The this talk consists of two parts.

In Part 1 I shall give an overview of a recent paper with Claude Warnick (former JRF at Queens’ College, now at Edmonton, Alberta)

The helical phase of chiral nematic liquid crystals as the Bianchi VII(0) group manifold

In part 2 I shall describe some unpublished work relating the geometry of nematics to Lorentzian metrics in $2+1$ spacetime dimensions. In particular I shall

- relate **Gravitational Kinks** in general relativity to **topological defects** in nematics

- and the surfaces of droplets in nematics to null hypersurface in $2+1$ dimensional Lorentzian Spactimes.
The motivation for both parts is to see to what extent the theoretical ideas and techniques developed in Differential Geometry and Einstein’s theory General Relativity may be applied to condensed matter systems. General Relativity may be thought of as combining optics with geometry to give a complete description of the interaction of light with gravity. It is well known that liquid crystals exhibit a rich set of geometrical configurations and optical properties.

Liquid Crytsals also exhibit a rich structure of tolopogical phenomenon and in the case of nematics, there is a close link with some basic topological ideas in Lorentzian geometry.
In particular chiral nematics, in which the order parameter is a direction field or director $n$ such that

$$n \cdot n = 1, \quad n \equiv -n$$

are bi-refringent and minimize energy by adopting geometrically non-trivial configurations.
Warnick and I found that the helical phase of a chiral nematic exhibits the same optical properties as a three dimensional spatially homogeneous but anisotropic curved space previously studied by general relativists under the name of a Bianchi $VII_0$ universe as a model for possible anisotropies in the expansion of our universe*. These anisotropies show up in the Cosmic Background Radiation, CMB currently being observed by the Planck satellite.

In principle this would allow one to construct an analogue model of such a universe, an anisotropic generalisation of James Clerk Maxwell’s well known Fish Eye Lens.
In Part 1 I shall

• introduce, in as simple a fashion as possible, some of the underlying geometrical ideas concerning curved spaces as used in general relativity. Of particular importance are their symmetries and this leads one to introduce what are called Lie Groups and Bianchi’s classification of three dimensional Lie Groups.

• Recall some basic properties of chiral nematic liquid crystals and their associated Joets-Ribotta metric

• Put the two together
Our story begins when Lobachevsky and Bolyai introduced a Non-Euclidean Geometry in which Euclid’s Fifth Postulate does not hold. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
Helmholtz classified all geometry's allowing the Free mobility of rigid bodies and obtained the three Congruence Geometries which are homogeneous (i.e. admit three not-necessarily commuting) translation symmetries and isotropic (i.e admit full $SO(3)$ rotational symmetry about every point).

- Spherical space $S^3, k = 1$
- Flat or Euclidean space $E^3, k = 0$
- Hyperbolic or Lobachevsky space $H^3, k = -1$
Straight lines in these geometries are **geodesics**, that is curves $\gamma$ extremizing length $\int_{\gamma} ds$, where $ds$ is the line element

$$ds^2 = n^2(x)dx \cdot dx = n^2(x)|dx|^2 = n^2(x)(dx^2 + dy^2 + dz^2).$$

The line element is obviously rotationally symmetric about the origin $x = 0$. It is less obvious (if $k = 0$), but nevertheless true even if $k \neq 0$ that a suitable translation exists (i.e. coordinate transformation $\tilde{x} = \tilde{x}(x)$) showing that the line element is rotationally symmetric about every point.
The analogy with Fermat’s Principle of Least Time is clear where \( n(x) \) corresponds to the refractive index of an isotropic optical medium. The case \( k = 1 \) gives Maxwell’s Fish Eye Lens this is the basis of the Luneburg Lens.

