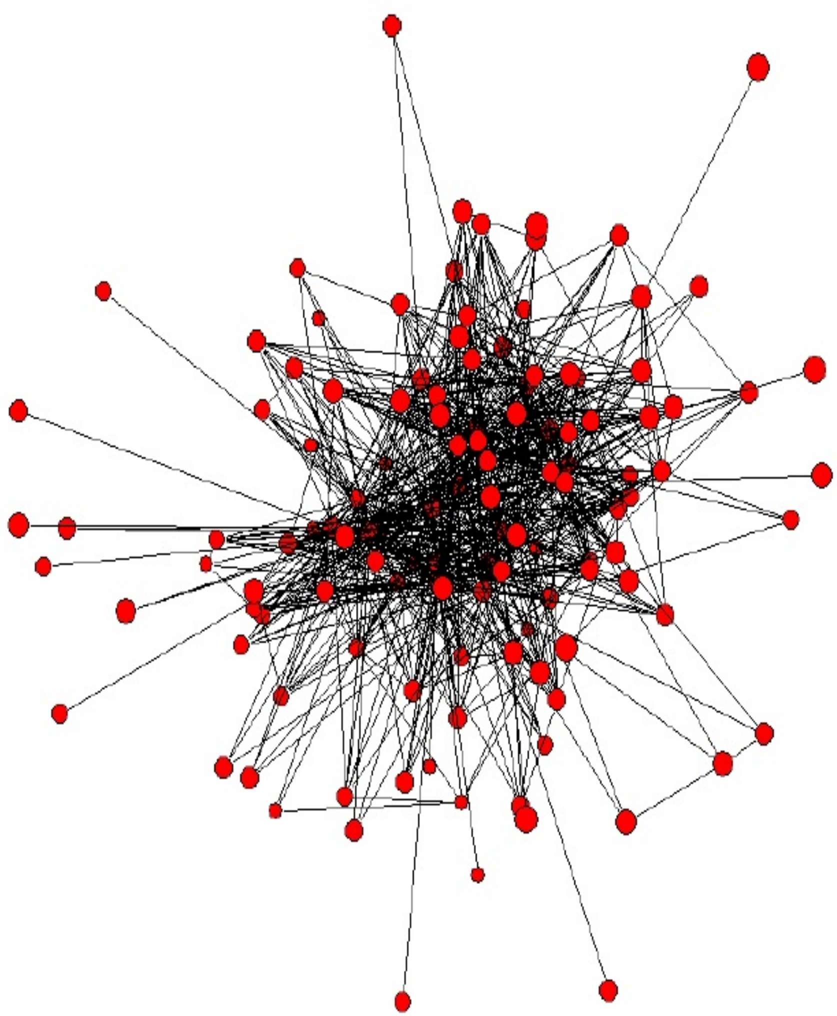
Systemic risk: a challenge for mathematical modelling

Rama CONT

Brazilian network: scale-free structure



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- R Cont, A Moussa, E B Santos (2010) Network structure and systemic risk in banking systems. To appear in: Handbook of Systemic Risk, Camb Univ Press.
- H Amini, R Cont, A Minca (2012) Stress testing the resilience of financial networks, *Intl Journal of Theoretical and applied finance*, Vol 15, No 1.
- H Amini, R Cont, A Minca (2011) Resilience to contagion in financial networks, SSRN.
- R Cont & L Wagalath (2011)Running for the exit: distressed selling and endogenous correlation in financial markets.

Systemic Risk

- Systemic risk may be defined as the risk that a significant portion of the financial system fails to function properly.
- The monitoring and management of systemic risk has become a major issue for regulators and market participants since the 2008 crisis.
- The financial crisis has simultaneously underlined

· the importance of contagion effects and systemic risk

· the lack of adequate indicators for monitoring systemic risk.

· the lack of adequate data for computing such indicators

Many initiatives under way: new regulations (Basel III), new financial architecture (derivatives clearinghouses), legislation on transparency in OTC markets, creation of Office of Financial Research (US), various Financial Stability Boards

BUT: methodological shortcomings, open questions

Systemic Risk

Various questions:

Mechanisms which lead to systemic risk

Metrics for systemic risk

Monitoring of systemic risk: data type/granularity ?

Management and control of systemic risk by regulators

Need for quantitative, operational answers

Mathematical / Quantitative Modeling can and should play a more important role in the study of systemic risk and in the current regulatory debate.

The ‘microprudential’ approach to financial stability

Traditional approach to risk management and bank regulation: focused on failure/non-failure (solvency, liquidity) of individual banks

Focuses on balance sheet structure of individual banks

Risk of each bank’s portfolio is measured using a statistical approach based on historical data: assumes that losses arises due to exogenous random fluctuations in risk factors (stock prices, exchange rates, interest rates, housing prices..)

Main tool for stabilization of system: capital requirements

Based on premise that ‘it is enough to supervise the stability of each bank to ensure stability of system’

Ignores links or interactions between market participants which can lead to market instabilities even when banks are ‘well capitalized’ (Hellwig 1995, 1998; Rochet & Tirole 1996; Freixas, Parigi & Rochet 2000)

Systemic Risk

Need to shift focus in risk measurement/ modeling of financial stability from institution-based viewpoint to a systemic viewpoint

Requires design and monitoring of system-wide indicators of stability and volatility

Regulatory recommendations and risk management recipes should be examined in the light of impact on systemic risk

Rules which improve the risk profile of an individual portfolio may simultaneously increase systemic risk if applied on a large scale

Example: portfolio diversification (Stiglitz 2010; Cont & Wagalath 2011, 2012; Greenwald et al 2008)

Systemic Risk in banking systems: channels of contagion

- Why do many financial institutions simultaneously default or suffer large losses ?
- 1. Correlation: large exposures to common risk factors can lead to large simultaneous losses across institutions
- 2. Counterparty Risk/ Balance sheet contagion: the default of one institution may lead to writedowns of assets held by its counterparties which may result in their insolvency.
- 3. Spirals of illiquidity: market moves and/or credit events may lead to margin calls which lead to default of institutions which lack sufficient short term funds.
- 4. Procyclical feedback effects: fire sales of assets due to deleveraging can further depreciate asset prices and lead to losses in other portfolios, generating endogenous instability

Channels of contagion: underlying network structure

- Each of these mechanism may be viewed as a contagion process on some underlying “network”, but the relevant “network topologies’’ and data needed to track them are different in each case:
- 1. Correlation: cross-sectional data on common exposures to risk factors/asset classes for tracking large-scale imbalances
- 2. Balance sheet contagion: network of interbank exposures, cross-holdings and liabilities + capital
- 3. Spirals of illiquidity: network of short-term liabilities (payables) and receivables + ‘liquidity reserves’
- 4. Fire sales/ feedback effects: data on *portfolio holdings* of financial institutions across asset classes + capital

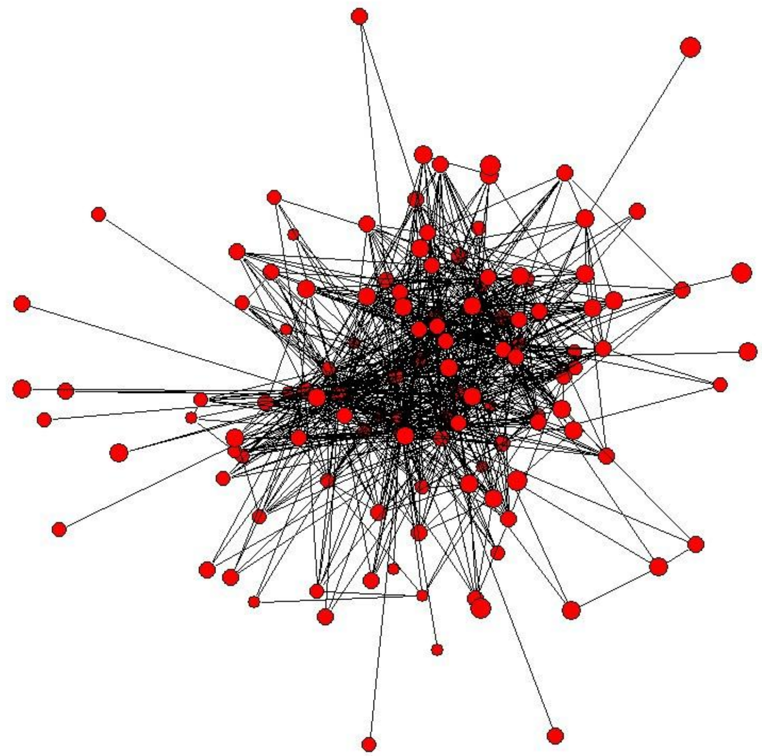
Counterparty networks: interbank exposures

- The relevant setting for studying balance sheet contagion (insolvency cascades) is a network – a weighted, directed graph- whose nodes are financial institutions and whose links represents interbank exposures :

$$E_{ij} = \text{exposure of } i \text{ to } j$$

= writedown in balance sheet of i when j defaults, taking into account liabilities+ cross-holdings.

- Data on interbank exposures reveal a complex, heterogeneous structure which is poorly represented by simple network models used in the theoretical literature.



Brazilian Interbank network (Cont, Moussa, Santos 2010)

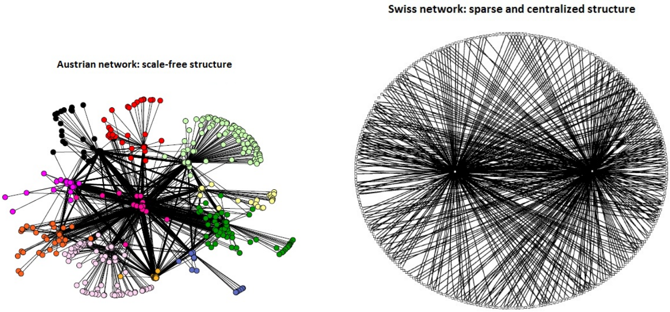


Figure: Network structures of real-world banking systems. Austria: scale-free structure (Boss et al2004), Switzerland: sparse and centralized structure (Müller 2006).

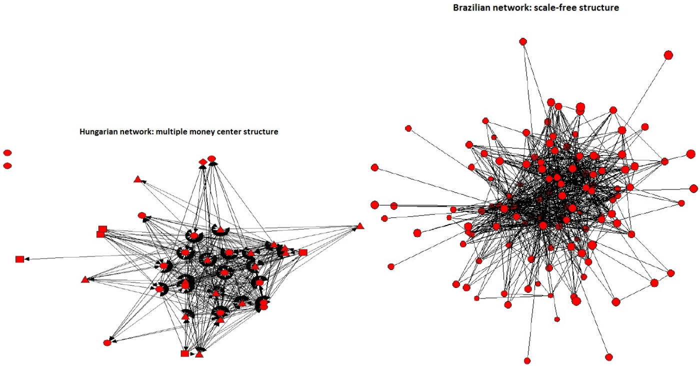


Figure: Network structures of real-world banking systems. Hungary: multiple money center structure (Lubloy et al 2006) Brazil: scale-free structure (Cont, Bastos, Moussa 2010).

