Elliptic Grothendieck-Teichmüller Theory

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Part I. Introduction

Hain-Matsumoto studied the category of mixed elliptic motives, MEM. Let U denote the unipotent radical of the Tannakian fundamental group $\pi_1(MEM)$ and $\mathfrak{u}$ its Lie algebra; it fits into a short exact sequence

$$1 \to \mathfrak{u}^{geom} \to \mathfrak{u}^{MEM} \to \mathfrak{g} \to 1,$$

where $\mathfrak{g}$ is the Lie algebra of the unipotent radical of $\pi_1(MTM)$. There is a section of this exact sequence corresponding to the choice of the Tate curve $E'_{\partial/\partial q}$, using which we can write $\mathfrak{u}^{MEM} \simeq \mathfrak{u}^{geom} \rtimes \mathfrak{g}$.

The Lie algebra $\mathfrak{u}^{MEM}$ acts on the fundamental Lie algebra of the Tate curve: thus we can write

$$\mathfrak{u}^{geom} \rtimes \mathfrak{g} \to \mathrm{Der}\,\mathrm{Lie}\,\pi_1^{unip}(E').$$

Taking the associated gradeds for the weight filtration, we obtain

$$Gr^w \mathfrak{u}^{geom} \rtimes \mathfrak{g} \to \mathrm{Der}\,\mathrm{Lie}(H)$$

where $\mathrm{Lie}(H) = Gr^w \mathrm{Lie}\pi_1^{unip}(E'_{\partial/\partial q})$. This $\mathrm{Lie}(H)$ is simply isomorphic to $\mathrm{Lie}[a, b]$, the free Lie algebra on two generators.

In this talk, we'll introduce four different candidates, or avatars, for the graded Lie algebra $Gr^w \mathfrak{u}^{MEM}$. They generalize four Lie algebras that are closely related in the genus zero situation:

$$\mathfrak{g} \hookrightarrow \mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{krv},$$

where

$$\begin{cases} \mathfrak{g} = \mathrm{Lie}\, \pi_1(MTM)^{unip} & \text{is the (free) motivic fundamental Lie algebra} \\ \mathfrak{grt} & \text{is the Grothendieck-Teichmüller Lie algebra} \\ \mathfrak{ds} (= \mathfrak{dmr}) & \text{is the double shuffle Lie algebra} \\ \mathfrak{krv} & \text{is the Kashiwara-Vergne Lie algebra.} \end{cases}$$

The key player here is the Lie algebra $\mathcal{E}$, discovered and studied by many people (Nakamura-Ihara, Levin-Racinet, Calaque-Enriquez-Etingof, Hain-Matsumoto, Pollack).

Let $\mathcal{E} \subset \mathrm{DerLie}[a, b]$ be the Lie subalgebra of derivations

$$\mathcal{E} = \mathrm{Lie}[ad(\epsilon_0)^i \epsilon_{2n}] \subset \mathrm{Der\,Lie}[a, b],$$

where

$$\epsilon_0(a) = b, \ \epsilon_0(b) = 0$$

and for $n \geq 2$,

$$\epsilon_{2n}(a) = ad(a)^{2n}(b), \quad \epsilon_{2n}(b) = \sum_{j=0}^{n-1} [ad(a)^j(b), ad(a)^{2m-j-2}(b)].$$

Every derivation in $\mathcal{E}$ maps $[a, b] \mapsto 0$.

Representation in $\mathrm{Lie}[a,b]$ **of** $Gr^w\mathfrak{u}^{MEM}$

The first avatar is the one studied by Hain and Matsumoto, who showed that the image of $Gr^w\mathfrak{u}^{geom}$ is equal to $\mathcal{E}\subset \mathrm{Der}\,\mathrm{Lie}[a,b]$. One can conjecture that the map is injective, so the first candidate for an explicit description of $Gr^w\mathfrak{u}^{MEM}$ is simply the image

$$\mathcal{E}\rtimes\mathfrak{g}\subset \mathrm{Der}\,\mathrm{Lie}[a,b].$$

Elliptic double shuffle

The second avatar would be a semi-direct product

$$\mathfrak{plds}\rtimes\mathfrak{ds}.$$

Here, $\mathfrak{plds}$ denotes the *pseudo-linearized double shuffle* space defined below. Its elements are Lie polynomials $f(a,b)$, and we identify it with a subspace of $\mathrm{Der}\,\mathrm{Lie}[a,b]$ by associating $D_f(a)=f$ and $D_f([a,b])=0$ to f.

We will show later how the double shuffle Lie algebra $\mathfrak{ds}$ is viewed inside $\mathrm{Der}\,\mathrm{Lie}[a,b]$ by extending the genus zero action on $\mathrm{Lie}[x,y]$.