For a \textbf{locally anisotropic optical medium} the line element must be generalised

\[ ds^2 = g_{ij}(x)dx^i dx^j, \quad i = 1, 2, 3 \]

This corresponds to \textbf{Riemann’s generalisation of Non-Euclidean Geometry} to \textbf{Riemannian Geometry} which is also used in Einstein’s Theory of General Relativity.
Riemannian Geometry gives up isotropy so that at every point there are privileged directions such as in a crystal, it is still possible to retain homogeneity. In other words, we insist on invariance under three independent translations, but, just as with $S^3$ and $H^3$, we do not insist that all translations commute. If we did, we would be back to the case of ordinary (pre-quasi-)crystallography. We shall demand that composing two translations gives another translation * the translations form three parameter group, called a Three dimensional Lie Group. These groups were invented and studied by Lie and used by Killing for strengthening and generalising the classification by Helmholtz of Congruence Geometries and, as discovered by Bianchi, there are 9 classes.

*This is actually not true for $H^3$ but it is true for $S^3$. 
The methods developed by Lie, Killing and Bianchi are now standard in General Relativity when discussing the symmetries of spacetimes.
The Frank-Oseen Free energy for a chiral nematic in the one constant approximation is, up to a boundary term

\[ F[n] = \frac{1}{2} \int \left( |\nabla q_n|^2 - \lambda (n \cdot n - 1) \right) d^3 x, \quad \nabla^q_{ij} n_j = \nabla n_j + q \epsilon_{ijk} n_k. \]

The bulk term would be minimized if \( \nabla^q_{ij} n_j = 0 \). But since \( \nabla^q_{i} \) and \( \nabla^q_{j} \) do not commute for \( i \neq j \) this is inconsistent: the system is frustrated and adopts a compromise configuration such as in the helical phase in which

\[ n(x) = i \cos(pz) + j \sin(pz) \]
The differential operator $\nabla^q_i v_j = \nabla_i v_j + q\epsilon_{ijk}v_k$ acting on a vector field $v_j$ is a known to relativists as a metric preserving covariant derivative with torsion. That is $\nabla_i$ is the usual preserves the flat Euclidean metric $\delta_{ij}$ and $\epsilon_{ijk}$ is the affine connection which since it is totally skew-symmetric is pure torsion and also preserves $\delta_{ij}$. $\nabla^q_i$ provides a rule for parallel transport of vectors along an arbitrary curve.

Einstein and others have made use of such connections in constructing theories unifying gravity with the other forces. They arise in modern supergravity theories.

The failure of the operator to commute means that the connection has curvature.
On a three sphere $S^3 \equiv SU(2)$ of appropriate radius with $\nabla_i$ the Levi-Civita connection preserving the spherical metric and $\epsilon_{ijk}$ the volume form, the connection implements left or right translation and the connection is flat.

This has been made use of * to model the blue phase.

A nematic liquid crystal seen through cross polarisers. It appears dark in places where the director is oriented along one of the polarizer axes. The points where the dark areas converge are disclinations.
Chiral nematic in its helical phase or Grandjean texture seen through cross polarisers. The director is parallel to the substrate plane and the axis perpendicular to it. The white lines are disclinations.
The helical phase is invariant under the three group of transformations consisting of translations in the horizontal coordinates \((x, y)\) parametrized by \((\nu, \eta)\) and a screw rotation along the vertical direction \(z\) parametrized by \(\zeta\)

\[
\begin{align*}
    x &\rightarrow x \cos(p\zeta) - y \sin(p\zeta) + \nu \\
    y &\rightarrow y \cos(p\zeta) + x \cos(p\zeta) + \eta \\
    z &\rightarrow z + \zeta
\end{align*}
\]

This group is of type \(VII_0\) in Bianchi’s classification. If we forget what happens to \(z\) we see that it corresponds to translations and rotations of the \(x, y\) plane that is the Euclidean group \(E(2)\) *.

*strictly speaking to its universal cover \(E(2)\) unless \(\zeta\) is taken to periodic period \(\frac{2\pi}{p}\)
It is sometimes convenient to represent the this group by a matrix \( M(x, y, z) \) depending on parameters \( (x, y, z) \) which takes the origin \((0, 0, 0)\) to the point \((x, y, z)\)

\[
M(x, y, z) = \begin{pmatrix}
\cos pz & -\sin pz & x \\
\sin pz & \cos pz & y \\
0 & 0 & 1
\end{pmatrix}
\]

