What an Old Problem Has Still to Say: the Infamous K_{13} Case

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Summary

Splay-Bend Elasticity

Continuum Limit

Surface Order on Nematic Shells

Splay-Bend Elasticity

For a ***uniaxial nematic*** phase

$$\mathbf{Q} = S\left(\boldsymbol{n} \otimes \boldsymbol{n} - \frac{1}{3}\mathbf{I}\right)$$

S scalar order parameter

$\boldsymbol{n}$ nematic director

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elastic energy density

Letting $S \equiv 1$, we can write the elastic free-energy per unit volume in the form

$$W_{\mathrm{e}}(\boldsymbol{n},\nabla\boldsymbol{n}) = \frac{1}{2}K_1(\operatorname{div}\boldsymbol{n})^2 + \frac{1}{2}K_2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2 + \frac{1}{2}K_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 + \frac{1}{2}(K_2+K_{24})\left[\operatorname{tr}(\nabla\boldsymbol{n})^2 - (\operatorname{div}\boldsymbol{n})^2\right] + K_{13}\operatorname{div}[(\operatorname{div}\boldsymbol{n})\boldsymbol{n}]$$

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K_2	twist constant
K_3	bend constant
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one-constant approximation

$$K_1 = K_2 = K_3 = K \quad \& \quad K_{24} = 0 \quad K_{13} = 0 \quad \Rightarrow \quad W_{\mathrm{e}} = \tfrac{K}{2}|\nabla\boldsymbol{n}|^2$$

free-energy functional

$$\mathscr{F}_{\mathrm{e}}[\boldsymbol{n}] = \int_{\mathscr{B}} W_{\mathrm{e}}(\boldsymbol{n}, \nabla\boldsymbol{n})dV$$

$\mathscr{B}$ region in space

V volume measure

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Null Lagrangians

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$\partial\mathscr{B}$ boundary of $\mathscr{B}$

$\boldsymbol{\nu}$ outer unit normal to $\partial\mathscr{B}$

∇_{s} surface gradient

$\mathrm{div}_{\mathrm{s}}$ surface divergence

A area measure

$$\int_{\mathscr{B}} \operatorname{div}[(\operatorname{div} \boldsymbol{n})\boldsymbol{n}]dV = \int_{\partial\mathscr{B}} (\operatorname{div} \boldsymbol{n})\boldsymbol{n} \cdot \boldsymbol{\nu} dA$$

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pitfall

In the presence of ***strong*** boundary conditions, which prescribe the nematic director over the whole $\partial\mathscr{B}$, the derivatives $(\nabla_{\mathrm{s}}\boldsymbol{n})\boldsymbol{e}$ of $\boldsymbol{n}$ in all surface directions $\boldsymbol{e}$ are prescribed, whereas the normal derivative $(\nabla_{\mathrm{s}}\boldsymbol{n})\boldsymbol{\nu}$ is completely ***free***.

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- OLDANO & BARBERO (1985) found by example that K_{13} makes $\mathscr{F}_{\mathrm{e}}$ ***unbounded*** from below.

first attempted remedies

- Adding to W_{e} terms quadratic in the second gradient $\nabla^2 \boldsymbol{n}$ that comply with both ***frame-indifference*** and ***nematic symmetry***.

(Barbero, Madhusudana & Oldano 1989)
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  It formally solves the unboundedness of $\mathscr{F}_{\mathrm{e}}$, but in a rather ad hoc manner.

towards proving that $K_{13} = 0$

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- Yokoyoma (1997) proved that a properly phrased density functional theory requires that $K_{13} = 0$, in accord with the well-posedness of the phenomenological theory, and that $K_{24} \geqq \frac{1}{2}(K_1 - K_2)$.