The Brazil financial system: a directed scale-free network

- Exposures are reportted daily to Brazilian central bank.
- Data set of all consolidated interbank exposures (incl. swaps)+ Tier I and Tier II capital (2007-08).
- $n \simeq 100$ holdings/conglomerates, $\simeq 1000$ counterparty relations
- Average number of counterparties (degree)= 7
- Heterogeneity of connectivity: in-degree (number of debtors) and out-degree (number of creditors) have heavy tailed distributions

  $$\frac{1}{n}\#\{v, \mathrm{indeg}(v) = k\} \sim \frac{C}{k^{\alpha_{in}}} \qquad \frac{1}{n}\#\{v, \mathrm{outdeg}(v) = k\} \sim \frac{C}{k^{\alpha_{out}}}$$

  with exponents $\alpha_{in}, \alpha_{out}$ between 2 and 3.
- Heterogeneity of exposures: heavy tailed Pareto distribution with exponent between 2 and 3.

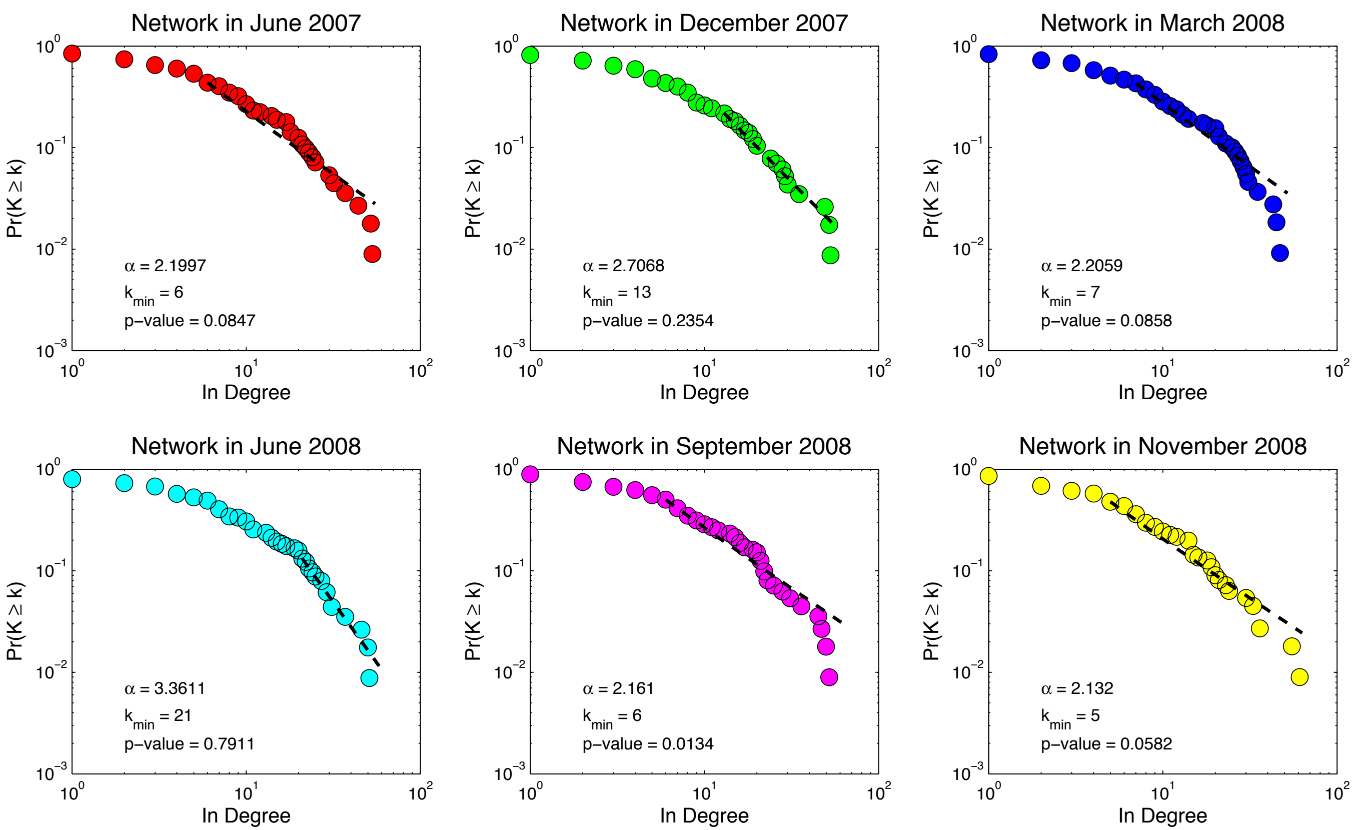


Figure 3: Brazilian financial network: distribution of in-degree.

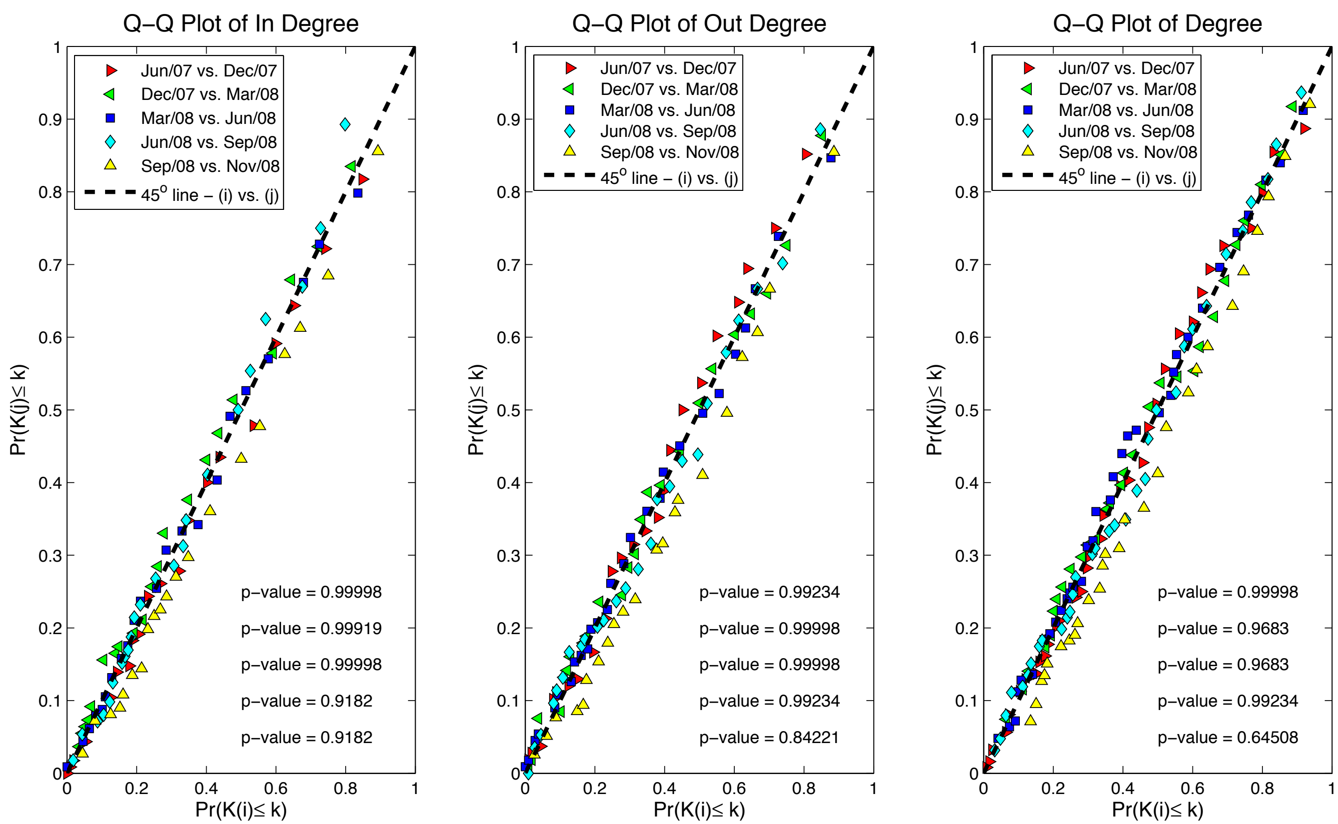


Figure 4: Brazilian financial network: stability of degree distributions across dates.

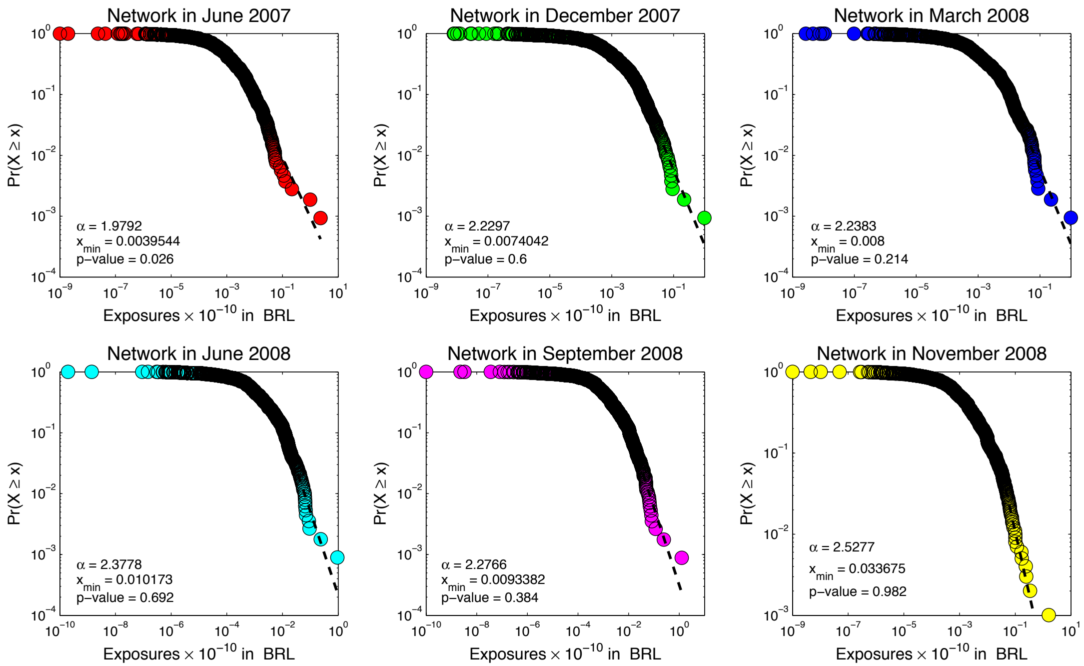


Figure 6: Brazilian network: distribution of exposures in BRL.

Some Questions

- How does the default of a bank affect its counterparties, counterparties of counterparties,… (domino effect)?
- Which are the banks whose defaults generates the largest systemic loss? : identification of SIFIs
- Can the default of one or few institutions generate a macro-cascade / large-scale instabiity of network?
- How do the answers to the above depend on network structure? Which features of network structure determine its stability/ resilience to contagion?

