Introduction to E_n-operads & Little Discs Operads

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March 5, 2013

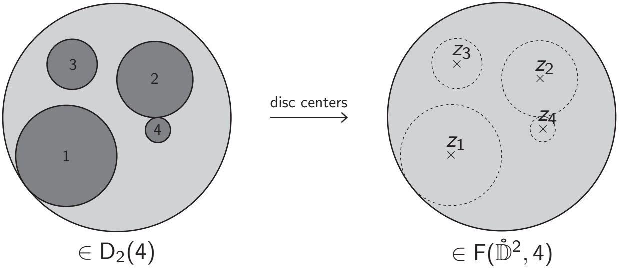
Introduction

- The operad of little n-discs was introduced in topology in order to model structures attached to n-fold loop spaces (Boardman-Vogt, May).
- The little discs operads are also used to define a hierarchy of homotopy commutative structures, from fully homotopy associative but non-commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$).
- In homotopical algebra, one deals with operads weakly-equivalent to the reference model of little n-discs, and authors use the noun of E_n-operad to refer to this notion.

Plan

1. The little discs and little cubes models of E_n-operads
2. The Barratt-Eccles operad
3. The surjection operad
4. Koszul duality of E_n-operads
5. Miscellaneous models of E_n-operads

§1. The little discs and little cubes models of E_n-operads

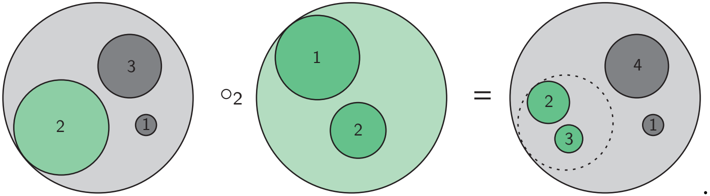


- The little n-discs spaces $D_n(r)$ consist of collections of r little n-discs with disjoint interiors inside a fixed unit n-disc $\mathbb{D}^n$ (see Figure).
- The configuration spaces $F(\mathring{\mathbb{D}}^n, r)$ consist of collections of r distinct points in the open disc $\mathring{\mathbb{D}}^n$ (see Figure).
- There is an obvious homotopy equivalence $D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$.

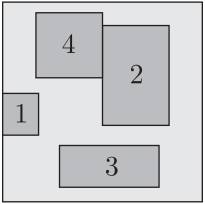
- The symmetric group Σ_r acts on $\mathrm{D}_n(r)$ by permutation of the little disc indices (and on the configuration space similarly).
- The little n-discs spaces (unlike the configuration spaces) inherit operadic composition operations

$$\circ_k : \mathrm{D}_n(r) \times \mathrm{D}_n(s) \to \mathrm{D}_n(r + s - 1)$$

given by the following substitution process



- The little n-discs operad D_n is the structure defined by the collection of little n-discs spaces $\mathrm{D}_n = \{\mathrm{D}_n(r), r \in \mathbb{N}\}$ equipped with this symmetric operadic composition structure.



- The little n-cubes operad C_n is a variant of the operad of r little n-discs, where we consider configurations of cubes in a fixed unit cube $\mathbb{I}^n = [0,1]^n$ instead of discs (see Figure).
- The symmetric and composition structure of this operad is defined as in the little discs setting.
- The little cubes spaces $\mathsf{C}_n(r)$ also inherit obvious homotopy equivalences $\mathsf{C}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{I}}^n, r)$.

Reminder:

- The little n-cubes operad (and the little n-discs operad similarly) encodes operations acting on n-fold loop spaces $\Omega^n X$.
- Theorem (May's approximation theorem): *If X is a connected base space, then we have a weak-equivalence $\iota : \mathbb{S}_*(\mathsf{C}_n, X) \xrightarrow{\sim} \Omega^n \Sigma^n X$, where $\mathbb{S}_*(\mathsf{C}_n, X)$ is a base version of the free C_n-algebra generated by X.*
- Theorem (Boardman-Vogt', May's recognition theorem): *If Y is a connected (or more generally a group-like) base space equipped with an action of the operad C_n, then we have $Y \sim \Omega^n X$ for some n-fold loop space $\Omega^n X$.*