K_{13} *cyclic resurrections*

Recently, an extension of Nehring & Saupe's theory to a mean-field treatment that would incorporate the effects of molecular anisotropy into the dispersion forces interactions, has reproduced a mutation of K_{13}, though both its sign and its dependence on temperature make it different from the other elastic constants predicted by theory, and so open to suspicion. (Cestari, Bosco & Ferrarini 2009) (Ferrarini 2010)

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undistorted state (at zero temperature) & homogeneity

$$W_{13} := K_{13}\,\mathrm{div}[(\mathrm{div}\,\boldsymbol{n})\boldsymbol{n}] = K_{13}[(\mathrm{div}\,\boldsymbol{n})^2 + \underbrace{\nabla(\mathrm{div}\,\boldsymbol{n})\cdot\boldsymbol{n}}_{\text{linear}}]$$

If we assume that the ***undistorted*** state is attained when $\boldsymbol{n}$ is ***uniform*** in space, then in the limit of small distortions the linear component of W_{13} would prevail over all other elastic terms, making the undistorted state unstable, unless $K_{13} = 0$. (Yokoyama 1997)

Ericksen inequalities

$$W_{\mathrm{e}} \geqq 0 \quad \Leftrightarrow \quad K_1 \geqq \frac{1}{2}\left(K_2 + K_{24}\right), \quad K_2 \geqq |K_{24}|, \quad K_3 \geqq 0 \quad \& \quad K_{13} = 0$$

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illuminating passage

"Even though the two approaches [phenomenological and molecular] lead to nearly the same result, there is a fundamental difference between both ways of treating elastic properties. In the phenomenological method, we regard the ***free*** energy which is the interesting quantity for the determination of equilibrium structures. The molecular method deals only with the ***interaction*** energy, which is assumed to be additive, and takes no account of the entropy."

(Nehring & Saupe 1971)

Continuum Limit

Inspired by the work of NEHRING & SAUPE (1971), we consider a simple molecular interaction model as a heuristic, conceptual tool to identify the appropriate ***measure*** of elastic distortion.

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interaction potential

$$V(r, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = V_{\text{iso}}(r) + V_{\text{aniso}}(r, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2)$$

r inter-molecular distance

$\boldsymbol{\ell}_1$ $\boldsymbol{\ell}_2$ molecular spins

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We imagine that at ***equilibrium*** the first neighbors of the spin $\boldsymbol{\ell}$ at $p \in \mathscr{B}$ lie on a ***sphere*** of radius r_{e}.

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$\boldsymbol{\ell}_{\boldsymbol{u}}$ spin at $p_{\boldsymbol{u}}$

average distortion energy

$$w_{\mathrm{e}} := -\frac{3}{2} U_{\mathrm{aniso}} \varrho \left(\frac{\sigma}{r_{\mathrm{e}}} \right)^{6} \langle (\boldsymbol{\ell} \cdot \boldsymbol{\ell}_{\boldsymbol{u}})^{2} \rangle_{\mathbb{S}^2}$$

ϱ volume number density

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- The local elastic distortion is minimized by $\nabla \boldsymbol{n} = \mathbf{0}$.
- The elastic free energy density is a quadratic function of $\nabla \boldsymbol{n}$.

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$\boldsymbol{\ell}$ molecular director

$\mathscr{S}$ closed orientable surface

$\boldsymbol{\nu}$ outer unit normal

$$\boldsymbol{\ell} \cdot \boldsymbol{\nu} \equiv 0$$

order tensor

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order bounds

$$0 \leqq \lambda \leqq \tfrac{1}{2}$$

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$\boldsymbol{\ell}_1$ $\boldsymbol{\ell}_2$ molecular spins

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$$\boldsymbol{\ell}_{\boldsymbol{u}} = \boldsymbol{\ell} + r_\mathrm{e}(\nabla_\mathrm{s}\boldsymbol{\ell})\boldsymbol{u} + \frac{1}{2}r_\mathrm{e}^2(\nabla_\mathrm{s}^2\boldsymbol{\ell}) \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + o(r_\mathrm{e}^2)$$

molecular surface distortion

We imagine that at ***equilibrium*** the first neighbors of the spin $\boldsymbol{\ell}$ at $p \in \mathscr{S}$ lie on a ***geodesic circle*** of radius r_{e}.

$$p_{\boldsymbol{u}} := p + r_{\mathrm{e}}\boldsymbol{u} \quad \boldsymbol{u} \in \mathbb{S}^1$$