Previous work: many simulation studies+ analytical results for *average* cascade size on *homogeneous networks* *(*Watts (2002), Gai & Kapadia (2011),…)

Here: **analytical results on resilience and cascade size (not just average) for general, heterogeneous networks**

Measuring systemic risk: why exposures are important inputs

Market-based indicators have been recently proposed for quantifying

- contagion effects: CoVaR (quantile regression of past bank portfolio losses, Adrian & Brunnenmeier 2009)
- the (global) level of systemic risk in the financial system (Lehar 2005, Bodie, Gray, Merton 2008, IMF 2009, Huang, Zhu & Zhou 2010, Acharya et al 2010,..)

Useful for analyzing past/current economic data and should be part of any risk dashboard/ systemic risk tool kit.

Value as forward-looking diagnosis tools? any predictive ability?

Also: market-implied measures capture *market-perceived systemic risk*. Did market prices capture the systemic risk of AIG prior to its collapse?

Network approaches are based on **exposures** which represent potential **future** losses, which can give quite a different picture from past losses.

Even if we believe the Efficient Market hypothesis, market indicators need not reflect exposures, which are not public information.

Regulators, on the other hand, have access to non-public information on exposures and should use such information for stress testing and for computing systemic risk indicators.

Default contagion in a financial network

Two different approaches

1) Equilibrium approach : clearing vector (Eisenberg & Noé 1998)

   Given a matrix of 'liabilities'/exposures E and a vector of 'capital buffers' c, find a stream of cash flows which either clears liabilities or results in default. In case of default, with recovery proportional to liability.

Equilibrium defined as a fixed point.

'Liabilities'/ exposures are realized as cash flows at equilibrium. Endogenous recovery rates.

2) Stress testing approach (cascade approach):

Given a matrix of exposures E and a vector of 'capital buffers' c, investigate impact of the default of a given node i by studying the cascade of domino effects it generates through propagation of losses across counterparties.

Not an equilibrium: models actual outcome of a default (stress test of network)

Recovery rates are exogenous (typically, zero).

Measuring the systemic impact of a default

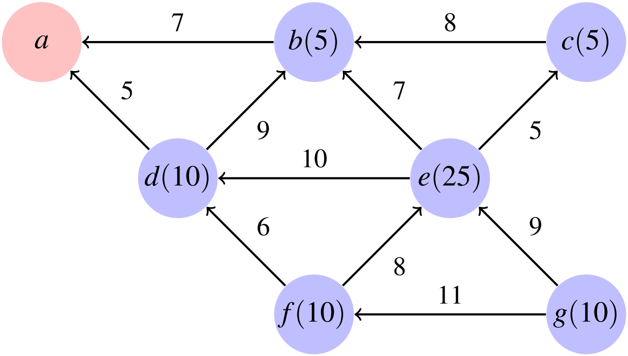
Objective: quantify the losses generated across the network by the initial default of a given financial institution.

Defaults can occur through

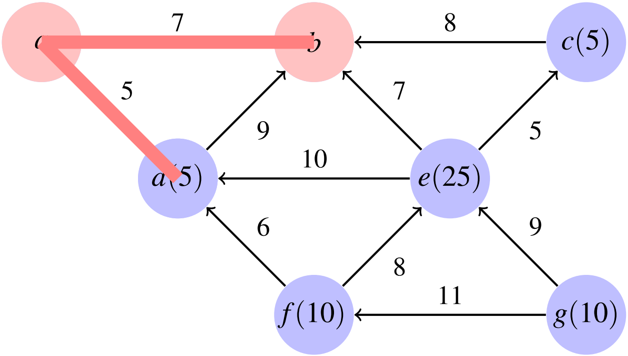
1. (correlated) market shocks to balance sheets
   $c_i \mapsto \max(c_i + \epsilon_i, 0)$
2. counterparty risk: default of i may lead to default of j if
   $c_j < E_{ji}$
3. lack of liquidity: if margin calls/ derivative payouts π_{ij} exceed available liquidity $l_i + \sum_j \pi_{ij}(c + \epsilon, E) < 0$

In cases 2 and 3 this can generate a 'domino effect' and initiate a cascade of defaults.

Example



Example



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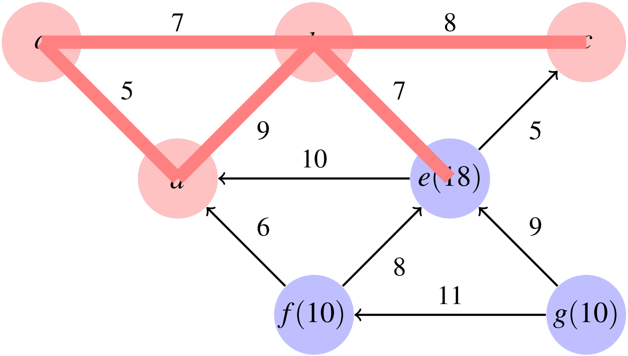
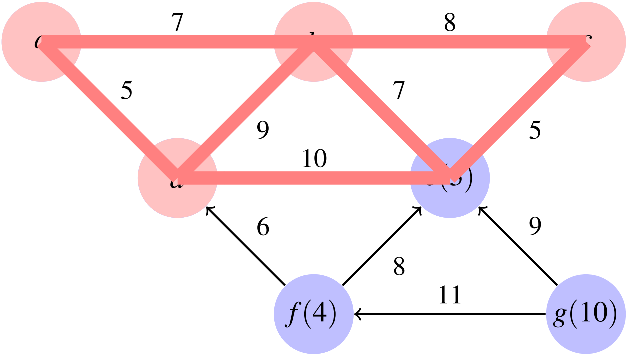
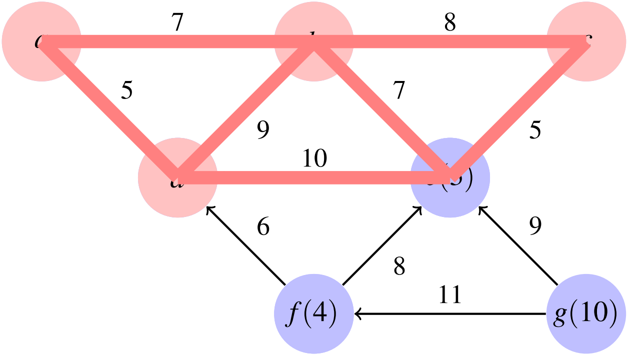


Example



Example

Contagion lasts 3 rounds.
Fundamental defaults: $\{i \mid c^0(i) = 0\} = \{a\}$.
Contagious defaults: $\{i \mid c^0(i) > 0 \;\&\; c^T(i) = 0\} = \{b, c, d\}$.
Total number of defaults: = 4.



Definition (Loss cascade)

Consider an initial configuration with capital levels $(c(j), j \in V)$. We define the sequence $(c_k(j), j \in V)_{k \geq 0}$ as

$$c_0(j) = c(j) \qquad \text{and} \quad c_{k+1}(j) = \max(c_0(j) - \sum_{\{i, c_k(i)=0\}} (1 - R_i) E_{ji}, 0), \quad (1)$$

where R_i is the recovery rate at the default of institution i.
$(c_{n-1}(j), j \in V)$, where $n = |V|$ is the number of nodes in the network, then represents the remaining capital once all counterparty losses have been accounted for. The set of insolvent institutions is then given by

$$\mathbb{D}(c, E) = \{j \in V : c_{n-1}(j) = 0\} \quad (2)$$

Default impact

Definition (Default Impact)

The *Default Impact* $DI(i, c, E)$ of a financial institution $i \in V$ is defined as the total loss in capital in the cascade triggered by the default of i:

$$DI(i, c, E) = \sum_{j \in V} c_0(j) - c_{\mathrm{final}}(j), \tag{3}$$

where $(c_{\mathrm{final}}(j), j \in V)_{k \geq 0}$ is the final level of capital at the end of the cascade with initial condition $c_0(j) = c(j) \quad \text{for} \quad j \neq i \quad \text{and} \quad c_0(i) = 0$.

Default Impact does not include the loss of the institution triggering the cascade, but focuses on the loss this initial default inflicts to the rest of the network: it thus measures the loss due to contagion.

Variants

If one adopts the point of view of deposit insurance, then the relevant measure is the sum of deposits across defaulted institutions:

$$DI(i, c, E) = \sum_{j \in \mathbb{D}(c,E)} Deposits(j).$$

Alternatively one can focus on lending institutions (e.g. commercial banks), whose failure can disrupt the real economy. Defining a set $\mathbb{C}$ of such **core** institutions we can compute

$$DI(i, c, E) = \sum_{j \in \mathbb{C}} c_0(j) - c_{\text{final}}(j)$$

Default impact in a macroeconomic stress scenarios: the Contagion index (Cont, Moussa, Santos 2010)

- Idea: measure the joint effect of economic shocks and contagion by measuring the Default Impact of a node in a macroeconomic stress scenario
- Apply a common shock Z (in % capital loss) to all balance sheets, where Z is a negative random variable
- "Stress scenario" = low values/quantiles of Z
- Compute Default Impact of node *k* in this scenario:

$$DI(\,k, c(1+Z)\,, E)$$

- Average across stress scenarios:

$$CI(k)=E[\,DI(\,k, c(1+Z)\,,E)\mid Z < z_q]$$

Forward-looking, based on exposures and stress scenarios

Heterogeneous stress scenarios

Macroeconomic shocks affect bank portfolios in a highly correlated way, due to common exposures of these portfolios.
Moreover, in market stress scenarios fire sales may actually exacerbate such correlations.
In many stress-testing exercises conducted by regulators, the shocks applied to various portfolios are actually scaled version of the same random variable i.e. perfectly correlated across portfolios.
A generalization is to consider co-monotonic shocks generated by a common factor Z:

$$\epsilon(i, Z) = c(i) f_i(Z) \tag{4}$$

f_i are strictly increasing with values in $(-1, 0]$, representing % loss in capital.
A *macroeconomic stress scenario* corresponds to low quantiles α of Z: $\mathbb{P}(Z < \alpha) = q$ where $q = 5\%$ or 1% for example.

The Contagion Index

Definition (Contagion Index)

The Contagion Index $CI(i, c, E)$ (at confidence level q) of institution $i \in V$ is defined as its expected Default Impact in a macroeconomic stress scenario:

$$CI(i, c, E) = \mathbb{E}\left[DI(i, c + \epsilon(Z), E) | Z < \alpha\right] \quad (5)$$

where the vector $\epsilon(Z)$ of capital losses is defined by (**??**) and α is the q-quantile of the systematic risk factor Z: $\mathbb{P}(Z < \alpha) = q$.

Z represents the magnitude of the macroeconomic shock
In the examples given below, we choose for α the 5% quantile of the common factor Z.