- Definition: A weak-equivalence of operads (in topological spaces) is an operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ of which components $\phi(r) : \mathsf{P}(r) \to \mathsf{Q}(r)$ are weak-equivalences (of topological spaces). The notation $\xrightarrow{\sim}$ will be used to mark any distinguished class of weak-equivalences in a category (e.g. in the category of topological spaces, in the category of operads).
- Definition: An E_n-operad (in topological spaces) is an operad E_n connected to the operad of little n-discs D_n (or little n-cubes) by a chain of weak-equivalences of operads

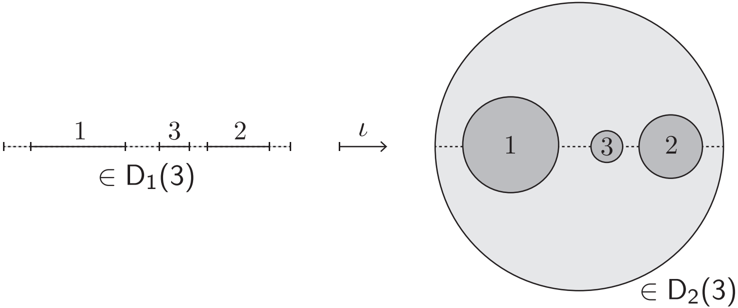
$$\mathsf{E}_n \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdot \ldots \cdot \xrightarrow{\sim} \mathsf{D}_n \, .$$

- Not such an easy exercise: There is such a chain of operad weak-equivalences between the little n-discs and the little n-cubes operads. The little discs and the little cubes models accordingly define the same notion of E_n-operad.
- Problem: How to recognize E_n-operads?

- For each r, we have a topological inclusion $\iota : \mathsf{D}_{n-1}(r) \hookrightarrow \mathsf{D}_n(r)$ mapping any collection of little $(n-1)$-discs

$$c_i(\mathbb{D}^{n-1}) = P_i + R_i \cdot \mathbb{D}^{n-1}, \quad i = 1, \dots, r,$$

to the collection of little n-discs with the same radius R_i and the same center P_i in the equatorial hyperplane of the unit n-discs $\mathbb{D}^n$, as in the following picture:



- These maps define an inclusion of topological operads $\iota : \mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$.

- Thus the operads of little discs form a nested sequence of operads in topological spaces:

$$\mathsf{D}_1 \hookrightarrow \mathsf{D}_2 \hookrightarrow \cdots \hookrightarrow \mathsf{D}_n \hookrightarrow \cdots .$$

  We also set $\mathsf{D}_\infty = \operatorname{colim}_n \mathsf{D}_n$.
- Proposition:
  - *For each r, we have $\mathsf{D}_1(r) \sim \Sigma_r$, and the little 1-discs operad D_1 is weakly-equivalent to the set operad of associative monoids* As*, regarded as a discrete operad in the category of topological spaces.*
  - *For each r, we have $\mathsf{D}_\infty(r) \sim pt$, and the operad D_∞ is weakly-equivalent to the set operad of associative and commutative monoids* Com*, regarded as a discrete operad in the category of topological spaces.*

Theorem (F. Cohen):

- *For $n > 1$, we have $\mathtt{H}_*(\mathsf{D}_n) = \mathsf{Gerst}_n$, where Gerst_n (the nth Gerstenhaber operad) is the operad generated by:*
  - *a cup product operation $\mu = \mu(x_1, x_2) \in \mathsf{Gerst}_n(2)$ in degree 0, such that $\mu(x_2, x_1) = \mu(x_1, x_2)$,*
  - *a Lie bracket operation $\lambda = \lambda(x_1, x_2) \in \mathsf{Gerst}_n(2)$, in degree $n - 1$, such that $\lambda(x_2, x_1) = (-1)^n \lambda(x_1, x_2)$,*

  and equipped with the associativity, Jacobi, and Poisson distribution relations as generating relations.
- *The operad Gerst_n is also equipped with a unitary operation $e \in \mathsf{Gerst}_n(0)$, such that $\mu(e, x_1) = \mu(x_1, e) = x_1$ and $\lambda(e, x_1) = \lambda(x_1, e) = 0$, and the identity $\mathtt{H}_*(\mathsf{D}_n) = \mathsf{Gerst}_n$ holds in the category of unitary Hopf operads.*
- *The map $\iota_* : \mathtt{H}_*(\mathsf{D}_{n-1}) \to \mathtt{H}_*(\mathsf{D}_n)$ induced by the operad embedding $\iota : \mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$ is determined on generating operations by $\iota_*(\mu) = \mu$ and $\iota_*(\lambda) = 0$.*