$p_{\boldsymbol{u}}$ point on the geodesic circle

$\boldsymbol{u}$ geodetic unit tangent

$$\boldsymbol{\ell}_{\boldsymbol{u}} = \boldsymbol{\ell} + r_{\mathrm{e}}(\nabla_{\mathrm{s}}\boldsymbol{\ell})\boldsymbol{u} + \frac{1}{2}r_{\mathrm{e}}^2(\nabla_{\mathrm{s}}^2\boldsymbol{\ell}) \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + o(r_{\mathrm{e}}^2)$$

∇_{s} surface gradient

$\boldsymbol{\ell}_{\boldsymbol{u}}$ spin at $p_{\boldsymbol{u}}$

elastic surface energy density

$$w_\mathrm{s} := -\frac{3}{2}U_\mathrm{aniso}\varrho\left(\frac{\sigma}{r_\mathrm{e}}\right)^6\langle(\boldsymbol{\ell}\cdot\boldsymbol{\ell}_{\boldsymbol{u}})^2\rangle_{\mathbb{S}^1}$$

ϱ surface number density

elastic surface energy density

$$w_{\mathrm{s}} := -\frac{3}{2} U_{\mathrm{aniso}} \varrho \left(\frac{\sigma}{r_{\mathrm{e}}}\right)^{6} \langle (\boldsymbol{\ell} \cdot \boldsymbol{\ell}_{\boldsymbol{u}})^{2} \rangle_{\mathbb{S}^1}$$

ϱ surface number density

Evaluating the average,

$$\langle \boldsymbol{u} \otimes \boldsymbol{u} \rangle_{\mathbb{S}^1} = \frac{1}{2} (\mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}) \qquad (\nabla_{\mathrm{s}} \boldsymbol{\ell})^{\mathsf{T}} \boldsymbol{\ell} = \mathbf{0}$$

$\boldsymbol{\nu}$ outer unit normal to $\mathscr{S}$

elastic surface energy density

$$w_{\mathrm{s}} := -\frac{3}{2}U_{\mathrm{aniso}}\varrho\left(\frac{\sigma}{r_{\mathrm{e}}}\right)^{6}\langle(\boldsymbol{\ell}\cdot\boldsymbol{\ell}_{\boldsymbol{u}})^{2}\rangle_{\mathbb{S}^{1}}$$

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$\boldsymbol{\nu}$ outer unit normal to $\mathscr{S}$

$$w_{\mathrm{s}} = \frac{k}{2}|\nabla_{\mathrm{s}}\boldsymbol{\ell}|^{2} + \text{constant} \qquad k := \frac{3}{2}\varrho U_{\mathrm{aniso}}r_{\mathrm{e}}^{2}\left(\frac{\sigma}{r_{\mathrm{e}}}\right)^{6}$$

elastic surface energy density

$$w_{\mathrm{s}} := -\frac{3}{2} U_{\mathrm{aniso}} \varrho \left(\frac{\sigma}{r_{\mathrm{e}}}\right)^6 \langle (\boldsymbol{\ell} \cdot \boldsymbol{\ell}_{\boldsymbol{u}})^2 \rangle_{\mathbb{S}^1}$$

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$\boldsymbol{\nu}$ outer unit normal to $\mathscr{S}$

$$w_{\mathrm{s}} = \frac{k}{2} |\nabla_{\mathrm{s}} \boldsymbol{\ell}|^2 + \text{constant} \qquad k := \frac{3}{2} \varrho U_{\mathrm{aniso}} r_{\mathrm{e}}^2 \left(\frac{\sigma}{r_{\mathrm{e}}}\right)^6$$

A similar conclusion was also arrived at via Monte Carlo simulations on tori by Selinger, Konya, Travesset & Selinger (2011).