Contagion index: simulation-based computation

- Simulate independent values of Z
- Compute Default Impact of node *k* in each scenario as

$$DI(\, k, c+ \varepsilon(Z),E)$$

- Average across stress scenarios given by $Z<\alpha$

$$CI(k)=E[\, DI(\, k, c+\varepsilon(Z),E)|\, Z<\alpha]$$

Forward-looking, based on exposures and stress scenarios

Depends on:

- network structure through DI
- Joint distribution F of $\varepsilon(Z)=(\varepsilon_1(Z),\varepsilon_2(Z),\ldots\varepsilon_n(Z))$

Contagion index: empirical results for the Brazilian banking system

In the examples below, we model Z as a negative random variable with a heavy-tailed distribution F and an exponential function for f_i:

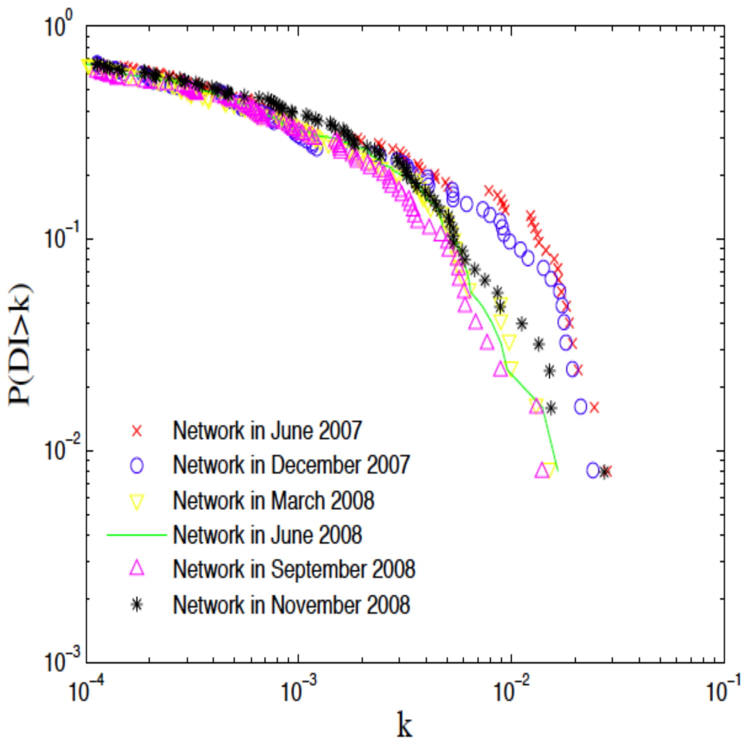
$$\epsilon(i, Z) = c(i)\,(\exp(\sigma_i Z) - 1) \tag{8}$$

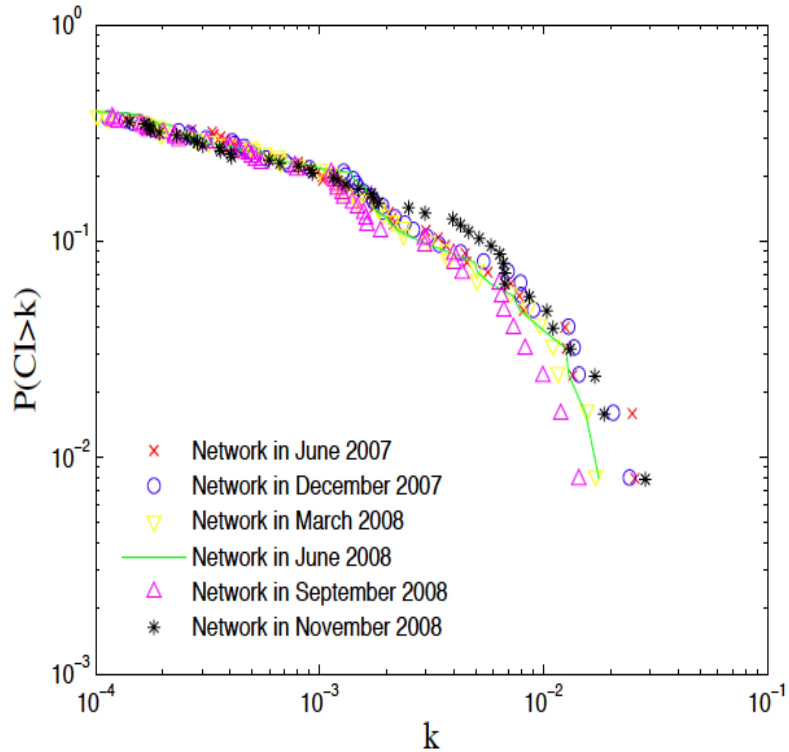
where σ_i is a scale factor which depends on the creditworthiness, or probability of default p_i, of institution i. For example, a possible specification is to choose σ_i such that p_i corresponds to the probability of losing 90% of the Tier 1 capital in a market stress scenario:

$$\sigma_i = -\frac{\log(10)}{F^{-1}(p_i)}. \tag{9}$$

Default probabilities are obtained from historical default rates given by Standard & Poors ratings for the firms at the date corresponding to the simulation.

Contagion index: empirical results for the Brazilian banking system





Empirical results for the Brazilian banking system

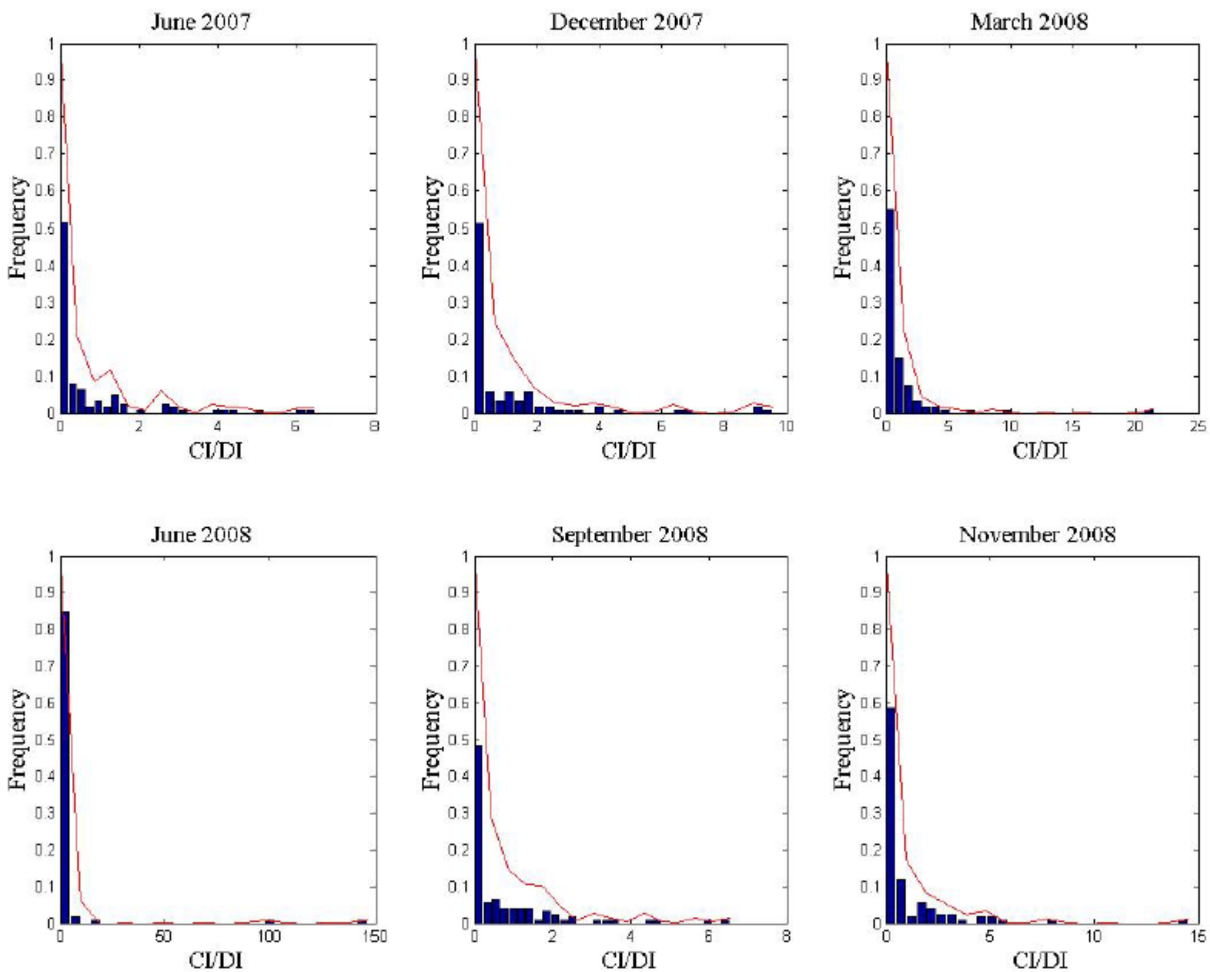


Figure 10: Default impact vs Contagion Index: the Contagion Index can be up to fifteen times larger than the Default Impact for some nodes.

Role of capital ratios

- Homogeneity: $\forall \lambda > 0, DI(i, \lambda c, \lambda E) = \lambda DI(i, c, E)$.
  Consequence : natural normalization is to express CI, DI as % of total capital
- Monotonicity in capital ratio: Default Impact and Contagion index, as % of initial capital, are (componentwise) increasing functions of ratio of exposures to capital $E(i,j)/c(i)$:

$$\forall i, j \in V, \frac{E(i,j)}{c(i)} > \frac{E'(i,j)}{c'(i)} \Rightarrow \forall k \in V, \frac{DI(k,c,E)}{\sum_i c(i)} \geq \frac{DI(k,c',E')}{\sum_i c'(i)}$$

- BUT: Default Impact and Contagion index are NOT monotone functions of the (usual) capital ratios! One can have

$$\forall i \in V, \frac{\sum_j E(i,j)}{c(i)} > \frac{\sum_j E'(i,j)}{c'(i)} \quad \text{and} \frac{DI(k,c,E)}{\sum_i c(i)} < \frac{DI(k,c',E')}{\sum_i c'(i)}$$

Capital-efficiency of networks

- The lack of monotonicity of the Contagion Index with respect to total capital or capital ratios leads to the question: given a network of exposures and capital allocation, is there a better scheme of capital requirements/allocations which reduces systemic risk (Contagion Indices) without increasing the total level of capital requirements?
- A capital allocation c in the network of exposures E is said to more globally capital-efficient than c' if

$$\sum_i c'(i) > \sum_i c(i) \quad \text{and} \quad \forall k \in V, CI(k, c', E) \leq CI(k, c, E)$$

Such examples exist! But they also arise in empirical data...

Monitoring nodes or monitoring links?
A new look at capital requirements

Current prudential regulation uses as main tool monitoring and lower bounds for capital ratios defined as $c(i)/A(i)$

where $A(i)$= sum of exposures of i+ other assets of i= $\Sigma_j E_{ij} + a(i)$

Typically a uniform lower bound is imposed on capital ratios for all institutions, *regardless of their size/ systemic risk.*

Capital ratios do not quantify the *concentration* of exposures.

On the other hand:

Simulations show the crucial role of contagious exposures ("weak links") with

$$E_{ij} > c(i) + \varepsilon_i(Z)$$

In other fields (epidemiology, computer network security,..) immunization strategies focus on

- Monitoring or immunizing the most 'systemic' nodes

-strengthening weak links as opposed to uniform or random monitoring.

This pleads for monitoring links representing large relative exposures relative to capital (large value of $E_{ij}/c(i)$)

In a heterogeneous network, this can make a big difference!