One may check that multiplication of these matrices corresponds to composition of translations. Translations in \( x \) and \( y \) commute with each other but not with a screw rotation.
According to Joets and Ribotta * The inverse speed or slowness of a ray moving in direction \( t = \frac{dx}{d|\mathbf{x}|} \) with ordinary \( n_o \) and extraordinary \( n_e \) refractive indices is given by

\[
\sqrt{n_o^2 (t \cdot \mathbf{n})^2 + n_e^2 (t - n (n \cdot t))^2},
\]

and Fermat’s Principle becomes

\[
\delta \int_\gamma \sqrt{n_o^2 (t \cdot \mathbf{n})^2 + n_e^2 (t - n (n \cdot t))^2} |dx| = 0.
\]

The corresponding Joets-Ribotta metric is

\[
ds_o^2 = n_e^2 dx^2 + (n_0^2 - n_e^2) (n \cdot dx)^2.
\]

*A. Joets and R. Ribotta, A geometrical model for the propagation of light rays in an anisotropic inhomogeneous medium *Optics Communications* **107** (1994) 200-204
It should be clear that the Joets-Ribotta metric is invariant under the Bianchi type \(VII_0\) group. To see this one may check that the three differentials (or 1-forms)

\[
\begin{align*}
\lambda^1 &= \cos(pz)dx + \sin(pz)dy = n \cdot dx, \quad d\lambda^1 = \lambda^3 \wedge \lambda^2, \\
\lambda^2 &= \cos(pz)dy - \sin(pz)dx, \quad d\lambda^2 = -\lambda^3 \wedge \lambda^1, \\
\lambda^3 &= pdz, \quad d\lambda^3 = 0.
\end{align*}
\]

are invariant under all three translations and therefore so is the Joets-Ribotta optical metric

\[
ds^2 = n_o^2(\lambda^1)^2 + n_e^2(\lambda^2)^2 + \frac{n_e^2}{p^2}(\lambda^3)^2.
\]

Note that if \(n_e = n_0 = 1\) we recover the flat Euclidean metric

\[
d|x|^2 = (\lambda^1)^2 + (\lambda^2)^2 + \frac{1}{p^2}(\lambda^3)^2
\]
Note that the exterior derivative $d$ is a generalised curl, such that

$$d^2 = 0$$

and $\wedge$ is a generalised cross product so that

$$\text{curl } \lambda^1 = \lambda^3 \times \lambda^2$$

or

$$\partial_i \lambda^1_j - \partial_j \lambda^1_i = \lambda^3_i \lambda^2_j - \lambda^3_j \lambda^2_i$$

As we shall see, the Cartan’s exterior calculus is a very efficient tool for dealing with Maxwell’s equations.
We are now in a position to discuss in detail the optics of the helical phase. There are three increasingly sophisticated levels of treatment.

- Ray Theory (Geometrical Optics)
- Scalar Wave Theory (Ignores Polarization)
- Maxwell’s Equations

In all three cases we can make progress because of the high degree of symmetry since by Noether’s Theorem we have sufficient constants of the motion and separation of variables is also possible. We can easily handle the complexity of Maxwell’s equations using Differential Forms, an extension of Vector Calculus developed by Elie Cartan.
The two commuting horizontal translations give rise to two conserved quasi-momenta, \(k_x\) and \(k_y\) and we find:

\[
\begin{align*}
\dot{n}_e^2x &= \frac{k_x}{2} \left(1 + \frac{n_e^2}{n_o^2}\right) - \frac{k}{2} \left(1 - \frac{n_e^2}{n_o^2}\right) \cos(2pz - \psi), \\
\dot{n}_e^2y &= \frac{k_y}{2} \left(1 + \frac{n_e^2}{n_o^2}\right) - \frac{k}{2} \left(1 - \frac{n_e^2}{n_o^2}\right) \sin(2pz - \psi), \\
\dot{n}_e^4z^2 &= \omega^2n_e^2 - \frac{k^2}{2} \left(1 + \frac{n_e^2}{n_o^2}\right) + \frac{k^2}{2} \left(1 - \frac{n_e^2}{n_o^2}\right) \cos(2pz - \theta).
\end{align*}
\]