continuum limit

Thus $\nabla_{\mathrm{s}}\boldsymbol{n}$ is the natural measure of local distortion in the continuum limit and W_{s} should be taken as a quadratic invariant function of the tensor $\nabla_{\mathrm{s}}\boldsymbol{n}$

$$
\begin{aligned}
W_{\mathrm{s}}(\boldsymbol{n}, \nabla_{\mathrm{s}}\boldsymbol{n}) :=& \frac{1}{2}k_1(\operatorname{div}_{\mathrm{s}}\boldsymbol{n})^2 + \frac{1}{2}k_2(\boldsymbol{n}\cdot\operatorname{curl}_{\mathrm{s}}\boldsymbol{n})^2 + \frac{1}{2}k_3|\boldsymbol{n}\times\operatorname{curl}_{\mathrm{s}}\boldsymbol{n}|^2 \\
&+ \frac{1}{2}(k_2+k_{24})\left[\operatorname{tr}(\nabla_{\mathrm{s}}\boldsymbol{n})^2 - (\operatorname{div}_{\mathrm{s}}\boldsymbol{n})^2\right]
\end{aligned}
$$

continuum limit

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$$W_{\mathrm{s}}(\boldsymbol{n},\nabla_{\mathrm{s}}\boldsymbol{n}) := \frac{1}{2}k_1(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{n})^2 + \frac{1}{2}k_2(\boldsymbol{n}\cdot\mathrm{curl}_{\mathrm{s}}\,\boldsymbol{n})^2 + \frac{1}{2}k_3|\boldsymbol{n}\times\mathrm{curl}_{\mathrm{s}}\,\boldsymbol{n}|^2 + \frac{1}{2}(k_2+k_{24})\left[\mathrm{tr}(\nabla_{\mathrm{s}}\boldsymbol{n})^2 - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{n})^2\right]$$

k's surface elastic moduli

continuum limit

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\end{aligned}
$$

k's surface elastic moduli

undistorted state

$$
\boldsymbol{n}\cdot\boldsymbol{\nu}\equiv 0 \quad\Rightarrow\quad (\nabla_{\mathrm{s}}\boldsymbol{n})^{\mathsf{T}}\boldsymbol{\nu}+(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n}\equiv\mathbf{0}
$$

$\nabla_{\mathrm{s}}\boldsymbol{\nu}$ curvature tensor

continuum limit

Thus $\nabla_s \boldsymbol{n}$ is the natural measure of local distortion in the continuum limit and W_s should be taken as a quadratic invariant function of the tensor $\nabla_s \boldsymbol{n}$

$$W_s(\boldsymbol{n}, \nabla_s \boldsymbol{n}) := \frac{1}{2} k_1 (\operatorname{div}_s \boldsymbol{n})^2 + \frac{1}{2} k_2 (\boldsymbol{n} \cdot \operatorname{curl}_s \boldsymbol{n})^2 + \frac{1}{2} k_3 |\boldsymbol{n} \times \operatorname{curl}_s \boldsymbol{n}|^2 + \frac{1}{2} (k_2 + k_{24}) \left[\operatorname{tr}(\nabla_s \boldsymbol{n})^2 - (\operatorname{div}_s \boldsymbol{n})^2\right]$$

k's surface elastic moduli

undistorted state

$$\boldsymbol{n} \cdot \boldsymbol{\nu} \equiv 0 \quad \Rightarrow \quad (\nabla_s \boldsymbol{n})^{\mathsf{T}} \boldsymbol{\nu} + (\nabla_s \boldsymbol{\nu}) \boldsymbol{n} \equiv \mathbf{0}$$

$\nabla_s \boldsymbol{\nu}$ curvature tensor

$$\nabla_s \boldsymbol{n} = \mathbf{0} \quad \Rightarrow \quad (\nabla_s \boldsymbol{\nu}) \boldsymbol{n} = \mathbf{0}$$

W_s is ***not*** minimized by letting $\nabla_s \boldsymbol{n} = \mathbf{0}$

director representation

$$\boldsymbol{n} = \cos\alpha\,\boldsymbol{u} + \sin\alpha\,\boldsymbol{u}_\perp$$

director representation

$$\boldsymbol{n} = \cos\alpha\,\boldsymbol{u} + \sin\alpha\,\boldsymbol{u}_\perp$$

$$\boldsymbol{u}_\perp := \boldsymbol{\nu} \times \boldsymbol{u} = \mathbf{N}\boldsymbol{u}$$