Targeted capital requirements

Using the Brazilian network data, we compare the 5% Tail condition expectation of the cross sectional distribution of the Contagion Index, in 3 cases:

(a) a minimum capital ratio is applied to all financial institutions in the network (*non-targeted capital requirements*),

(b) a minimum capital ratio applied only to the 5% most systemic institutions (*targeted capital requirements*),

(c) a minimum capital -to-exposure ratio is applied to the 5% most systemic institutions (disaggregated and targeted capital requirements),

We conclude that, given the heterogeneity of banks in terms of size, connectivity and systemic importance,

- targeting the most contagious institutions is more effective in reducing systemic risk than increasing capital ratios uniformly across all institutions, and
- capital requirements should not simply focus on the aggregate size of the balance sheet but depend on their concentration/distribution across counterparties: a minimal capital-to-exposure ratio can be a more effective way of controling contagious exposures.

Focusing on weak links: targeted capital requirements

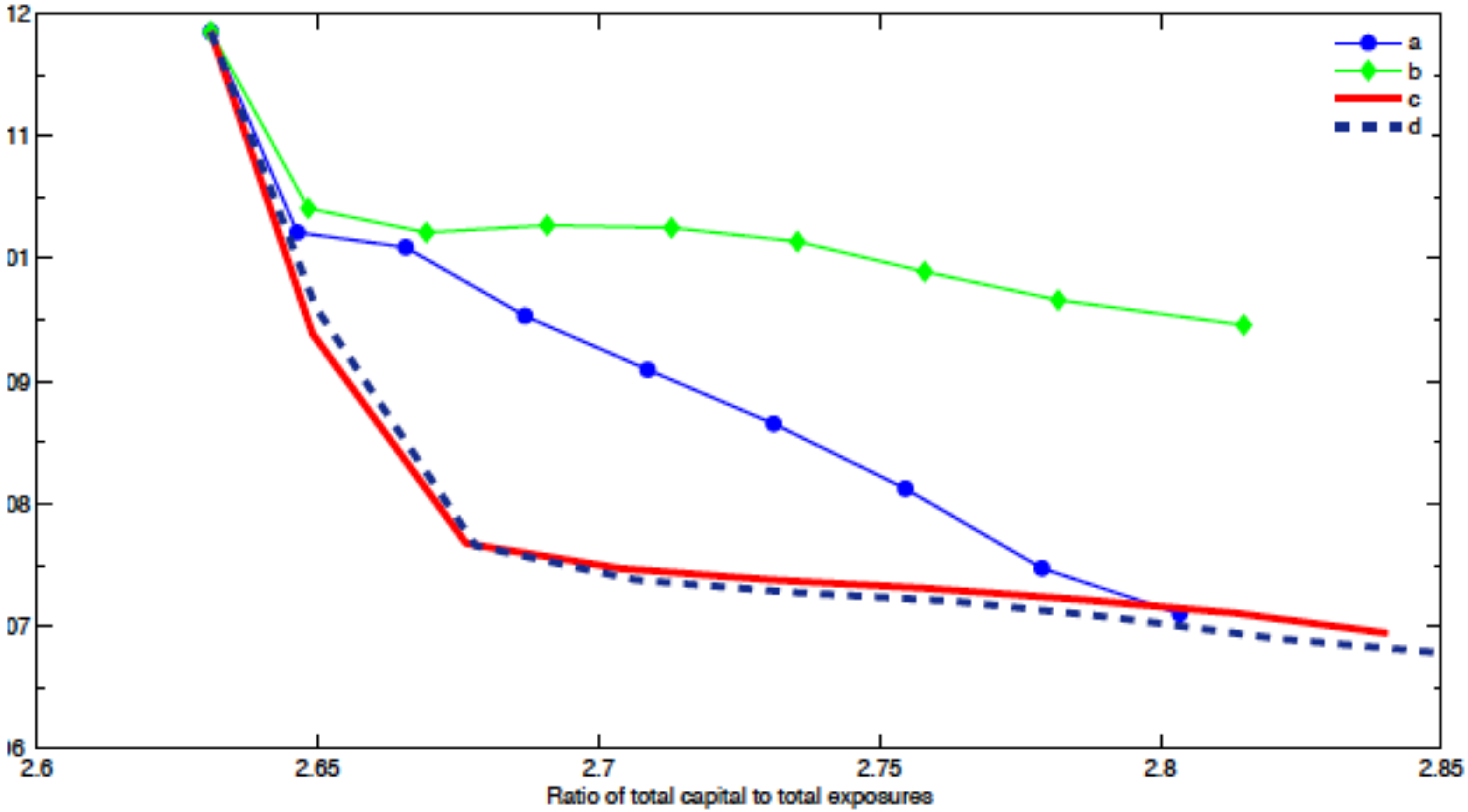


Figure 13.17 Comparison of various capital requirement policies: (a) imposing a minimum capital ratio for all institutions in the network, (b) imposing a minimum capital ratio only for the 5% most systemic institutions, (c) imposing a minimum capital-to-exposure ratio for the 5% most systemic institutions.

Role of macro-shocks and diversification

Stress scenarios triggered by large values of a risk factor Z:

$$\epsilon(i, Z) = c(i) f_i(Z) \tag{6}$$

$f_i(Z)$ represents the exposure of bank i to this risk factor.

- Monotonicity wrt macro-shocks: greater $|f_i|$ leads to greater values of Contagion Index.
- Contribution of macro shocks to $CI(k, c, E)$ is limited to he set $\{i, f_i(Z).f_k(Z) > 0\}$: tis set is smallest in totally segmented markets, and its size increases with diversification.
- Worst case: in a totally 'globalized'/ diversified market $\{i, f_i(Z).f_k(Z) > 0\} = V$
- Consequence: large-scale diversification increases exposure to systemic risk!
- Diversification reduces the 'volatility'/ marginal risk measure of bank portfolios in non-stress scenarios but.. increases the probability of joint losses in stress scenarios generated by the common risk factor(s) so increases the possibility of contagion.

Contagion in large counterparty networks: analytical results

- Amini, Cont, Minca (2010): mathematical analysis of the onset and magnitude of contagion in a large counterparty network (n->∞)
- Main point: contagion may become large-scale if

$$\sum_{j,k} \frac{\mu(j,k)\, jk}{\lambda} q(j,k) > 1$$

where

μ(j,k)= proportion of nodes with with j debtors, k creditors

λ = average number of counterparties

q(j,k) : fraction of *overexposed* nodes with (j,k) links,

= fraction of nodes with degree (j,k) such that at least ONE exposure exceeds capital

Analysis of cascades in large networks

We describe the topology of a large network by the joint distribution $\mu_n(j,k)$ of in/out degrees and assume that μ_n has a limit μ when graph size increases in the following sense:

1. $\mu_n(j,k) \to \mu(j,k)$ as $n \to \infty$: the proportion of vertices of in-degree j and out-degree k tends to $\mu(j,k)$).
2. $\sum_{j,k} j\mu(j,k) = \sum_{j,k} k\mu(j,k) =: m \in (0,\infty)$ (finite expectation property);
3. $m(n)/n \to m$ as $n \to \infty$ (averaging property).
4. $\sum_{i=1}^{n} (d_{n,i}^{+})^2 + (d_{n,i}^{-})^2 = O(n)$ (second moment property).

A random network model for asymptotics

To embed out networks in an ensemble of networks with increasing size, we use the *configuration model*

Given a sequence of in/out degrees $d^+_{n,i}$ and out-degrees $d^-_{n,i}$ and exposure matrices (E^n_{ij}), we generate a random ensemble of networks with the same degree sequence by randomly permuting the exposures across links going out of each node

This construction generates random networks with the *same* degree sequences and *same* distribution of exposures, which can be both specified from data.

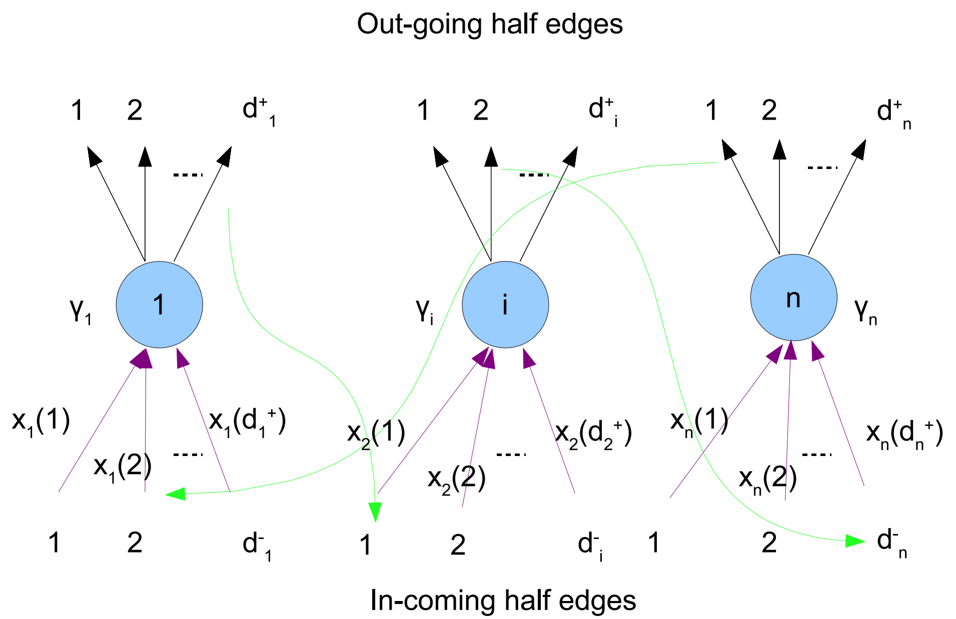


Figure 14: Random configuration model: random matching of in-coming half-edges with weighted out-going half-edges.

Contagious links: $i \to j$ is a contagious link if the default of i generates the default of j.

For each node i and permutation $\tau \in \Sigma_{d^+(i)}$, we define

$$\Theta(i,\tau) := \min\{k \geq 0, c_i < \sum_{j=1}^{k} E^n_{i,\tau(j)}\}$$

$\Theta(i,\tau)$ = number of counterparty defaults which will generate the default of i if defaults happen in the order prescribed by τ:

$$p_n(j,k,\theta) := \frac{\#\{(i,\tau) \mid \tau \in \underbrace{\Sigma_j}, \ d_i^{(n)+} = j, \ d_i^{(n)-} = k, \ \Theta(i,\tau) = \theta\}}{n\mu_n(j,k)j!}.$$

$n\mu_n(j,k)jp_n(j,k,1)$ is the total number of contagious links that enter a node with degree (j,k).

The value $p_n(j,k,1)$ gives the proportion of contagious links ending in nodes with degree (j,k).