§2. The Barratt-Eccles operad

- For $n = 1$, we have

$$\mathrm{D}_1 \sim \mathrm{As} \Rightarrow \mathbb{S}_*(\mathrm{As}, X) \sim \mathbb{S}_*(\mathrm{D}_1, X) \sim \Omega\Sigma X,$$

  where $\mathbb{S}_*(\mathrm{As}, X)$ is nothing but the James construction: the free associative monoid generated by X modulo the relation base point $\equiv$ monoid unit.
- For $n = \infty$, we do not have such a relation with the free commutative monoid. The construction $\mathbb{S}_*$ actually preserves weak-equivalences only when the action of symmetric groups on operads is nice enough.
- The simplicial Barratt-Eccles operad was introduced by Barratt-Eccles in order to get an analogue of the James construction for infinite loop spaces. This operad will be denoted by W.

Definition (Barratt-Eccles operad):

- The simplicial sets $\mathrm{W}(r)$ defining the Barratt-Eccles operad have
$$\mathrm{W}(r)_d = \{(w_0, \ldots, w_d) \in \Sigma_r \times \cdots \times \Sigma_r\}$$
as set of d-dimensional simplices, and are equipped with:
  - the face operators $d_i : \mathrm{W}(r)_d \to \mathrm{W}(r)_{d-1}$, $i = 0, \ldots, d$, such that
$$d_i(w_0, \ldots, w_d) = (w_0, \ldots, \widehat{w_i}, \ldots, w_d),$$
  - and the degeneracy operators $s_j : \mathrm{W}(r)_d \to \mathrm{W}(r)_{d+1}$, $j = 0, \ldots, d$, such that
$$s_j(w_0, \ldots, w_d) = (w_0, \ldots, w_j, w_j, \ldots, w_d).$$
- The symmetric group Σ_r acts diagonally on $\mathrm{W}(r)$, for each r, and the composition operations $\circ_k : \mathrm{W}(r) \times \mathrm{W}(s) \to \mathrm{W}(r+s-1)$ are defined factorwise by the operadic composition of permutations, formed within the associative operad $\mathrm{As}(r) = \Sigma_r$.

- Theorem (Barratt-Eccles): *Each space* $\mathrm{W}(r)$ *is contractible so that* W *defines an instance of an* E_∞*-operad, and we have*

$$\mathtt{S}_*(\mathrm{W}, X) \sim \Omega^\infty \Sigma^\infty X$$

for any connected (group-like) simplicial set X*.*

- Theorem (J.H. Smith, C. Berger):
  1. *The Barratt-Eccles operad* W *has a filtration by supoperads* W_n *so that* $\mathrm{W}_1 = \mathtt{sk}_0 \mathrm{W} = \mathrm{As}$*, and the sequence*

  $$\mathrm{W}_1 \hookrightarrow \mathrm{W}_2 \hookrightarrow \cdots \hookrightarrow \mathrm{W}_n \hookrightarrow \cdots \hookrightarrow \mathrm{W}_\infty = \mathrm{W}$$

  is weakly-equivalent to the nested sequence of the little n-discs operads (as a sequence of operads in spaces).
  2. *Each* W_n *is therefore an* E_n*-operad, and we also have*

  $$\mathtt{S}_*(\mathrm{W}_n, X) \sim \Omega^n \Sigma^n X$$

  for any connected (group-like) simplicial set X*.*

- Construction (filtration of the Barratt-Eccles operad):
  - For each $w \in \Sigma_r$, we set $w|_{ij}$ for the permutation of the pair (i,j) defined by removing the values $k \notin \{i,j\}$ from the sequence $(w(1),\dots,w(r))$.
  - The simplicial set $\mathsf{W}_n(r)$ consists of the simplices $(w_0,\dots,w_d)$ so that the sequence $(w_0|_{ij},\dots,w_d|_{ij})$ has a variation number $\mu_{ij} < n$, for each pair $\{i,j\} \subset \{1,\dots,r\}$.
- Proposition: *The simplicial sets* W_n *form a suboperad of* W*, and we have*

$$\mathsf{W}_1 = \mathtt{sk}_0\,\mathsf{W}$$

as asserted in the theorem.