where \(\tan \psi = k_y/k_x\), \(k = \sqrt{k_x^2 + k_y^2}\) and \(\tan \theta = 2k_xk_y/(k_x^2 - k_y^2)\). Let us first consider (1). Introducing new constants \(\alpha, \beta\) and defining \(\zeta = pz - \theta/2\), we find \(\zeta\) satisfies the quadrantant pendulum

\[
\dot{\zeta}^2 - \frac{p^2}{n_e^4}(\alpha + \beta \cos(2\zeta)) = 0
\]
The pendulum has two different types of behaviour, depending on the constants $\alpha$ and $\beta$. If $\alpha > |\beta|$, then $|\zeta|$ and hence $|z|$ will increase without bound. This corresponds to a pendulum swinging through complete revolutions. If $\alpha < |\beta|$, then $\zeta$ will oscillate about $2n\pi$ for some integer $n$. This corresponds to the standard libratory motion of a pendulum. Thus we find two behaviours for the rays. Either the rays can penetrate in the $z$-direction or else they are trapped to move between two planes perpendicular to the $z$-axis. Finally, we can consider the other equations of motion, (1) and (1). We can interpret these as saying that the tangent vector of the ray oscillates around an average direction. For rays which are not bounded in $z$, the result is a ‘cork-screw’ curve, similar to a helix. The ‘tightness’ of the spiral is determined by how close $n_e^2/n_o^2$ is to 1.
For wave theory we consider the scalar wave equation

\[ \frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial \Psi}{\partial x^j} \right), \quad g = \det g_{ij}, \quad g^{ik} g_{kj} = \delta^i_j \]

which is covariant w.r.t. to the Joets-Ribotta metric. In the W.K.B. approximation this gives the same rays as above. We find if \( \Psi = e^{i(k_xx + k_y y - \omega t)} F(z) \), then

\[ \frac{1}{n_e^2} \frac{d^2 F}{dz^2} + \left( \omega^2 - \frac{1}{n_o^2} (k_x \cos(pz) + k_y \sin(pz))^2 - \frac{1}{n_e^2} (k_x \sin(pz) - k_y \cos(pz))^2 \right) F = 0. \]

or with the previous notation

\[ \frac{d^2 F}{d\zeta^2} + \left( \alpha + \beta \cos(2\zeta) \right) F = 0, \]

which is Mathieu’s equation.
By the Floquet-Bloch theorem, the general solution is of the form

\[ F = c_1 e^{i\mu \zeta} f(\zeta) + c_2 e^{-i\mu \zeta} f(-\zeta) \]

where \( f(\zeta) = f(\zeta + 2\pi) \) and \( \mu \) depends on \( \alpha \) and \( \beta \). Expanding \( f(\zeta) \) as a Fourier series, we deduce the Laue-Bragg conditions that an incoming wave with wave vector \( k_{\text{in}} \) is reflected/diffracted with wave vector \( k_{\text{out}} \) where

\[ p(k_{\text{out}} - k_{\text{in}})_z = m \in \mathbb{Z}. \] (1)

A detailed calculation of the optical band structure may now be obtained using Hill Determinants.
\[
\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0, \\
\text{curl } \mathbf{H} = +\frac{\partial \mathbf{D}}{\partial t}, \quad \text{div } \mathbf{D} = 0,
\]

or

\[
F = -E_i dt \wedge dx^i + \frac{1}{2} \varepsilon_{ijk} B_i dx^j \wedge dx^k \\
G = H_i dt \wedge dx^i + \frac{1}{2} \varepsilon_{ijk} D_i dx^j \wedge dx^k
\]

\[dF = 0 = dG\]
However to “close the system”, one must relate $F$ to $G$ by means of a “constitutive equation”. Nematic liquid crystals are uniaxial: at each point, we can assume that the permittivity and permeability tensors $\epsilon$ and $\mu$ admit locally a basis in which the tensors have the form

$$
\epsilon = \begin{pmatrix}
\epsilon_\parallel & 0 & 0 \\
0 & \epsilon_\perp & 0 \\
0 & 0 & \epsilon_\perp
\end{pmatrix}, \quad 
\mu = \begin{pmatrix}
\mu_\parallel & 0 & 0 \\
0 & \mu_\perp & 0 \\
0 & 0 & \mu_\perp
\end{pmatrix}.
$$