Proposition 1 (Asymptotic fraction of defaults)**.** *Under the above assumptions:*

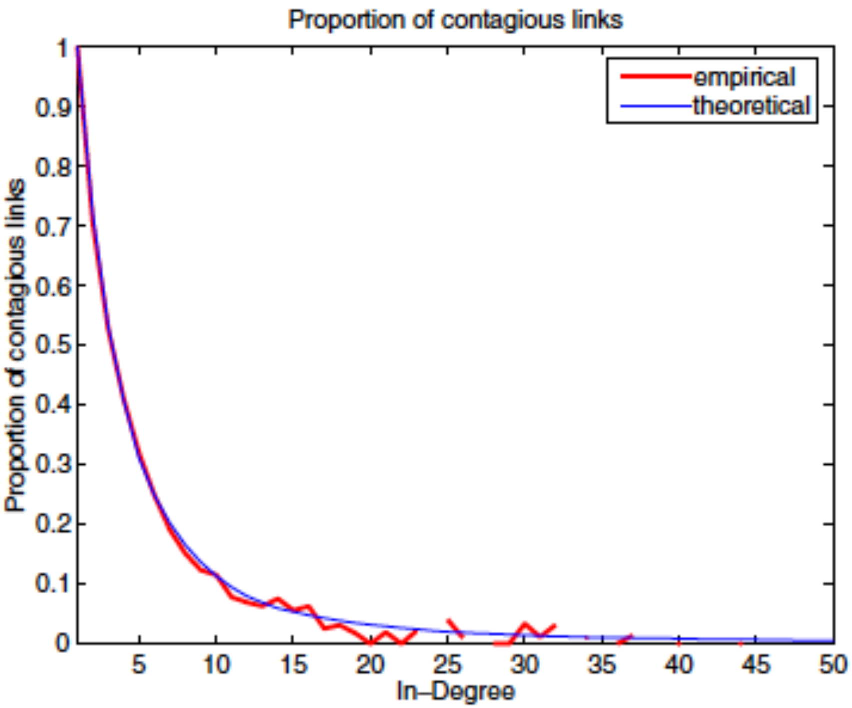
1. *If $\pi^* = 1$, i.e. if $I(\pi) > \pi$ for all $\pi \in [0, 1)$, then an initial default of a finite subset leads to global cascade where asymptotically all nodes default.*

$$\frac{|D(A, c_n, E_n)|}{n} \xrightarrow{p} 1$$

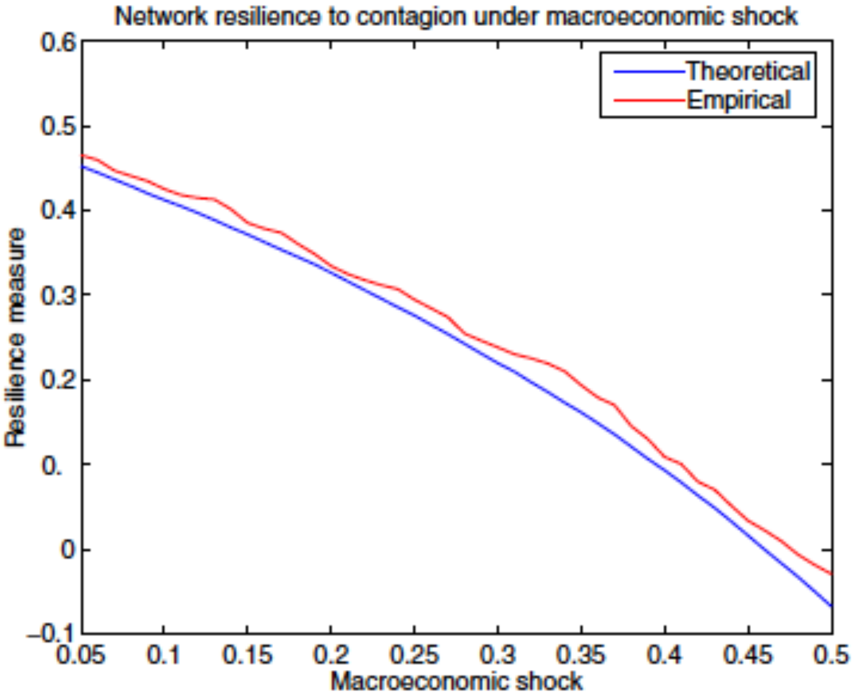
2. *If $\pi^* < 1$ and furthermore π^* is a stable fixed point of I, then the asymptotic fraction of defaults*

$$\frac{|D(A, c_n, E_n)|}{n} \xrightarrow{p} \sum_{j,k} \mu(j, k) \sum_{\theta=0}^{j} p(j, k, \theta)\beta(j, \pi^*, \theta).$$

The relevance of asymptotics



(a)



(b)

Resilience to contagion This leads to a condition on the network which guarantees absence of contagion:

Proposition 2 (Resilience to contagion)**.** *Denote* $p(j,k,1)$ *the proportion of contagious links ending in nodes with degree* (j,k)*. If*

$$\sum_{j,k} k \frac{\mu(j,k)\ j}{\lambda} p(j,k,1) < 1 \tag{11}$$

then with probability $\to 1$ *as* $n \to \infty$*, the default of a finite set of nodes cannot trigger the default of a positive fraction of the financial network.*

Resilience condition:

$$\sum_{j,k} k \frac{\mu(j,k)\ j}{\lambda} p(j,k,1) < 1 \qquad (12)$$

This leads to a *decentralized recipe* for monitoring/regulating systemic risk: monitoring the capital adequacy of each institution with regard to its *largest exposures.*

This result also suggests that one need not monitor/know the *entire* network of counterparty exposures but simply the *skeleton*/ subgraph of contagious links.

It also suggests that the regulator can efficiently contain contagion by focusing on fragile nodes -especially those with high connectivity- and their counterparties (e.g. by imposing higher capital requirements on them to reduce $p(j,k,1)$).

A measure for the resilience of a financial network

- Stress scenario: apply a common macro-shock Z, measured in % loss in asset value, to all balance sheets in network
- The fraction q(j,k,Z) of *overexposed* nodes with (j,k) links is then an increasing function of Z
- Network remains resilient as long as $\sum_{j,k} \frac{\mu(j,k)\,jk}{\lambda} q(j,k,Z) < 1$

DEFINITION: Network Resilience = maximal shock Z* network can bear while remaining resilient to contagion

Z* is solution of

$$\sum_{j,k} \frac{\mu(j,k)\,jk}{\lambda} q(j,k,Z) < 1$$

Given network data, Z* computed by solving single equation

Simulation-free stress testing of banking systems

- These analytical results may be used for stress-test the resilience of a banking system, *without* the need for large scale simulation.
- Stress scenario: apply a common macro-shock Z, measured in % loss in asset value, to all balance sheets in network
- Analytical result allow to compute fraction of defaults as function of Z
- Network remains resilient (no macro-cascade) as long as

$$\sum_{j,k} \frac{\mu(j,k)jk}{\lambda} q(j,k,Z) < 1 \Leftrightarrow Z < Z^*$$

An abrupt transition from resilience to non-resilience occurs when shock amplitude reaches Z*: cascade size/ number of defaultsas function of initial shock Z is discontinuous at Z=Z*

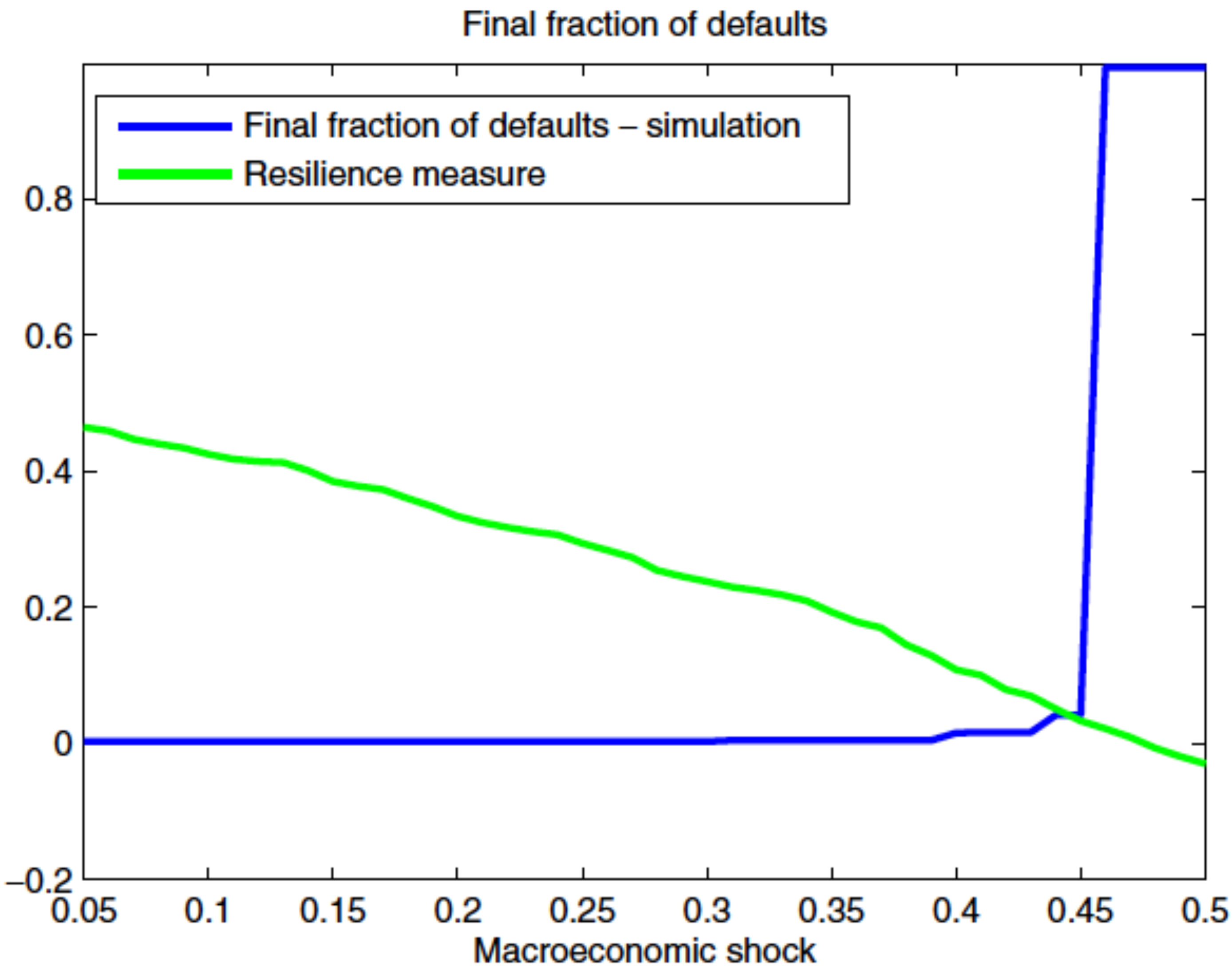


Fig. 7. Final fraction of defaults triggered by an initial fraction of defaults representing 0.1% of the total network.