$\mathbf{N}$ skew-symmetric tensor associated with $\boldsymbol{\nu}$

$(\boldsymbol{u}, \boldsymbol{u}_\perp, \boldsymbol{\nu})$ local co-ordinate basis

director representation

$$\boldsymbol{n} = \cos\alpha\,\boldsymbol{u} + \sin\alpha\,\boldsymbol{u}_\perp$$

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$\mathbf{N}$ skew-symmetric tensor associated with $\boldsymbol{\nu}$

$(\boldsymbol{u}, \boldsymbol{u}_\perp, \boldsymbol{\nu})$ local co-ordinate basis

parallel transport

The unit vector $\boldsymbol{n}$ is parallel transported along a curve $\mathscr{C}$ whenever it is seen as immobile in a frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ whose spin $\boldsymbol{\Omega}$ is everywhere ***tangential***, $\boldsymbol{\Omega} \cdot \boldsymbol{\nu} \equiv 0$.

director representation

$$\boldsymbol{n} = \cos\alpha\,\boldsymbol{u} + \sin\alpha\,\boldsymbol{u}_\perp$$

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$$\alpha' = \boldsymbol{w} \cdot \boldsymbol{t}$$

director representation

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$$\alpha' = \boldsymbol{w} \cdot \boldsymbol{t}$$

$\boldsymbol{t}$ unit tangent vector to $\mathscr{C}$

$'$ arc-length derivative

$\boldsymbol{w} := (\nabla_{\mathrm{s}}\boldsymbol{u}_\perp)^{\mathsf{T}}\boldsymbol{u}$ spin connection

parallel connection

The ***undistorted*** state is obtained by requiring that a given director $\boldsymbol{n}$ be parallel transported in all directions:

$$(\nabla_{\mathrm{s}}\boldsymbol{n})_{\mathrm{u}} := -\boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n}$$

parallel connection

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$$(\nabla_{\mathrm{s}}\boldsymbol{n})_{\mathrm{u}} := -\boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n}$$

gradient decomposition

$$\nabla_{\mathrm{s}}\boldsymbol{n} = \boldsymbol{n}_{\perp} \otimes (\nabla_{\mathrm{s}}\alpha - \boldsymbol{w}) - \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n}$$

$$\boldsymbol{w} := (\nabla_{\mathrm{s}}\boldsymbol{u}_{\perp})^{\mathsf{T}}\boldsymbol{u}.$$

parallel connection

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Equivalently, we can write

$$\nabla_{\mathrm{s}}\boldsymbol{n} = \mathrm{D}\boldsymbol{n} - \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n}$$

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Equivalently, we can write

$$\nabla_{\mathrm{s}}\boldsymbol{n} = \mathrm{D}\boldsymbol{n} - \boldsymbol{\nu}\otimes(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n}$$

covariant gradient

$$\mathrm{D}\boldsymbol{n} := (\mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu})\nabla_{\mathrm{s}}\boldsymbol{n} = \boldsymbol{n}_{\perp}\otimes(\nabla_{\mathrm{s}}\alpha - \boldsymbol{w})$$

surface irrotationality

$$\operatorname{curl}_{\mathrm{s}} \nabla_{\mathrm{s}} \alpha = \mathbf{N}(\nabla_{\mathrm{s}} \boldsymbol{\nu}) \nabla_{\mathrm{s}} \alpha$$

surface irrotationality

$$\operatorname{curl}_{\mathrm{s}} \nabla_{\mathrm{s}}\alpha = \mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\nabla_{\mathrm{s}}\alpha$$

$$\boldsymbol{a}\cdot\boldsymbol{\nu} \equiv 0 \qquad \boldsymbol{a} = \nabla_{\mathrm{s}}\alpha \quad \Leftrightarrow \quad \operatorname{curl}_{\mathrm{s}} \boldsymbol{a} = \mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{a}$$

surface irrotationality

$$\mathrm{curl_s}\, \nabla_{\mathrm{s}}\alpha = \mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\nabla_{\mathrm{s}}\alpha$$

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surface distortion

$$\mathrm{curl_s}\, \boldsymbol{w} = K\boldsymbol{\nu} + \mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{w}$$

