The Grothendieck-Teichmüller Lie algebra and other animals

I. The graded Grothendieck-Teichmüller Lie algebra

Definition of the n-strand braid Lie algebra.

Let $\mathrm{Lie}\, P_n$ be the Lie algebra given by:

Generators: x_{ij}, $1 \leq i < j \leq 5$

Relations: $[x_{ij}, x_{ij} + x_{ik} + x_{jk}] = 0$

$$\sum_{j \neq i} x_{ij} = 0$$

$$[x_{ij}, x_{kl}] = 0 \text{ if } \{i, j\} \cap \{k, l\} = \emptyset$$

In particular, $\mathrm{Lie}\, P_4$ is free on x_{12}, x_{23} and $\mathrm{Lie}\, P_5$ is generated by $x_{12}, x_{23}, x_{34}, x_{45}, x_{15}$.

Definition of Grothendieck-Teichmüller Lie algebra.

The algebra $\mathfrak{grt}$ is given by

$$\mathfrak{grt} = \Big\{ f \in \mathrm{Lie}_{\geq 3}[x, y] \,|$$

$$\text{(I)} \quad f(x, y) + f(y, x) = 0$$

$$\text{(II)} \quad f(x, y) + f(z, x) + f(y, z) = 0 \text{ if } x + y + z = 0$$

$$\text{(III)} \quad f(x_{12}, x_{23}) + f(x_{34}, x_{45}) + f(x_{51}, x_{12}) + f(x_{23}, x_{34}) + f(x_{45}, x_{51}) = 0 \quad \text{in} \quad \mathrm{Lie}\, P_5 \Big\}$$

Theorem 1. (Furusho) *Relation (III) implies (I) and (II), so is not needed in the definition.*

Elements of $\mathfrak{grt}$ are **Lie associators** and the pentagon relation (III) is the **Lie associator relation**.

Definition of the Poisson Lie bracket.

Every $f \in \mathrm{Lie}[x, y]$ yields a derivation D_f of $\mathrm{Lie}[x, y]$:

$$D_f(x) = 0, \quad D_f(y) = [y, f].$$

We can put a Lie bracket on $\mathrm{Lie}[x, y]$ called the Poisson or Ihara bracket via

$$\{f, g\} = [f, g] + D_f(g) - D_g(f),$$

corresponding to bracketing derivations:

$$[D_f, D_g] = D_{\{f,g\}}.$$

In this way, we can view

$$\mathfrak{grt} \subset \mathrm{DerLie}[x, y].$$

Theorem 2. (Ihara) $\mathfrak{grt}$ *forms a Lie algebra under the Poisson bracket.*

Equivalently, this means that under the inclusion

$$\mathfrak{grt} \hookrightarrow \mathrm{DerLie}[x, y]$$
$$f \mapsto \widetilde{D}_f$$

where setting $z = -x - y$,

$$\widetilde{D}_f(x) = [x, f(z, x)], \quad \widetilde{D}_f(y) = [y, f(z, y)],$$

$\mathfrak{grt}$ is a Lie subalgebra of $\mathrm{DerLie}[x, y]$.

II. Kashiwara-Vergne Lie algebra

Original Kashiwara-Vergne problem:

Characterize pairs $A, B \in \mathrm{Lie}[x, y]$ such that

$$x + y - ch(y, x) = (1 - e^{-\mathrm{ad}(x)})A + (e^{\mathrm{ad}(y)} - 1)B \in \mathrm{Lie}[x, y]$$

$$\mathrm{div}(A, B) = \frac{1}{2}\mathrm{tr}\big(b(x) + b(y) - b(\mathrm{ch}(x, y))\big) \in Tr_2$$

where

- $\mathrm{ch}(x, y) = x + y + \frac{1}{2}[x, y] + \cdots =$ Campbell-Hausdorff law

- $Tr_2 = \mathbb{Q}\langle x, y\rangle / \langle ab - ba\rangle$ (not commutative!) Trlic means mod cyclic permutation of the letters: $xxyy = yxxy = yyxx = xyyx$, $xyxy = yxyx$.

- $\mathrm{tr} : \mathbb{Q}\langle x, y\rangle \to Tr_2$ quotient map

- $b(x) = \sum_{k \geq 1} \frac{B_k}{k!} x^k$

- $\mathrm{div}(A, B) = \mathrm{tr}(A_x x + B_y y) \in Tr_2$,
  where $A = A_x x + A_y y$ and $B = B_x x + B_y y$.

The KV-problem was solved by Alekseev-Meinrenken in 2006.

Tangent KV-problem:

Find pairs $A, B \in \mathrm{Lie}[x, y]$ (homogeneous of degree n, say) such that

$$[x, A] + [y, B] = 0 \in \mathrm{Lie}[x, y] \tag{1}$$

$$div(A, B) = \mathrm{tr}\big(x^n + y^n - (x + y)^n\big) \in Tr_2. \tag{2}$$

Definition. Let $\mathfrak{kv}$ be the space of solutions.

