$\omega = 1.25 \times 10^{-4}\,\mathrm{s^{-1}},\ L_x = 160\,\mathrm{km},\ L_z = 10\,\mathrm{km},\ \psi_0 = 10\,\mathrm{m^2\,s^{-1}}$$

$$\Delta x = 10\,\mathrm{km},\ \Delta z = 250\,\mathrm{m}$$

Non-linear near-hydrostatic test case (Durran and Blossey, 2012)

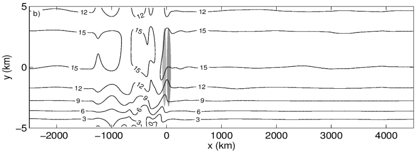


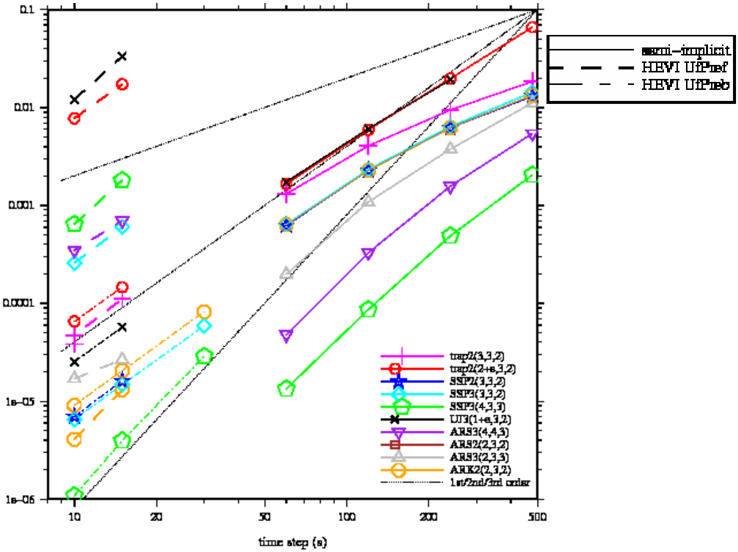


Reference solution at 33.3 hours
Explicit 4th-order RK, $\Delta t = 0.5$ s

Above:
L: b (colours); $\mathbf{u}$ vectors; ψ (contours);
R: u contours; ψ (colours):

Bottom: Durran and Blossey (2012)




0.1
0.01
0.001
0.0001
1e-05
1e-06
10
20
50
100
200
500
time step (s)
semi-implicit
HEVI UfPref
HEVI UfPreb
trap2(3,3,2)
trap2(2+e,3,2)
SSP2(3,3,2)
SSP3(3,3,2)
SSP3(4,3,3)
UJ3(1+e,3,2)
ARS3(4,4,3)
ARS2(2,3,2)
ARS3(2,3,3)
ARK2(2,3,2)
1st/2nd/3rd order

Summary

- Most HEVI schemes performed well in one of UFPreF or UFPreB
- Some schemes indicate loss of accuracy moving from semi-implicit to HEVI: SSP3(4,3,3) and ARS*
- UJ3$(1,3,2)$ performs well with only a single implicit solve
- ARK2(2,3,2) shows good error properties & in both UFPreF and UFPreB formulations

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- Some schemes indicate loss of accuracy moving from semi-implicit to HEVI: SSP3(4,3,3) and ARS*
- $UJ3(1,3,2)$ performs well with only a single implicit solve
- ARK2(2,3,2) shows good error properties & in both UFPreF and UFPreB formulations

Additionally:

- Here, only stability properties, i.e. $|A|_{\max}$ from $A = |A|\,\mathrm{e}^{-i\tilde{K}\Delta t}$
- Also considering $\tilde{K}\Delta t$ — phase and group velocity properties
- And errors accumulated over $\Delta t_{\mathrm{SI}} = M\Delta t_{\mathrm{HEVI}}$ for $M \sim 10$

$$\Rightarrow \quad A^M = |A|^M\,\mathrm{e}^{-iM\tilde{K}\Delta t}$$

Thanks for your attention!

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