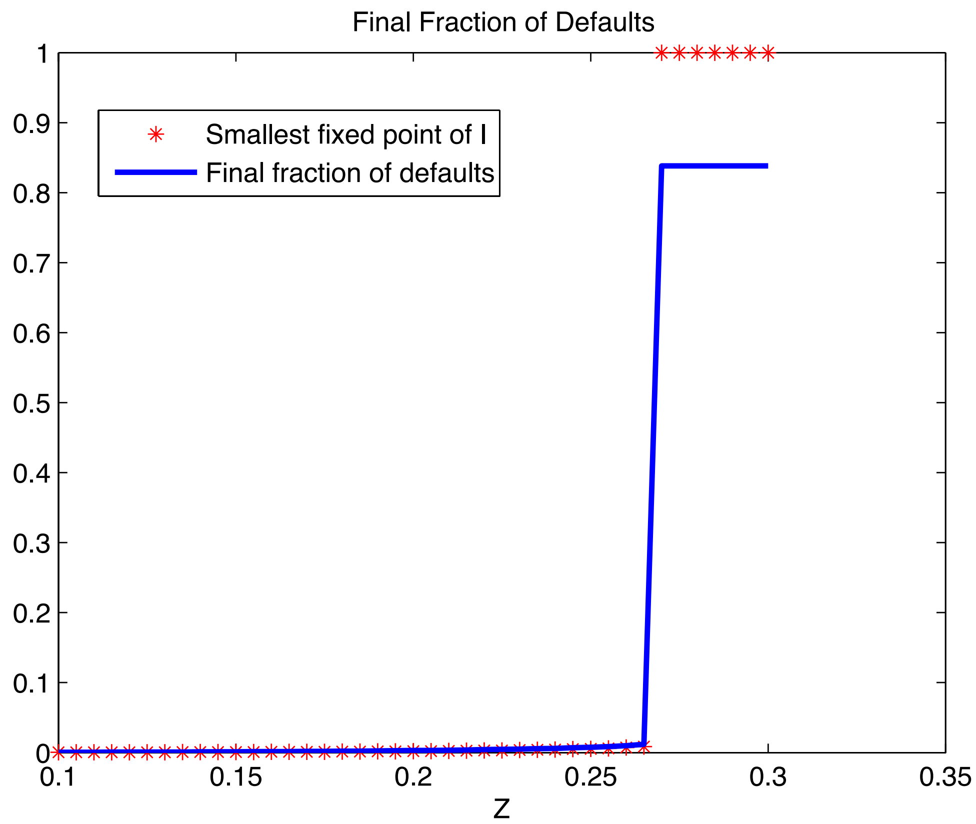


Figure 15: Final fraction of defaults as a function of common shock to balance sheets in a scale-free directed network with Pareto exposures

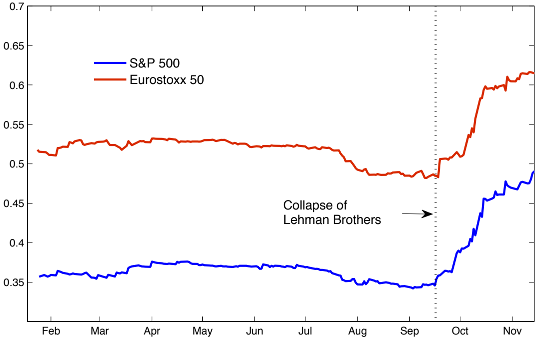
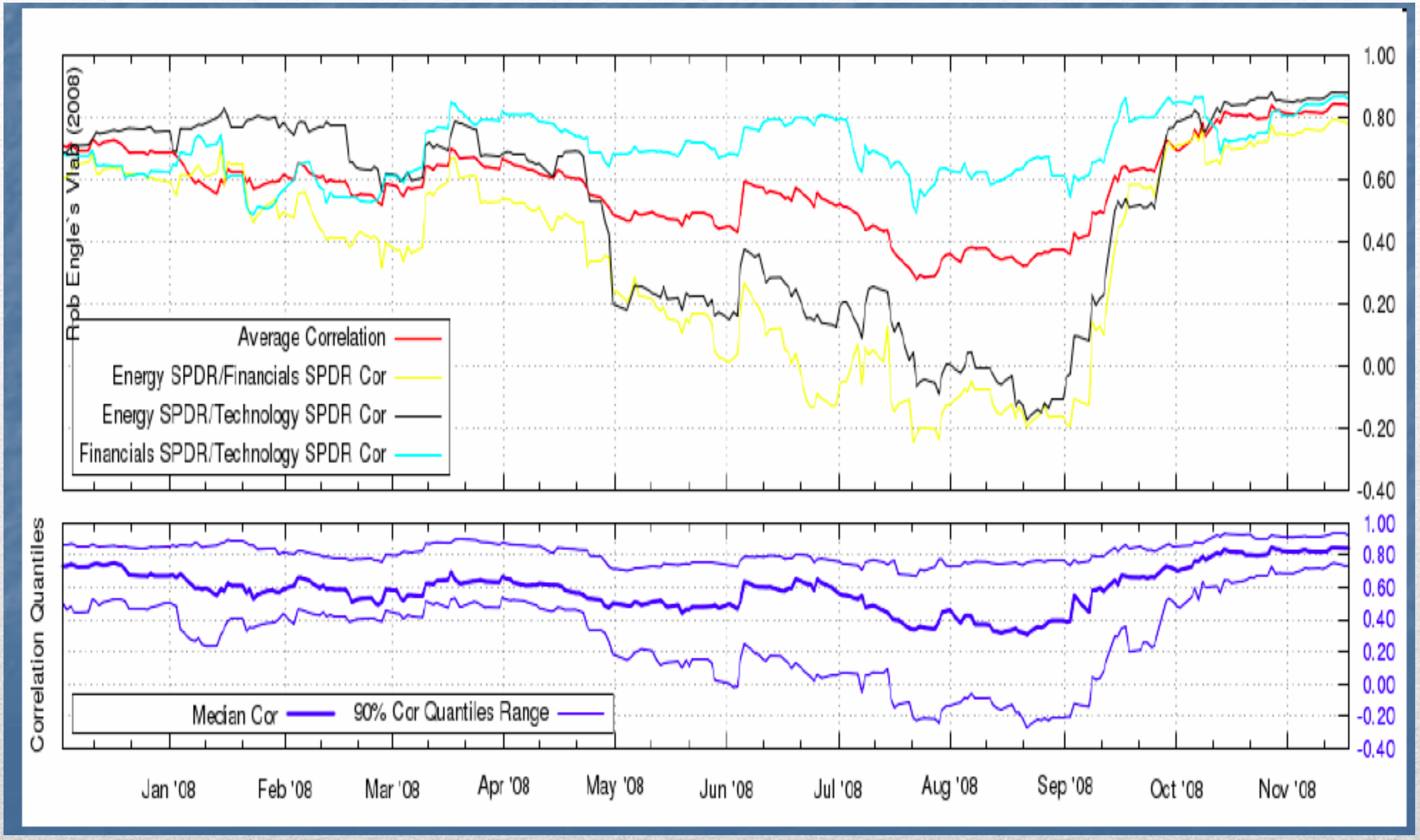


Figure: EWMA average correlation among SPDRs and Eurostoxx 50



Correlations among US stock returns, 2008

$$Y_t = \gamma_0 + \sum_{k=1}^{4} \gamma_{1k} Y_{t-k} + \gamma_{2k} I_{2006}\ ABX_{t-k} + \gamma_{3k} I_{2007}\ ABX_{t-k} + \epsilon_{it}$$

Y	ABX	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	$p(\gamma_2 = 0)$	$p(\gamma_3 = 0)$	Adj. R^2
One-Year Treasury	AAA	−0.36	−0.21	−0.11	−0.16	2.68	2.71	3.50	0.44	0.988	0.000	0.249
	AA	0.48	0.06	0.09	0.10	3.94	1.43	5.12	−0.73	0.993	0.000	0.322
	A	−0.53	0.49	−0.84	−0.67	0.47	0.77	6.47	−0.21	0.760	0.000	0.364
	BBB	−0.16	0.14	−0.20	−0.06	−0.05	−0.47	5.57	1.35	0.999	0.000	0.303
	BBB−	−0.08	0.41	−0.50	−0.06	0.22	−1.11	5.43	2.60	0.983	0.000	0.323
Ten-Year Treasury	AAA	0.25	−0.20	0.17	0.69	0.69	2.60	0.79	1.57	0.887	0.009	0.081
	AA	1.51	1.58	0.68	0.50	1.44	1.30	2.59	1.13	0.337	0.003	0.135
	A	−0.48	0.45	−1.07	−0.88	0.75	0.66	3.00	0.97	0.638	0.002	0.131
	BBB	−0.37	0.05	−0.77	0.31	0.63	0.47	3.48	0.56	0.931	0.000	0.155
	BBB−	−0.13	0.91	−1.26	0.29	1.37	0.33	2.61	1.46	0.716	0.001	0.152
S&P 500 Financials	AAA	−0.90	−0.09	0.26	0.26	3.17	−1.86	1.12	1.92	0.905	0.007	0.287
	AA	−0.90	−0.13	0.19	0.08	3.67	−2.61	3.48	1.00	0.903	0.000	0.363
	A	−1.28	−0.93	−0.70	−0.44	1.86	−0.50	3.76	0.71	0.462	0.000	0.359
	BBB	−0.50	−0.76	−0.94	−0.11	1.76	0.79	3.58	1.18	0.756	0.000	0.386
	BBB−	−0.34	−0.75	−0.44	0.32	2.04	1.84	2.35	0.06	0.861	0.000	0.355

Correlation between subprime ABX index returns and SP500 returns: negative before 2006, positive after 2007!

Longstaff (2009): The subprime credit crisis and contagion in financial markets.

Trading strategies and market impact

Most market participants do not have static portfolios but follow an *investment strategy*.
Each strategy implies entering/exiting positions through trades.
Trades impact market prices: Empirical studies (Obizhaeva 2008; Cont Kukanov Stoikov 2010) provide evidence for the linearity of this price impact at daily and intraday frequencie: a trade of size X moves prices by

$$\frac{\Delta S(t)}{S(t)} = r(t) + \frac{X}{\lambda}$$

$r(t)$: "fundamental" return, λ measure the market depth.
A large trade by a market participant (or many synchrnoized trades by small ones) may result in a cumulative market impact that substantially modifies price dynamics.

Deleveraging schedule

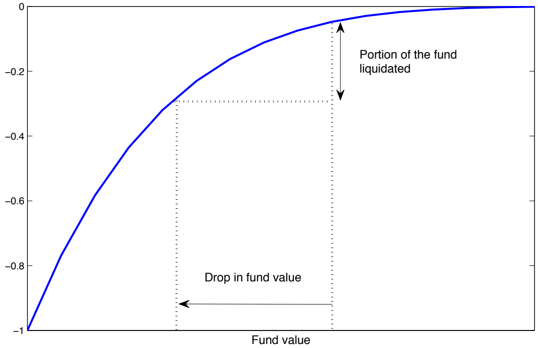


Figure: As fund value drops, manager/investors exit their positions: this is modeled by a 'liquidation schedule' $f(.)$: $X_i(t) = \alpha_i f(V(t)/V(0))$

Price impact of distressed selling

- Distressed selling activity impacts prices: market impact on asset i's return is equal to $\frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))$
- λ_i represent the *depth* of the market in asset i: a net demand of $\frac{\lambda_i}{100}$ shares for security i moves i's price by one percent.
- This impact is not 'random': it happens precisely when the fund has large losses.
- How does the price impact of distressed selling translate into volatility, correlation and portfolio risk ?