Theorem 4. (Alekseev-Torossian) *Under the inclusion* $\mathfrak{kv} \subset \mathrm{DerLie}[x, y]$ *via the map* $(A, B) \mapsto D_{A,B}$ *defined by*

$$D_{A,B}(x) = [x, A], \quad D_{A,B}(y) = [y, B],$$

$\mathfrak{kv}$ *forms a Lie subalgebra of* $\mathrm{DerLie}[x, y]$.

III. The double shuffle Lie algebra $\mathfrak{ds}$

Let $y_i = x^{i-1}y$ for $i \geq 1$, so any word in x, y ending in y can be written $y_{i_1} \cdots y_{i_r}$. For any $f \in \mathrm{Lie}_n[x, y]$, let $\pi_y(f)$ denote the projection onto its words ending in y (rewritten in the y_i), and set

$$\tilde{f} = \pi_y(f) + \frac{(-1)^{n-1}}{n}(f|x^{n-1}y)y_1^n.$$

Definition of the double shuffle Lie algebra.

The Lie algebra $\mathfrak{ds}$ is defined by

$$\mathfrak{ds} = \{f \in \mathrm{Lie}_{\geq 3}[x, y] \mid \Delta_*(f) = f \otimes 1 + 1 \otimes f\},$$

where Δ_* is the coproduct defined on $\mathbb{Q}\langle y_i \rangle$ by

$$\Delta_*(y_i) = \sum_{k+l=i} y_k \otimes y_l.$$

Theorem 3. (Racinet) *The vector space* $\mathfrak{ds}$ *is a Lie algebra under the Poisson bracket.*

Equivalent definition of $\mathfrak{ds}$:

Let the **stuffle product** $st((a_1,\dots,a_r),(j_1,\dots,j_s))$ for sequences of strictly positive integers be defined by

$$st((),(a_1,\dots,a_r)) = st((a_1,\dots,a_r),()) = (a_1,\dots,a_r),$$

$$st((a_1,\dots,a_r),(b_1,\dots,b_s)) =$$

$$\Big\{a_1\cdot st((a_2,\dots,a_r),(b_1,\dots,b_s)),$$

$$b_1\cdot st((a_1,\dots,a_r),(b_2,\dots,b_s)),$$

$$(a_1+b_1)\cdot st((a_2,\dots,a_r),(b_2,\dots,b_s))\Big\}.$$

Examples:

$$st((2),(3)) = \{(2,3),(3,2),(5)\}$$

$$st((2),(1,3)) = \{(2,1,3),(1,2,3),(1,3,2),(3,3),(1,5)\}.$$

For any sequence $\mathbf{a}=(a_1,\dots,a_r)$, write $y_{\mathbf{a}} = y_{a_1}\cdots y_{a_r}$. Then

$$\mathfrak{ds} = \{f\in \mathrm{Lie}_{\geq 3}[x,y] \mid \sum_{\mathbf{c}\in st(\mathbf{a},\mathbf{b})} (f|y_{\mathbf{c}}) = 0\ \forall\ (\mathbf{a},\mathbf{b})\},$$

where $(f|y_{\mathbf{c}})$ denotes the coefficient of the word $y_{\mathbf{c}}$ in f.

IV. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ **and Deligne-Ihara Lie algebra** DI_ℓ

Let $\pi = \pi_1(\mathbb{P}^1 - \{0,1,\infty\})$, $\pi^{(\ell)}$ denote the pro-ℓ completion.

Let $\left(\pi^{(\ell)}\right)^i$ denote the groups of the descending central sequence

$$\left(\pi^{(\ell)}\right)^0 = \pi^{(\ell)}, \quad \left(\pi^{(\ell)}\right)^i = \left(\pi^{(\ell)}, \left(\pi^{(\ell)}\right)^{i-1}\right).$$

Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and set

$$G^i_{\mathbb{Q}} = \mathrm{Ker}\left(G_{\mathbb{Q}} \to \mathrm{Out}\left(\pi^{(\ell)}/\left(\pi^{(\ell)}\right)^i\right)\right).$$

This is an ascending filtration

$$G^0_{\mathbb{Q}} \subset G^1_{\mathbb{Q}} \subset G^2_{\mathbb{Q}} \subset \cdots$$

We have

$$G^0_{\mathbb{Q}} = G_{\mathbb{Q}}, \quad G^1_{\mathbb{Q}} = \{\sigma \in G_{\mathbb{Q}} \mid \chi_\ell(\sigma) = 1\}$$

The successive quotients $G^i_{\mathbb{Q}}/G^{i-1}_{\mathbb{Q}}$ are $\mathbb{Z}_\ell$-modules.

Definition of the Deligne-Ihara Lie algebra.