Elliptic Kashiwara-Vergne

The third avatar would be a semi-direct product

$$\mathfrak{plkv}\rtimes\mathfrak{kv}.$$

Conjecture: *We have*

$$\mathfrak{plkv}\simeq\mathfrak{plds}\simeq\mathcal{P}=\{\epsilon(a)|\epsilon\in\mathcal{E}\},$$

and $\mathfrak{ds}$ *acts on* $\mathfrak{plds}$ *and* $\mathfrak{kv}$ *acts on* $\mathfrak{plkv}$ *by bracketing derivations of* $\mathrm{Lie}[a,b]$.

If true, we could set $\mathfrak{ds}_{ell}=\mathfrak{plds}\rtimes\mathfrak{ds}$ and $\mathfrak{kv}_{ell}=\mathfrak{plkv}\rtimes\mathfrak{kv}$. It might be easier to prove these results for $\mathfrak{kv}$ than for $\mathfrak{ds}$ because the definitions are given by many less linear equations.

Elliptic Grothendieck-Teichmüller

Finally, the fourth avatar is the Grothendieck-Teichmüller semi-direct product

$$\mathfrak{grt}_{ell}=\mathfrak{r}_{ell}\rtimes\mathfrak{grt}$$

defined and studied by B. Enriquez, who showed in particular that

$$\mathcal{P}\subset\mathfrak{r}_{ell}.$$

Before giving the definitions, we will apply some of the ideas, principally the definition of $\mathfrak{pls}$, to prove that the interesting relations of the type computed by A. Pollack inside $\mathcal{E}$ hold in general depth 3.

Part I. Pollack's relations in depth 2,3

Pollack investigated interesting relations inside $\mathcal{E}$.

Depth 2. He displayed all depth 2 relations in $\mathcal{E}$: they are of the form

$$[\epsilon_4, \epsilon_{10}] - 3[\epsilon_6, \epsilon_8] = 0,$$

$$2[\epsilon_4, \epsilon_{14}] - 7[\epsilon_6, \epsilon_{12}] + 11[\epsilon_8, \epsilon_{10}] = 0,$$

and proved the following result:

$$\sum_{k=1}^{[(n-2)/4]} a_{2k}[\epsilon_{2k+2}, \epsilon_{n-2k}] = 0$$

if and only if $\sum a_{2k}(X^{2k} - X^{n-2k-2})$ is the reduced even period polynomial of a cusp form of even weight n (degree $n-2$ and constant terms removed).

This is equivalent to an analogous result on Poisson brackets of $ad(x)^{2i}(y)$:

$$\sum_{k=1}^{[(n-2)/4]} a_{2k}\{ad(x)^{2k}(y), ad(x)^{n-2k-2}(y)\} = 0$$

if and only if the corresponding polynomial is a reduced even period polynomial.

Note: The Poisson bracket $\{f, g\}$ of two Lie polynomials is defined by

$$\{f, g\} = [f, g] + D_f(g) - D_g(f)$$

where $D_f(x) = 0$, $D_f(y) = [y, f]$.

Depth 3. Following Pollack, set

$$h^d_{p,q} = \sum_{i+j=d-2} (-1)^i \binom{d-2}{i} [ad(\epsilon_0)^i(\epsilon_{p+2}), ad(\epsilon_0)^j(\epsilon_{q+2})].$$

For each $d \geq 2$, Pollack investigated certain linear combinations of $h^d_{p,q}$ with odd (resp. even) period polynomial coefficients when d is odd (resp. even). He showed that these linear combinations are of Θ-*depth 3*, where the Θ-depth is simply the LCS filtration on the subalgebra

$$\mathbf{L} = \mathrm{Lie}[ad(a)^i ad(b)^j([a,b])] \subset \mathrm{Lie}[a,b].$$

Examples of Pollack's special elements:

$$\begin{cases} 4h^3_{2,10} - 25h^3_{4,8} + 21h^3_{6,6} \\ 36h^3_{2,14} - 245h^3_{4,12} + 539h^3_{6,10} - 660h^3_{8,8} \\ h^4_{2,q} \quad \text{for all even } q, \\ h^4_{4,10} - 3h^4_{6,8} \\ 2h^4_{4,14} - 7h^4_{6,12} + 11h^4_{8,10} \cdots \end{cases}$$

But he noticed something more with respect to these linear combinations: he found some actual relations in $\mathcal{E}$, beginning with linear combinations of these.