The complete graph posets:

- Let $\mathcal{K}(r)$ be the set of pairs (μ, σ) such that:
  - μ is a collection of non-negative integers $\mu_{ij} \in \mathbb{N}$ indexed by pairs $\{i, j\} \subset \{1, \ldots, r\}$,
  - σ is a permutation of $(1, \ldots, r)$.
- This set is equipped with a partial order relation $\leq$ such that

$$(\mu, \sigma) \leq (\nu, \tau)$$

if we have

$$(\mu_{ij} < \nu_{ij}) \quad \text{or} \quad (\mu_{ij}, \sigma|_{ij}) = (\nu_{ij}, \tau|_{ij})$$

for each pair $\{i, j\} \subset \{1, \ldots, r\}$.

The cell decomposition of the Barratt-Eccles operad:

- To any simplex $(w_0, \dots, w_d)$ in the Barratt-Eccles operad $\mathrm{W}(r)$, we associate the poset element

$$\kappa(w_0, \dots, w_d) = (\mu, \sigma) \in \mathcal{K}(r)$$

  such that:
  - μ_{ij} is the variation number of the sequence $(w_0|_{ij}, \dots, w_d|_{ij})$,
  - $\sigma = w_d$,
- and we consider the simplicial set decomposition

$$\mathrm{W}(r) = \bigcup_{\kappa \in \mathcal{K}(r)} \mathrm{W}(\kappa)$$

  such that

$$\mathrm{W}(\kappa)_d = \{(w_0, \dots, w_d) | \kappa(w_0, \dots, w_d) \leq \kappa\}.$$

- Observation: *The complete graph posets form an operad. The structure of this operad models the image of the cell decomposition* $\mathrm{W}(r) = \bigcup_{\kappa\in\mathcal{K}(r)} \mathrm{W}(\kappa)$ *under the structure operations attached to the Barratt-Eccles operad. To be precise:*
  - *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{W}(r) \arrow[r, "w"] & \mathrm{W}(r) \\ \mathrm{W}(\kappa) \arrow[u, hook] \arrow[r, dotted] & \mathrm{W}(s\kappa) \arrow[u, hook] \end{tikzcd}$$

    for any action of a permutation $s \in \Sigma_r$ *on* $\mathrm{W}(r)$,
  - *and a commutative diagram*

$$\begin{tikzcd} \mathrm{W}(r) \times \mathrm{W}(s) \arrow[r, "\circ_k"] & \mathrm{W}(r+s-1) \\ \mathrm{W}(\alpha) \times \mathrm{W}(\beta) \arrow[u, hook] \arrow[r, dotted] & \mathrm{W}(\alpha \circ_k \beta) \arrow[u, hook] \end{tikzcd}$$

    for any operadic composition operation $\circ_k$ *in the Barratt-Eccles operad.*

- Lemma (C. Berger):
  1. *The simplicial set* $\mathrm{W}(\kappa)$, *associated to any* κ, *is contractible.*
  2. *The collection of simplicial sets* $\{\mathrm{W}(\kappa) | \kappa \in \mathcal{K}(r)\}$ *forms a cofibrant diagram, for each* $r \in \mathbb{N}$.
- The little discs operad D_∞ inherits a similar nice cell decomposition (C. Berger, Brun-Fiedorowicz-Vogt) for a poset operad $\mathcal{K}'$ which differs from the complete graph operad, but is equivalent to this one in the sense that we have an equivalence of topological operads $|\,\mathcal{K}\,| \sim |\,\mathcal{K}'\,|$ at the level of geometric realizations.
- Berger's recognition theorem follows from the existence of this equivalence of poset operads. □