The space

$$\Big(\bigoplus_{i\geq 1} G^i_{\mathbb{Q}}/G^{i-1}_{\mathbb{Q}}\Big)$$

is a Lie algebra over $\mathbb{Z}_\ell$, with Lie bracket coming from commutators $\sigma\tau\sigma^{-1}\tau^{-1}$ in $G_{\mathbb{Q}}$. We write DI_ℓ for the tensor product of this Lie algebra with $\mathbb{Q}_\ell$.

V. Fundamental Lie algebra of mixed Tate motives

Let us give the easiest possible way of grasping the Tannakian category of *mixed Tate motives over* $\mathbb{Z}$ (ignoring loads of deep math).

Let $\mathfrak{f} = \mathrm{Lie}[s_3, s_5, s_7, \ldots]$ be the free Lie algebra with one generator in each odd weight.

Let $U = exp(\mathfrak{f})$ be the associated pro-unipotent group.

The grading on $\mathfrak{f}$ induces a $\mathbb{G}_m$-action on $\mathfrak{f}$ given by

$$\lambda(x) = \lambda^n(x) \ \ \forall \ \ x \in \mathfrak{f}_n.$$

The $\mathbb{G}_m$-action on $\mathfrak{f}$ induces a $\mathbb{G}_m$-action on U.

The category MTM is equivalent to the category of representations of the pro-algebraic group $G = U \rtimes \mathbb{G}_m$. The mixed Tate motives that we study in general come from *geometric constructions*, such as linearized π_1's or relative cohomology groups of moduli spaces.

The group G is known as the *fundamental group* of the category MM, and $\mathfrak{f}$ is its *fundamental Lie algebra.*

This Lie algebra $\mathfrak{f}$ acts on $\mathrm{Lie}[x, y]$ because $\mathrm{Lie}[x, y]$ arises as a mixed Tate motive as it is the linearization of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$.

Connections between the five Lie algebras

Main Conjecture. *The four* $\mathbb{Q}$*-Lie algebras* $\mathfrak{grt}$, $\mathfrak{ds}$, $\mathfrak{kv}$ *are all isomorphic to* $\mathfrak{f}$, *and tensored with* $\mathbb{Q}_\ell$, *they are also all isomorphic with* DI_ℓ.

Progress to date: We have injections

$$\mathfrak{f} \hookrightarrow \mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{kv}$$

and

$$DI_\ell \simeq \mathfrak{f} \otimes \mathbb{Q}_\ell.$$

Injection $\mathfrak{grt} \hookrightarrow \mathfrak{ds}$ (Furusho 2010) Proof is based on double polylogarithms that satisfy a "lifted" double shuffle and also a pentagon relation.

Injection $\mathfrak{grt} \hookrightarrow \mathfrak{kv}$ (Alexseev-Torossian 2009?) Proof is based on combinatorial calculation to show that $\mathfrak{grt}$ satisfies $\mathfrak{kv}$ div relation. Was shown long ago by Drinfeld that $\mathfrak{grt}$ satisfies $\mathfrak{kv}$ special relation. (Pentagon fundamental.)

Injection $\mathfrak{ds} \hookrightarrow \mathfrak{kv}$ (Schneps 2010) Proof is based on a reformulation (!) of the defining properties of solutions to the tangent KV problem, and combinatorial properties deduced from double shuffle (work of Ecalle).

Injection $\mathfrak{f} \hookrightarrow \mathfrak{grt}$ (Brown 2010) Proof works by comparing MTM to the subcategory MZ of *motivic multiple zetas*, defined by Goncharov, who proved that the motivic multiple zetas satisfy the associator relations. This translates directly as saying that there is a surjection $\mathcal{U}\mathfrak{grt}^* \to MZ$. Brown proved that $MTM = MZ$; thus $\mathrm{Lie}\pi_1(MTM) = \mathrm{Lie}\pi_1(MZ) = \mathfrak{f}$, so the surjection leads in the dual to the desired injection.

Isomorphism $DI_\ell \simeq \mathfrak{f}$ (follows from Brown 2010) Proof works as follows. Hain-Matsumoto proved that DI_ℓ is generated by one element in each odd rank. Ihara proved that

$$DI_\ell \hookrightarrow \mathfrak{grt}_\ell,$$

because the injection

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT} \simeq \mathrm{Out}^*_{S_5}\widehat{\Gamma}_{0,5}$$

descends to a Lie homomorphism

$$DI_\ell \hookrightarrow \mathfrak{grt} \simeq \mathrm{Der}^s_{S_n}(\mathrm{Lie}\, P_5).$$

The fact that motivic multizetas MZ satisfy the associator relations shows that the dual of the associated Lie coalgebra $\mathfrak{nmz}^* \hookrightarrow \mathfrak{grt}$. The images in $\mathfrak{grt}$ are the same, and Brown proved that $\mathfrak{nmz}^* \simeq \mathfrak{f}$.