K Gaussian curvature of $\mathscr{S}$

surface irrotationality

$$\operatorname{curl}_{\rm s} \nabla_{\rm s}\alpha = \mathbf{N}(\nabla_{\rm s}\boldsymbol{\nu})\nabla_{\rm s}\alpha$$

$$\boldsymbol{a}\cdot\boldsymbol{\nu} \equiv 0 \qquad \boldsymbol{a} = \nabla_{\rm s}\alpha \quad \Leftrightarrow \quad \operatorname{curl}_{\rm s}\boldsymbol{a} = \mathbf{N}(\nabla_{\rm s}\boldsymbol{\nu})\boldsymbol{a}$$

surface distortion

$$\operatorname{curl}_{\rm s}\boldsymbol{w} = K\boldsymbol{\nu} + \mathbf{N}(\nabla_{\rm s}\boldsymbol{\nu})\boldsymbol{w}$$

K Gaussian curvature of $\mathscr{S}$

energy splitting

$$|\nabla_{\rm s}\boldsymbol{n}|^2 = \underbrace{|\nabla_{\rm s}\alpha - \boldsymbol{w}|^2}_{\text{distortion energy}} + \underbrace{\boldsymbol{n}\cdot(\nabla_{\rm s}\boldsymbol{\nu})^2\boldsymbol{n}}_{\text{curvature field}}$$

surface irrotationality

$$\mathrm{curl_s}\,\nabla_\mathrm{s}\alpha = \mathbf{N}(\nabla_\mathrm{s}\boldsymbol{\nu})\nabla_\mathrm{s}\alpha$$

$$\boldsymbol{a}\cdot\boldsymbol{\nu}\equiv 0 \qquad \boldsymbol{a}=\nabla_\mathrm{s}\alpha \quad \Leftrightarrow \quad \mathrm{curl_s}\,\boldsymbol{a}=\mathbf{N}(\nabla_\mathrm{s}\boldsymbol{\nu})\boldsymbol{a}$$

surface distortion

$$\mathrm{curl_s}\,\boldsymbol{w} = K\boldsymbol{\nu} + \mathbf{N}(\nabla_\mathrm{s}\boldsymbol{\nu})\boldsymbol{w}$$

K Gaussian curvature of $\mathscr{S}$

energy splitting

$$|\nabla_\mathrm{s}\boldsymbol{n}|^2 = \underbrace{|\nabla_\mathrm{s}\alpha - \boldsymbol{w}|^2}_{\text{distortion energy}} + \underbrace{\boldsymbol{n}\cdot(\nabla_\mathrm{s}\boldsymbol{\nu})^2\boldsymbol{n}}_{\text{curvature field}}$$

$$K\neq 0 \quad\Rightarrow\quad \boldsymbol{w}\neq\nabla_\mathrm{s}\alpha \quad\Rightarrow\quad \text{distortion energy} \neq 0$$

no saddle-splay analogue

$$\nabla_{\mathrm{s}}\boldsymbol{n} = \mathrm{D}\boldsymbol{n} - \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n} \quad \Rightarrow \quad \mathrm{tr}(\nabla_{\mathrm{s}}\boldsymbol{n})^2 - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{n})^2 \equiv 0$$

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$$\nabla_{\mathrm{s}}\boldsymbol{n} = \mathrm{D}\boldsymbol{n} - \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n} \quad \Rightarrow \quad \mathrm{tr}(\nabla_{\mathrm{s}}\boldsymbol{n})^2 - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{n})^2 \equiv 0$$

$$W_{\mathrm{s}}(\boldsymbol{n}, \nabla_{\mathrm{s}}\boldsymbol{n}) = \frac{1}{2}k_1(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{n})^2 + \frac{1}{2}k_2(\boldsymbol{n} \cdot \mathrm{curl}_{\mathrm{s}}\,\boldsymbol{n})^2 + \frac{1}{2}k_3|\boldsymbol{n} \times \mathrm{curl}_{\mathrm{s}}\,\boldsymbol{n}|^2$$

no saddle-splay analogue

$$\nabla_s \boldsymbol{n} = \mathrm{D}\boldsymbol{n} - \boldsymbol{\nu} \otimes (\nabla_s \boldsymbol{\nu})\boldsymbol{n} \quad \Rightarrow \quad \mathrm{tr}(\nabla_s \boldsymbol{n})^2 - (\mathrm{div}_s\, \boldsymbol{n})^2 \equiv 0$$