§3. The surjection operad

- Reminder: The surjection operad S is an operad in dg-modules such that $\mathsf{S}(r)$ is generated in degree d by surjective maps $u : \{1, \dots, r+d\} \to \{1, \dots, r\}$ satisfying the non-degeneracy condition $u(i) \neq u(i+1)$, for $i = 1, \dots, r+d-1$.
- Observations:
  - *Each surjection $u \in \mathsf{S}(r)_d$ determines a prism*

$$\tau_u : \Delta^{d_1} \times \cdots \times \Delta^{d_r} \to \mathsf{W}(r),$$

where we set $d_i = |u^{-1}(i)| - 1$ for $i = 1, \dots, r$. For instance:

$$\begin{tikzcd} (0,1,0) \arrow[r] & (1,1,0) & & (1,3,2) \arrow[r] & (3,1,2) \\ (0,0,0) \arrow[u] \arrow[r] \arrow[ur] & (1,0,0) \arrow[u] & \arrow[r, mapsto, "\tau_{(1,2,3,1,2)}"] & {} & (1,2,3) \arrow[u] \arrow[r] \arrow[ur] & (2,3,1) \arrow[u] \end{tikzcd}$$

  - *These prisms define a cell decomposition of the space $\mathsf{W}(r)$.*

- Fact: Each orientation of the prism τ_u, determined by the choice of a fundamental simplex among its maximal simplices, gives rise to a local Eilenberg-Zilber map $\nabla : \mathtt{C}_*^{prism}(\Delta^{d_1} \times \cdots \times \Delta^{d_r}) \to \mathtt{C}_*^{simp}(\mathsf{W}(r))$ assigning a sum of simplices in the Barratt-Eccles operad $\mathsf{W}(r)$ to the surjection $u \in \mathsf{S}(r)_d$.
- Reminder: The simplicial cellular complex $\mathtt{C}_*^{simp}(X)$ of a simplicial set X is identified with the standard normalized complex $\mathtt{N}_*(X)$ associated to X. The collection of normalized complexes $\mathtt{N}_*(\mathsf{P}) = \{\mathtt{N}_*(\mathsf{P}(r)), r \in \mathbb{N}\}$ associated to the components of an operad in simplicial sets P forms an operad in dg-modules.
- Theorem (Berger-BF): *There exist Alexander-Whitney maps $\Delta : \mathtt{N}_*(\mathsf{W}(r)) \to \mathsf{S}(r)$, left inverse to a global Eilenberg-Zilber map $\nabla : \mathsf{S}(r) \to \mathtt{N}_*(\mathsf{W}(r))$, so that the dg-modules $\mathsf{S}(r)$ form a quotient operad of the chain Barratt-Eccles operad $\mathsf{E} = \mathtt{N}_*(\mathsf{W})$.*

- Observation: *The operad* S *actually inherits a filtration* $\mathsf{S}_1 \subset \cdots \subset \mathsf{S}_n \subset \cdots \subset \mathsf{S}_\infty = \mathsf{S}$ *from the Barratt-Eccles operad so that* S_n *defines an* E_n*-operad in dg-modules.*
- Observation: *The operad* S *acts on the normalized complex of simplicial sets, which therefore also inherits an action of the chain Barratt-Eccles operad* $\mathsf{E} = \mathtt{N}_*(\mathsf{W})$ *by restriction of structure through the Alexander-Whitney map* $\Delta : \mathsf{E} \to \mathsf{S}$.
- Theorem (McClure-Smith): *The operad* S_2 *is isomorphic to the operad, acting on the Hochschild cochain complex, which is generated by the cup-product* $x_1 \cup x_2$ *and the Getzler-Kadeishvili brace operations* $x_1\{x_2, \ldots, x_r\}$.