In particular, in depth 3, he found a relation

$$4h^3_{2,10} - 25h^3_{4,8} + 21h^3_{6,6} + \frac{345}{8}[\epsilon_6,[\epsilon_6,\epsilon_4]] - \frac{231}{20}[\epsilon_4,[\epsilon_{10},\epsilon_4]] = 0,$$

and an analogous relation

$$36h^3_{2,14} - 245h^3_{4,12} + 539h^3_{6,10} - 660h^3_{8,8} + \cdots = 0.$$

He conjectured (or at least suggested) that in depth 3, each "odd period-polynomial linear combination" forms the beginning of an actual relation in $\mathcal{E}$.

Theorem. (S-Baumard) *Pollack's conjecture is true in depth 3.*

Sketch of proof (4 slides)

Ecalle's commutative variable notation.

A *mould* is a family of functions $m = \{m_r(u_1, \ldots, u_r)\}_{r\geq 0}$. In a *polynomial mould*, the m_r are polynomial. A mould is *finite* if $\exists N$ such that $m_r = 0$ for $r > N$.

To every Lie polynomial $f \in \mathrm{Lie}[a,b]$, write $f = \sum_{r\geq 1} f^r$ to cut it into parts of b-depth r, and write

$$f^r = \sum_{\mathbf{i}=(i_0,\ldots,i_r)} c_{\mathbf{i}} a^{i_0} b \cdots b a^{i_r} \in \mathrm{Lie}_n^r[a,b].$$

Associate three moulds $vimo(f)$, $ma(f)$, $mi(f)$ to f:

$$\begin{cases} vimo(f^r) = \sum_{\mathbf{i}} c_{\mathbf{i}} z_0^{i_0} \cdots z_r^{i_r} \\ ma(f^r) = vimo(0, u_1, u_1+u_2, \ldots, u_1+\cdots+u_r) \\ mi(f^r) = vimo(0, u_r, \ldots, u_1). \end{cases}$$

It is easy to prove that $vimo(z_0 + a, \ldots, z_r + a) = vimo(z_0, \ldots, z_r)$ and thus any one of f, $vimo(f)$, $ma(f)$ and $mi(f)$ easily yields all the others.

A mould m is said to be *alternal* if each m_r satisfies the r-shuffle relations

$$\sum_{\mathbf{w}\in sh((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r))} m_r(\mathbf{w}) = 0.$$

A polynomial $f \in \mathbb{Q}\langle a,b\rangle$ is said to be *bialternal* if $ma(f)$ is alternal (i.e. $f \in \mathrm{Lie}[a,b]$) and $mi(f)$ is alternal.

Define the *linearized double shuffle space* to be the space of bialternal polynomial moulds

$$\mathfrak{lds} = \{f \in \mathbb{Q}\langle a,b\rangle \,|\, f \text{ is bialternal}\}.$$

Note: Ecalle considers general bialternal moulds, not just polynomial.

Let us now define the *pseudo-alternality* conditions. Suppose $mi(f)$ is alternal, and for each $r \geq 1$ (drop subscripts h^r, f^r), set

$$mi(h) = (u_1 - u_2)(u_2 - u_3) \cdots (u_{r-1} - u_r) u_r u_1 mi(f).$$

Then each alternality (shuffle) condition on $mi(f)$ induces a condition on $mi(h)$. For example in depth 2 we have

$$mi(f)(u_1, u_2) + mi(f)(u_2, u_1) = 0 \qquad \text{(alternality)}$$

and $mi(h) = (u_1 - u_2)u_1 u_2 mi(f)$ directly yields

$$mi(h)(u_1, u_2) - mi(h)(u_2, u_1) = 0. \qquad \text{(pseudo-alternality)}$$

In depth 3, the shuffle relation

$$mi(f)(u_1, u_2, u_3) + mi(f)(u_2, u_1, u_3) + mi(f)(u_2, u_3, u_1) = 0 \qquad \text{(alt)}$$

and $mi(h) = (u_1 - u_2)(u_2 - u_3)u_3 u_1 mi(f)$ yields

$$u_2(u_3 - u_1) mi(h)(u_1, u_2, u_3) + u_1(u_2 - u_3) mi(h)(u_2, u_1, u_3)$$
$$+ u_3(u_1 - u_2) mi(h)(u_2, u_3, u_1) = 0. \qquad \text{(pseudo-alt)}$$

In this way, every alternality condition on $mi(f)$ yields a pseudo-alternality condition on $mi(h)$. We say that $f \in \mathbb{Q}\langle a, b\rangle$ is *pseudo-bialternal* if for each depth $r \geq 1$, $ma(f^r)$ is alternal (i.e. $f \in \mathrm{Lie}[a, b]$), and $mi(f^r)$ is pseudo-alternal.