Price dynamics

$$\begin{tikzcd}[column sep=huge] t_k \arrow[d] & & t_{k+1} \arrow[d] \\ \boxed{S_k, V_k} \arrow[r, "\text{exogenous}", "\text{economic factors}"'] & \boxed{S^*_{k+1}, V^*_{k+1}} \arrow[r, "\text{distressed selling}", "\text{short selling}"'] & \boxed{S_{k+1}, V_{k+1}} \end{tikzcd}$$

Return of asset class $k=$ sum of fundamental component + price impact

$$\frac{S^i_{k+1} - S^i_k}{S^i_k} = \sqrt{\tau}\xi^i_{k+1} + \frac{\alpha_i}{\lambda_i}\left(f(\frac{V_k}{V_0} + \sum_{i=1}^{n} \frac{\alpha_i S^i_k}{V_0}\sqrt{\tau}\xi^i_{k+1}) - f(\frac{V_k}{V_0}) \right) \tag{1}$$

where $V_k = \sum_{i=1}^{n} \alpha_i S^i_k$ is the fund value.

A simulation example: the LTCM effect

Consider a large fund invested in 3 asset classes, whose returns are 'fundamentally uncorrelated': fundmental covariances are assumed ZERO.
The fund quickly reduced by 20% its positions over a one month period, generating a volume which is 10% of market depth during this period.
What is the impact on market correlation/ volatility of the assets?
What is the volatility/correlation experienced by the fund during its liquidation period?

'Fundamentally uncorrelated' assets can correlate during liquidation!

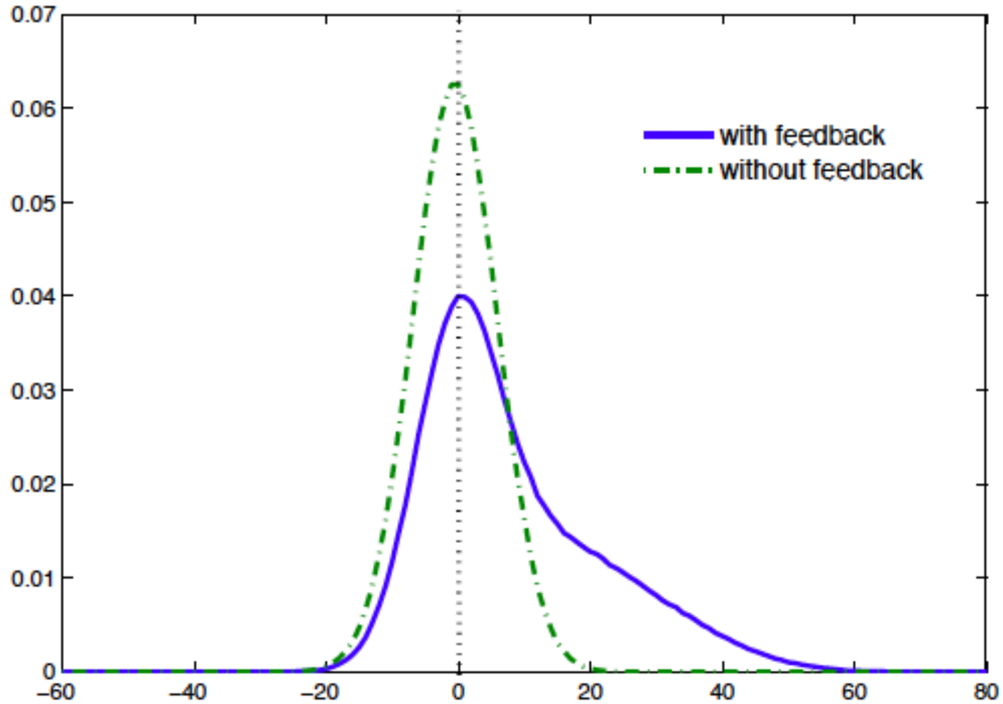
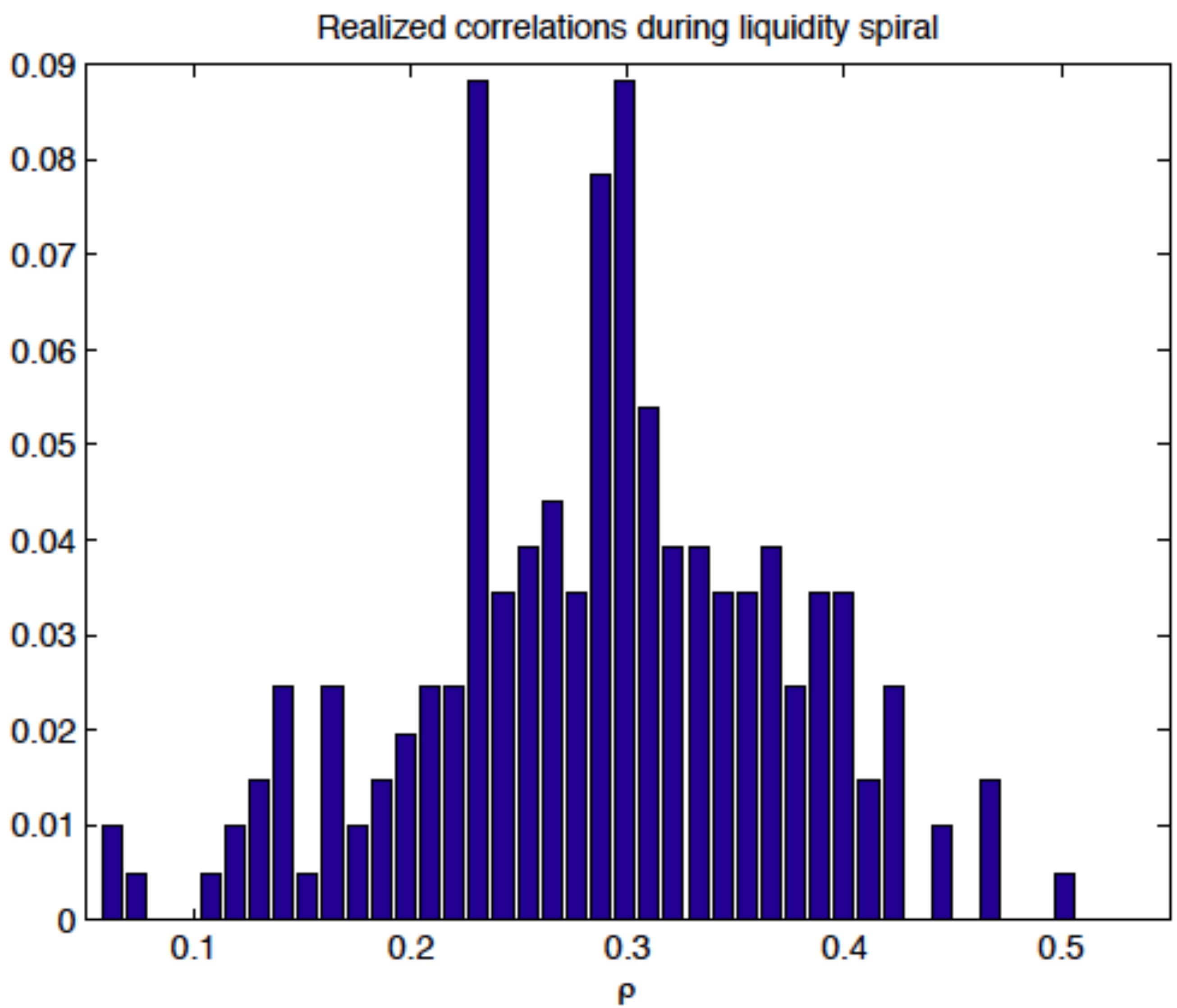


Figure: Distribution of realized correlation between the two securities


Realized correlations during liquidity spiral
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
ρ

Continuous-time limit: diffusion model for liquidation value

Theorem

When deleveraging schedule is smooth $f \in C_b^3$ and risk factors verify $\mathbb{E}(|\epsilon_i|^4) < \infty$, the continuous-time limit of the price dynamics is given by a diffusion model $P_t = (P_t^1, \ldots P_t^n)^t$ where

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \qquad 1 \leq i \leq n$$

$$\mu_i(P_t) = \frac{\alpha_i}{2\lambda_i} f''\left(\frac{V_t}{V_0}\right) \frac{< \pi_t, \Sigma\pi_t >}{V_0^2}; \sigma_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{\lambda_i} f'\left(\frac{V_t}{V_0}\right) \frac{(A^t \pi_t)_j}{V_0}$$

- $\pi_t = (\alpha_1 P_t^1, \ldots, \alpha_n P_t^n)^t$ *is the (dollar) allocation of the fund*
- $V_t = \sum_{1 \leq i \leq n} \alpha_i P_t^i$ *is the value of the fund*

Realized covariance

Proposition

The realized covariance $C_s^{i,j}$ between returns of i and j is given by the sum of a fundamental covariance and a *liquidity-dependent excess covariance term*

$$C_t^{i,j} = \Sigma_{i,j} + \frac{\alpha_j}{\lambda_j} f'(\frac{V_t}{V_0}) \frac{(\Sigma\pi_t)_i}{V_0} + \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \frac{(\Sigma\pi_t)_j}{V_0}$$

$$+ \frac{\alpha_i\alpha_j}{\lambda_i\lambda_j} (f')^2 (\frac{V_s}{V_0}) \frac{< \pi_s, \Sigma\pi_s >}{V_0^2}$$

where $\pi_s = (\alpha_1 P_s^1, ..., \alpha_n P_s^n)^t$ are portfolio wieghts.

Realized covariance is *scenario-dependent*, liquidity-dependent and depends on the ratio (Liquidation Size)/Market Depth.

- Network models provide useful framework for analyzing contagion of default via insolvency/ illiquidity
- Financial networks are highly heterogeneous (exposures,connectivity, size): simple, homogeneous networks may provide wrong insights on systemic risk.
- Pay attention to the risk measures, not just the model: due to strong heterogeneity, assessments of contagion risk based on cross-sectional averages do not reflect the contribution of contagion to systemic risk.
- Asymptotic analysis of large networks allows to derive rigorous, explicit mathematical results about the relation between network structure and resilience to contagion for networks with arbitrary topology, which explain results of large-scale simulations.
- Heterogeneity entails that targeted capital requirements – focusing on the most systemic institutions - are more effective for reducing systemic risk.
- Disaggregating capital ratios: monitoring concentration of exposures (as % of capital) is more important than capital ratios based on total asset size.

Some references

- R Cont, A Moussa, E B Santos (2010) Network structure and systemic risk in banking systems. To appear in: Handbook of Systemic Risk, Camb Univ Press.
- H Amini, R Cont, A Minca (2012) Stress testing the resilience of financial networks, *Intl Journal of Theoretical and applied finance*, Vol 15, No 1.
- H Amini, R Cont, A Minca (2011) Resilience to contagion in financial networks, SSRN.
- R Cont & L Wagalath (2011) Running for the exit: distressed selling and endogenous correlation in financial markets.
- R Cont, L Wagalath (2012) Fire sales forensics: measuring endogenous risk.