The Hopf algebra of dissection polylogarithms

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1. (Motivic) Aomoto polylogarithms

2. The computation of the motivic coproduct

3. Dissection polylogarithms

Definition

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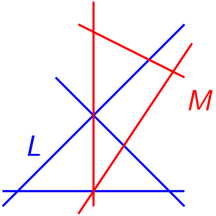

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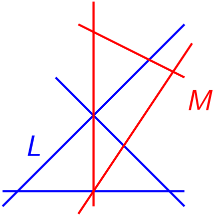

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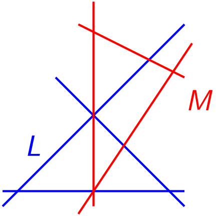

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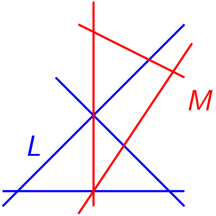

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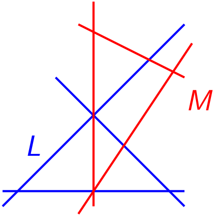

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$$\rightsquigarrow I(L; M) = \int_{\Delta_M} \omega_L$$

Examples

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Example (Multiple zeta values)

$$\zeta(n_1, \cdots, n_r) = (-1)^r \, \mathbb{I}(0; \underbrace{1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{1, 0, \cdots, 0}_{n_r}; 1)$$

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The motivic Hopf algebra

Let $\mathcal{H}$ be the Hopf algebra of functions on U: $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$

$$\mathrm{MTM} \cong Comod(\mathcal{H}_\bullet)$$

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What can we recover from the motivic Aomoto polylogarithms?

$I^{\mathcal{H}}(L;M) \rightsquigarrow I(L;M)$ up to the ambiguity of the choice of Δ_M inside $\mathbb{P}^n(\mathbb{C}) \setminus L$.

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- We want a combinatorial formula for $\Delta(I^{\mathcal{H}}(L;M))$ in terms of $I^{\mathcal{H}}(\cdots)$.
- More degeneracy = fewer terms.

What is known?

- The low dimensional cases: $n \leq 3$ (Goncharov, Zhao).
- Pairs of simplices in general position.
- Iterated integrals (Goncharov).
- Dissection polylogarithms (D.).

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Theorem (Goncharov, 2001)

$$\Delta(\mathbb{I}^{\mathcal{H}}(u_0; u_1, \cdots, u_n; u_{n+1})) = \sum_{\substack{0 \leq k \leq n \\ 0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = n+1}}$$

$$\mathbb{I}^{\mathcal{H}}(u_0; u_{s_1}, \cdots, u_{s_k}; u_{n+1}) \otimes \prod_{i=0}^{k} \mathbb{I}^{\mathcal{H}}(u_{s_i}; u_{s_i+1}, \cdots, u_{s_{i+1}-1}; u_{s_{i+1}})$$

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Computing $\Delta(I^{\mathcal{H}}(L; M)) \longleftrightarrow$ finding functorial bases for $\mathrm{gr}_{2k}^W H(L; M)$.

A relative Brieskorn-Orlik-Solomon theorem

Theorem (Brieskorn-Orlik-Solomon)

Let $L = L_1 \cup \cdots \cup L_N$ be a union of linear hyperplanes in $\mathbb{C}^n$. Then we have an isomorphism of graded algebras

$$H^\bullet(\mathbb{C}^n \setminus L) \cong \Lambda^\bullet(e_1, \cdots, e_N)/\mathcal{R}$$

where $\mathcal{R}$ is generated by the relations:

$$\sum_{i=1}^k (-1)^i e_{s_1} \wedge \cdots \wedge \widehat{e_{s_i}} \wedge \cdots \wedge e_{s_k} = 0$$

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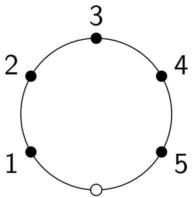
Theorem (D.)

For $k = 0, \cdots, n$ we have an explicit functorial presentation

$$\mathrm{gr}^W_{2k} H^n(\mathbb{P}^n \setminus L, M \setminus M \cap L) \cong \left(\Lambda^k(e_0, \cdots, e_n) \otimes \Lambda^{n-k}(f_0, \cdots, f_n)\right) / \mathcal{R}'$$

Dissection diagrams



Dissection diagram of degree n

We start with a polygon with $n+1$ vertices, with a special vertex called the *root*.

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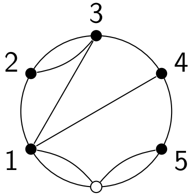

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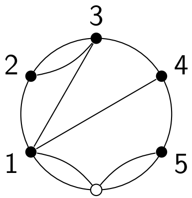

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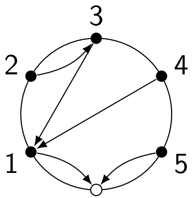

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Hence *the chords form a rooted tree*.