If the axis of the material lies along $\mathbf{n}$, we have

$$
\epsilon_{ij} = \epsilon_\perp \delta_{ij} + (\epsilon_\parallel - \epsilon_\perp) n_i n_j, \quad 
\mu_{ij} = \mu_\perp \delta_{ij} + (\mu_\parallel - \mu_\perp) n_i n_j.
$$
At the W.K.B. level we obtain in general two metrics of Joets-Ribotta form

\[ ds_B^2 = -\frac{dt^2}{\mu_\perp} + \varepsilon_\parallel dx^2 + (\varepsilon_\perp - \varepsilon_\parallel)(n \cdot dx)^2, \quad B \cdot n = 0 \quad (2) \]

\[ ds_E^2 = -\frac{dt^2}{\varepsilon_\perp} + \mu_\parallel dx^2 + (\mu_\perp - \mu_\parallel)(n \cdot dx)^2, \quad E \cdot n = 0 \]

If \( \mu_\perp = \mu_\parallel \), the metric \( ds_E^2 \), whose geodesics are the ordinary rays is flat. The metric \( ds_B^2 \) then coincides with the Joets-Ribotta metric with

\[ \varepsilon_\perp \mu_\perp = n_o^2, \quad \varepsilon_\parallel \mu_\perp = n_e^2. \]

and its geodesics are the extraordinary rays.
For the helical phase we define components $\tilde{B}_i$ and $\tilde{E}_i$ by

\[
F = \tilde{E}_i \lambda^i \wedge dt + \frac{1}{2} \epsilon_{ijk} \tilde{B}_i \lambda^j \wedge \lambda^j
\]

and use Cartan’s exterior calculus to find that we may separate variables with the ansatz

\[
\tilde{E}_i = e^{i(k_xx + k_yy - \omega t)} f_i(z), \quad \tilde{B}_i = e^{i(k_xx + k_yy - \omega t)} g_i(z).
\]

The components $f_3(z), g_3(z)$ are given by a linear combination of other components, so that Maxwell’s equations reduce to a system of differential equations of the form:

\[
F'(z) + (\alpha + \beta_1 e^{2ipz} + \beta_2 e^{-2ipz}) F(z) = 0.
\]

Here $F(z) = (f_1(z), f_2(z), g_1(z), g_2(z))^t$ is a component column vector.
To conclude Part 1. we have have

• Revealed the underlying symmetry group of the helical phase of chiral nematics as the Bianchi group of type $VII_0$ aka $\tilde{E}(2)$

• Shown that the ray optical properties, as encoded in the Joets-Ribotta metric, give rise to a homogeneous and anisotropic 3-space used (in its time dependent form) in cosmology to model anisotropies in the CMB.

• Integrated the ray equations of motion using the symmetries of the metric.

• Justified the ray approximation by separating variables for the wave equation and passing to the WKB approximation.

• Shown, by applying the Bloch-Floquet theorem to the resulting Mathieu equation, how scalar wave theory gives rise to Bragg-von-Laue diffraction.

• Justified the Ray and Wave theory approximations by solving the full Maxwell equations, using the Cartan's exterior calculus and the Bianchi $VII_0$ Maurer-Cartan relations.
In Part 2 I shall

• Relate Finkelstein and Misner’s classification of **Gravitational Kinks** to **Topological Defects in nematics.**
and discuss droplets.
Gravitational Kinks

The Topology of a Lorenztian metric may be (partially) captured by a direction field $n^i$. Given a Riemannian metric $g^R_{ij}$, and a unit direction field $n^i$ such that $g^R_{ij}n^i n^j = 1$ we may construct a Lorentzian metric $g^L_{ij}$ via

$$
\begin{align*}
g^L_{ij} &= g^R_{ij} - \frac{1}{\sin^2 \alpha} n_i n_j, \\
g^R_{ij} &= g^R_{ij} - \frac{1}{\cos^2 \alpha} n^i n^i, \\
n_i &= g^R_{ij} n^j
\end{align*}
$$