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curvature field

$$\nabla_s \boldsymbol{\nu} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

$\boldsymbol{e}_1,\ \boldsymbol{e}_2$ principal directions of curvature

$\sigma_1,\ \sigma_2$ principal curvatures

no saddle-splay analogue

$$\nabla_{\mathrm{s}}\boldsymbol{n} = \mathrm{D}\boldsymbol{n} - \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{n} \quad \Rightarrow \quad \mathrm{tr}(\nabla_{\mathrm{s}}\boldsymbol{n})^2 - (\mathrm{div}_{\mathrm{s}}\, \boldsymbol{n})^2 \equiv 0$$

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curvature field

$$\nabla_{\mathrm{s}}\boldsymbol{\nu} = \sigma_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

$\boldsymbol{e}_1, \ \boldsymbol{e}_2$ principal directions of curvature

$\sigma_1, \ \sigma_2$ principal curvatures

$$\boldsymbol{n} = \cos\vartheta\, \boldsymbol{e}_1 + \sin\vartheta\, \boldsymbol{e}_2$$

$$
\begin{aligned}
W_{\mathrm{s}} =& \frac{1}{2} k_1 (\boldsymbol{n}_{\perp} \cdot \nabla_{\mathrm{s}} \vartheta)^2 + \frac{1}{2} k_2 (\sigma_2 - \sigma_1)^2 (\boldsymbol{n} \cdot \boldsymbol{e}_1)^2 (\boldsymbol{n} \cdot \boldsymbol{e}_2)^2 \\
&+ \frac{1}{2} k_3 \{ [\sigma_1 (\boldsymbol{n} \cdot \boldsymbol{e}_1)^2 + \sigma_2 (\boldsymbol{n} \cdot \boldsymbol{e}_2)^2] + (\boldsymbol{n} \cdot \nabla_{\mathrm{s}} \vartheta)^2 \}
\end{aligned}
$$

$$W_{\rm s} = \frac{1}{2}k_1(\boldsymbol{n}_\perp \cdot \nabla_{\rm s}\vartheta)^2 + \frac{1}{2}k_2(\sigma_2 - \sigma_1)^2(\boldsymbol{n}\cdot\boldsymbol{e}_1)^2(\boldsymbol{n}\cdot\boldsymbol{e}_2)^2$$
$$+\frac{1}{2}k_3\{[\sigma_1(\boldsymbol{n}\cdot\boldsymbol{e}_1)^2 + \sigma_2(\boldsymbol{n}\cdot\boldsymbol{e}_2)^2] + (\boldsymbol{n}\cdot\nabla_{\rm s}\vartheta)^2\}$$

Ericksen inequalities

$$k_1 \geqq 0 \quad \& \quad k_3 \geqq 0$$

$$W_{\mathrm{s}} = \frac{1}{2} k_1 (\boldsymbol{n}_\perp \cdot \nabla_{\mathrm{s}} \vartheta)^2 + \frac{1}{2} k_2 (\sigma_2 - \sigma_1)^2 (\boldsymbol{n} \cdot \boldsymbol{e}_1)^2 (\boldsymbol{n} \cdot \boldsymbol{e}_2)^2$$
$$+ \frac{1}{2} k_3 \{ [\sigma_1 (\boldsymbol{n} \cdot \boldsymbol{e}_1)^2 + \sigma_2 (\boldsymbol{n} \cdot \boldsymbol{e}_2)^2] + (\boldsymbol{n} \cdot \nabla_{\mathrm{s}} \vartheta)^2 \}$$

Ericksen inequalities

$$k_1 \geqq 0 \quad \& \quad k_3 \geqq 0$$

curvature field equilibria

$$\underbrace{\sin 2\vartheta = 0}_{\text{curvature lines}} \quad \& \quad \underbrace{\cos 2\vartheta = \frac{k_3}{k_2 - k_3} \frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2}}_{\text{spiralling lines}} \in [-1, 1]$$

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Discussion

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