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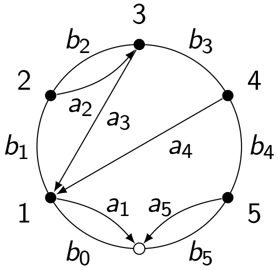

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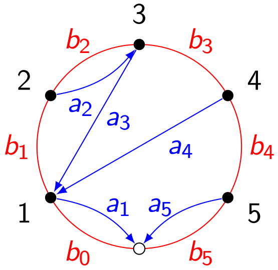

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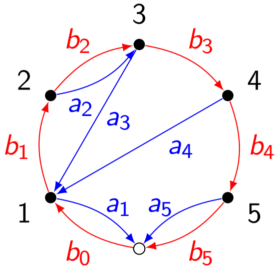

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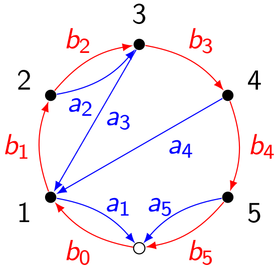

Generic decorations

For every simple cycle in the total (directed) graph, the (directed) sum of the decorations is $\neq 0$.

Dissection polylogarithms

D dissection diagram of degree n → $(L; M) \subset \mathbb{P}^n(\mathbb{Q})$ → Dissection polylogarithm
$I(D) := I(L; M)$
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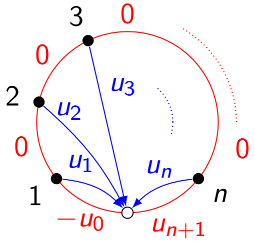

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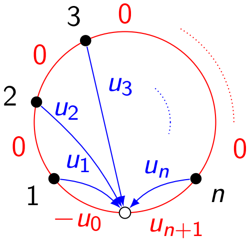
The special case of (generic) iterated integrals

Genericity condition: $u_i \neq u_j$ for $i \neq j$.

$$I(D) = \mathbb{I}(u_0; u_1, \cdots, u_n; u_{n+1}) = \int_{u_0 \leq t_1 \leq \cdots \leq t_n \leq u_{n+1}} \frac{dt_1}{t_1 - u_1} \cdots \frac{dt_n}{t_n - u_n}$$

$D =$ 

Reduction to iterated integrals

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For every dissection diagram D of degree n, the dissection polylogarithm $I(D)$ is a linear combination with $\mathbb{Z}$-coefficients of (generic) iterated integrals $\mathbb{I}(u_0; u_1, \cdots, u_n; u_{n+1})$ where the u_k's are linear combinations with $\mathbb{Z}$-coefficients of the decorations a_i and b_j of D.

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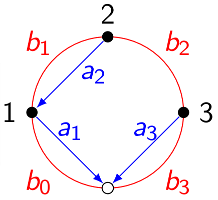

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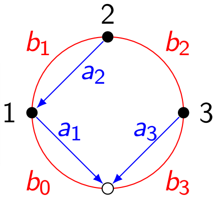

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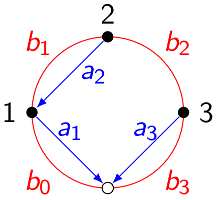

- One can describe an algorithm that reduces $I(D)$ to iterated integrals. But there is no *canonical* algorithm.
- The number of iterated integrals that appear is between 1 and $n!$.

The motivic coproduct of motivic dissection polylogarithms

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The motivic coproduct of the motivic dissection polylogarithms is given by the formula

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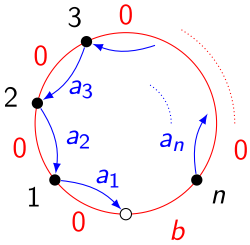
- For D = a corolla, we recover Goncharov's formula for the coproduct of (generic) iterated integrals.

Path polylogarithms



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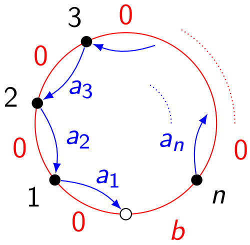
$$J(a_1, \cdots, a_n; b) = \int_{\Delta(0,b)} \frac{dt_1 \cdots dt_n}{(t_1 - a_1)(t_2 - t_1 - a_2)\cdots(t_n - t_{n-1} - a_n)}$$



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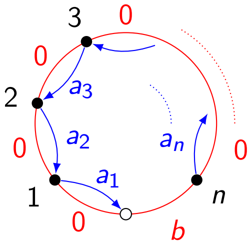

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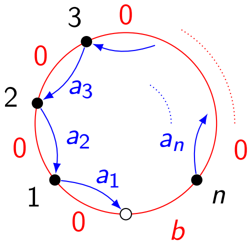

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Proposition

$$\Delta(J^{\mathcal{H}}(a_1, \cdots, a_n; b)) = \sum_{S \subset \{1, \cdots, n\}} J^{\mathcal{H}}(a(S); b) \otimes J^{\mathcal{H}}(a(\overline{S}); b - a_S)$$

$$(a_S = \textstyle\sum_{s \in S} a_s)$$