Conversely given $g^L_{ij}$ and $g^R_{ij}$ we may reconstruct $n_i$ up to a sign. Fixing the sign amounts to fixing a time orientation In what follows we will choose $g^R_{ij}$ to be the usual flat Euclidean metric.

$$
\begin{align*}
ds^2_L &= g^L_{ij} dx^i dx^j = dx^2 - \frac{1}{\cos^2 \alpha} (n \cdot dx)^2
\end{align*}
$$
Given a closed surface enclosing a domain \( D \), Finkelstein and Misner quantified the notion of \textit{tumbling light cones} the light cone tips over on \( \Sigma = \partial D \) by introducing a \textit{kink number} which counts how many times the light cone tips over on \( \Sigma = \partial D \). The outward unit normal \( \nu \) gives a 2-dimensional cross section of the four-dimensional bundle \( S(\partial D = \Sigma) \) of unit 3-vectors over \( \partial D = \Sigma \). In the orientable case, the director field gives another 2-dimensional cross section of \( S(\Sigma) \). The kink number \( \kink(\Sigma, g^L) \) is number of intersections of these two sections with attention paid to signs. In the non-orientable case, one considers the bundle of directions. If the Lorentzian metric is non-singular we have

\[
\chi(D) = \kink(\partial D, g^L).
\]

For planar domains \( \kink(\partial D, g^L) \) is the obvious winding number.
disclination line  \( n = (\cos(s\phi), \sin(s\phi), 0), \quad \phi = \arctan\left(\frac{y}{x}\right) \)

\( s \in \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2} \). If \( s \) is half integral, then then we just have a direction field, not a vector field.

\[ n \cdot \mathbf{d}x = \cos((s - 1)\phi)\,dr + \sin((s - 1)\phi)\,r\,d\phi, \]

\[ \alpha = \frac{\pi}{2}, \Rightarrow ds^2_L = g^L_{ij}dx^i dx^j = -\cos(2(s - 1)\phi)(dr^2 - r^2d\phi^2) - 2\sin(2(s - 1)\phi) \]

Moving around a circle \( r = \text{constant} \), the radial coordinate is timelike and the angular coordinate spacelike or vice versa depending upon the sign of \( \cos(2(s - 1)\phi) \) (tumbling light cones). \( \det g^L_{ij} = -r^2 \) and the components \( g^L_{ij} \) finite \( \Rightarrow \) metric non-singular if \( r > 0 \).
Of course other systems for which the order parameter is a direction or vector field also admit this interpretation. For example, ferro-magnets with magnetization vector $\mathbf{M}$. Consider Bloch Walls...
If parity symmetry holds then a typical free energy functional takes the form

\[ F[M] = \frac{1}{2} \int d^3x \left( \alpha_{ij} \partial_i M \cdot \partial_j M + \beta_{ij} M_i M_j \right) \]

In the uniaxial case with the easy direction along the third direction: \( \alpha_{ij} = \text{diag}(\alpha_1, \alpha_1, \alpha_2), \beta_{ij} = \text{diag}(\beta, \beta, 0) \). For a domain wall separating a region \( x << -1 \) and with \( M \) pointing along the positive 3rd direction, from the region \( x >> +1 \) where it points along the negative 3rd direction

\[ M = M(0, \sin \theta(x), \cos \cos \theta(x)), \quad M = \text{constant} \]

and finds that \( \theta \) must satisfy the quadrant pendulum equation, \( l = \)
\sqrt{\frac{\alpha_1}{\beta}}

\[ \theta^2 - \frac{1}{l^2} \sin^2 \theta = \text{constant}', \]
If we impose the boundary condition that $\theta \to 0$ as $x \to -\infty$ and $\theta \to \pi$ as $x \to +\infty$, then \( \text{constant}' = 0 \) and

$$\cos \theta = -\tanh\left(\frac{x}{l}\right)$$

The Lorenzian metric (if $\alpha = \frac{\pi}{2}$) is

$$ds^2 = g_{ij}^L dx^i dx^j dx^2 + \cos(2\theta)(dy^2 - dz^2) - 2\sin 2\theta dz dy,$$