§4. Koszul duality of E_n-operads

Reminder:

- The operad $\mathsf{Gerst}_n = \mathtt{H}_*(\mathsf{D}_n)$ is Koszul self dual (up to operadic suspension): we have a weak-equivalence $\kappa : \mathtt{B}^c(\Lambda^{-n}\mathsf{Gerst}_n^\vee) \xrightarrow{\sim} \mathsf{Gerst}_n$ for each $n > 0$.
- These Koszul duality weak-equivalences fit in a commutative diagram

$$\begin{tikzcd}
\mathtt{B}^c(\Lambda^{-1}\mathsf{As}^\vee) \arrow[r, dotted, "\sigma^*"] \arrow[d, "\sim"'] & \cdots \arrow[r, dotted, "\sigma^*"] & \mathtt{B}^c(\Lambda^{-n}\mathsf{Gerst}_n^\vee) \arrow[r, dotted, "\sigma^*"] \arrow[d, "\sim"'] & \cdots, \\
\mathsf{As} \arrow[r, "\iota_*"'] & \cdots \arrow[r, "\iota_*"'] & \mathsf{Gerst}_n \arrow[r, "\iota_*"'] & \cdots
\end{tikzcd}$$

where σ^* is induced by the operad morphism $\sigma_* : \mathsf{Gerst}_n \to \Lambda^{-1}\mathsf{Gerst}_{n-1}$ such that $\sigma_*(\mu) = 0$ and $\sigma_*(\lambda) = \lambda$ for the generating operations of the Gerstenhaber operad.

Theorem (BF):

- *Let* $\mathsf{E}_n = \mathtt{N}_*(\mathsf{W}_n)$ *be the operad in dg-modules associated to the nth layer of the little n-discs filtration of the Barratt-Eccles operad* W.
- *There are operad morphisms*

$$\begin{tikzcd}
\mathtt{B}^c(\Lambda^{-1}\,\mathsf{As}^\vee) \arrow[r, dotted, "\sigma^*"] \arrow[d, "\sim"'] & \cdots \arrow[r, dotted, "\sigma^*"] & \mathtt{B}^c(\Lambda^{-n}\,\mathsf{E}_n^\vee) \arrow[r, dotted, "\sigma^*"] \arrow[d, "\sim"'] & \cdots, \\
\mathsf{As} \arrow[r, "\iota"'] & \cdots \arrow[r, "\iota"'] & \mathsf{E}_n \arrow[r, "\iota"'] & \cdots
\end{tikzcd}$$

which in a sense realize the weak-equivalences of the Koszul duality of the Gerstenhaber operad at the homology level.

Conjecture (M. Ching): *This Koszul duality statement also holds at the topological level (in the category of spectra).*

Motivations:

- For be any space X satisfying mild nilpotence and finiteness assumptions, we have an identity

$$\mathtt{H}^*(\Omega^n X) = \mathtt{H}_*(\mathtt{B}^n(\mathtt{C}^*(X)),$$

where:

  - $\mathtt{B}^n(-)$ refers to a suitable n-fold iteration of the bar construction which we apply to the cochain algebra of our space $\mathtt{C}^*(X)$.

- For any E_n-algebra A, we have identities:

$$\mathtt{H}_*(\Sigma^{-n}\,\mathtt{B}^n(A)) = \mathtt{H}^{\mathsf{E}_n}_*(A) = \mathtt{H}_*(\Lambda^{-n}\,\mathsf{E}_n^\vee(A), \partial),$$

where:

  - $\mathtt{H}^{\mathsf{E}_n}_*(-)$ refers to the homology theory naturally associated to the category of E_n-algebras,
  - and $(\Lambda^{-n}\,\mathsf{E}_n^\vee(A), \partial)$ is the expression of the Koszul duality complex computing this homology $\mathtt{H}^{\mathsf{E}_n}_*(-)$.

§5. Miscellaneous models of E_n-operads

- Boardman-Vogt tensor products (Dunn)
- The operads governing braided monoidal categories (Fiedorowicz), and iterated monoidal categories (Baltenau-Fiedorowicz-Schwänzel-Vogt)
- Fulton-MacPherson compactifications (Getzler-Jones, ...)
- Batanin's *n*-operads (Batanin)
- ...

Plan for the follow-up:

1. Operads governing braided monoidal structures
2. Homotopy automorphisms of operads in topological spaces
3. The topological interpretation of the Grothendieck-Teichmüller group