This closely resembles our previous examples and clearly exhibits the phenomenon of tumbling light cones. We note, en passant that in principle the tensor $\alpha_{ij}$ could itself vary with position. If so, we might interpret it in terms of an effective metric $g_{ij}$ with inverse $g^{ij}$ and

$$g = \det g_{ij}$$

obeying

$$\alpha_{ij} = \sqrt{g} g^{ij}. \quad (5)$$
Example: Liquid Crystal Droplets

The normal $\nu_i = \partial_i S$ to the surface $S = \text{constant}$ of a droplet of anisotropic nematic phase inside a domain $D$ with unit outward normal $\nu$ surrounded by an isotropic phase satisfies the constant angle condition

$$\mathbf{n} \cdot \nu = \cos \alpha = \text{constant}.$$ 

That is

$$\nu \cdot \nu - \frac{1}{\cos^2 \alpha} (\nu \cdot \mathbf{n})(\nu \cdot \mathbf{n}) = 0 = g^{ij} \nu_i \nu_j = g^{ij} \partial_i S \partial_j S$$

The surface $\partial D$ of the droplet $\partial D$ is a null-hypersurface or wave surface (a solution of the zero rest mass Hamilton-Jacobi equation)
Taking the $z$-coordinate as time so time runs vertically upwards and making the ansatz

$$S = \frac{z}{\sin \alpha} + W(x, y), \quad \nabla W \cdot \nabla W = 1.$$ 

Simple solutions of this Eikonal equation are given by Sandpiles with

$$\frac{\pi}{2} - \alpha$$ the angle of repose.
Figs. 149 to 152.—Examples of plastic stress function for torsion represented by wooden models for various cross-sections.
Fig. 162.—Sand heap over equilateral triangle.

Fig. 163.—Sand heap over an ellipse.

Fig. 164a. Fig. 164b.
Figs. 164a and b.—Sand heaps over areas bounded by two circular arcs.
The **Eikonal equation** also governs Bitter Domains in a ferromagnetic film \(^*\) with \( n = \frac{M}{|M|} \) with normal \( \nu \) and boundary condition \( M \cdot \nu = 0 \).

\[
\nabla \cdot M = 0, \quad |M| = \text{constant}
\]

\[
\nabla \cdot n \Rightarrow n_x = \partial_y \psi, \quad n_y = -\partial_x \psi \quad |\nabla \psi| = 1.
\]

The axisymmetric solution of the Eikonal equation is the spiral wave surface swept out by the involute of a circle, a helical developable.

\[
S = \pm \frac{z}{\sin \alpha} + \pm a \left( \sqrt{\frac{r^2}{a^2} - 1} - \arctan \left( \sqrt{\frac{r^2}{a^2} - 1} \right) \right) \pm a \phi
\]
For the **helical phase** we make the ansatz

\[
S = F(z) + x \cos \theta + y \sin \theta
\]

\(F(z)\) solves the **quadrantal pendulum equation**

\[
\cos^2(\theta - pz) - \cos^2 \alpha = (\cos \alpha \frac{dF}{dz})^2 \Rightarrow F = \frac{1}{\cos \alpha} \int dz \sqrt{\cos^2(\theta - pz) - \cos^2 \alpha}
\]

The surface is ruled by horizontal straight lines making a constant angle \(\theta\) with the \(x\)-axis and is bounded by \(|pz - (\theta + n\pi)| < \alpha\), \(n \in \mathbb{Z}\) In other words it is **horizontal cylinder** or **tube**. The angle of the director \(\mathbf{n}\) makes with the fixed direction \((\cos \theta, \sin \theta, 0)\) cannot be less than \(\alpha\).
Let’s conclude. In Part 2 I have

- Used the director field \( \mathbf{n} \) to associate a Lorentzian (i.e. signature \(-+++\)) metric with every nematic configuration.

- Shown that the boundaries of droplets in an isotropic exterior correspond to null hypersurfaces.

- Related the classification of topological defects in nematics to exotic entities such as Finkelstein and Misner’s Gravitational Kinks.

- Interpreted constant slope boundaries of droplets of nematics as null hypersurfaces with respect to a Lorentzian metric determined by the director field.