Definition. The *pseudo-linearized double shuffle space* $\mathfrak{plds}$ is defined to be the space of pseudo-bialternal polynomial moulds, i.e. $f \in \mathbb{Q}\langle a, b\rangle$ such that f is pseudo-bialternal.

Conjecture: *We have*

$$\mathfrak{plds} = \mathcal{P} := \{\epsilon(a) | \epsilon \in \mathcal{E}\}.$$

Note (added 11/04/13): is there a map from $\mathfrak{plds}$ to Brown's $\mathfrak{pls}$ (polar $\mathfrak{lds}$) simply given by (pseudo-bialternal→bialternal) via

$$mi(h) \mapsto mi(f) = mi(h)/(u_1 - u_2)\cdots(u_{r-1} - u_r)u_r u_1?$$

Example: $\epsilon = ad(\epsilon_0)^2(\epsilon_4) \in \mathcal{E}$.

$$h = 2[aabbb] - 2[ababb]$$

$$mi(h) = 2u_3^2 - 8u_2u_3 + 12u_1u_3 + 2u_2^2 - 8u_1u_2 + 2u_1^2$$

$$\frac{mi(f) = mi(h)}{u_1 - u_2)(u_2 - u_3)u_1u_3.}$$

- The polynomial $mi(h)$ is pseudo-bialternal
- the rational function $mi(f)$ is bialternal (so in $\mathfrak{pls}$).

Proof of Pollack's conjecture in depth 3.

It is known (Goncharov+Brown) that all bialternal polynomials in depth 3 are Poisson brackets of the form

$$r(a,b) = \big\{ad(a)^{2i}(b), \big\{ad(a)^{2j}(b), ad(a)^{2k}(b)\big\}\big\}.$$

The proofs of all the following lemmas are elementary.

Lemma 1. *If $r(a,b)$ is a bialternal polynomial, then*

$$f = [a, r(a,[a,b])]$$

is pseudo-bialternal. Conversely, if a Lie polynomial $f(a,b)$ is pseudo-bialternal and if there exists r such that f can be written $f = [a, r(a,[a,b])]$, then r is bialternal.

Lemma 2. *If*

$$r(a,b) = \{ad(a)^{2i}(b), \{ad(a)^{2j}(b), ad(a)^{2k}(b)\}\}$$

and $f = [a, r(a,[a,b])]$, then

$$f(a,b) = [\epsilon_{2i}, [\epsilon_{2j}, \epsilon_{2k}]](a).$$

The next lemma proves the depth 3 case of the conjecture $\mathfrak{plds} = \mathcal{P}$.

Lemma 3. *A depth 3 polynomial $f(a,b)$ is pseudo-bialternal if and only if $f(a,b) = \epsilon(a)$ for some $\epsilon \in \mathcal{E}$ of depth 3, i.e. some linear combination of brackets of three ϵ_{2i}, $i \geq 0$.*

Lemma 4. *Let $f(a, b) \in \mathcal{P}$ be a linear combination of depth 3 terms of the form*

$$[\epsilon_{2i}, [\epsilon_0, \epsilon_{2j}]](a).$$

If $f(a, b)$ is of Θ-depth 3, then $f = [a, r(a, [a, b])]$ for some depth 3 Lie polynomial $r(a, b)$.

Now, let f be one of Pollack's odd period-poly elements in depth 3.

• Pollack showed that f is of Θ-depth 3.

• By Lemma 3, since $f \in \mathcal{P}$, it is pseudo-bialternal.

• Thus, by Lemma 4, there exists $r(a, b)$ such that $f = [a, r(a, [a, b])]$.

• By Lemma 1, this $r(a, b)$ is bialternal.

• By Goncharov's result, $r(a, b)$ is then a linear combination of terms of the form

$$\{ad(a)^{2i}(b), \{ad(a)^{2j}(b), ad(a)^{2k}(b)\}\}.$$

• Finally, by Lemma 2, we see that f is thus equal to a linear combination of terms of the form

$$[\epsilon_{2i}, [\epsilon_{2j}, \epsilon_{2k}]](a).$$

This proves Pollack's relations in depth 3.

Part II. Four avatars of $Gr^w\mathfrak{u}^{MEM}$

Recall from the beginning that Hain, Matsumoto, Pollack studied the image of $Gr^w\mathfrak{u}^{MEM}$ in $\mathrm{Der}\,\mathrm{Lie}[a,b]$. This image, in which various explicit calculations can be made, is the model on which we base our double shuffle and Kashiwara-Vergne versions, and with which we compare the Grothendieck-Teichmüller version defined by B. Enriquez.

Double shuffle

We will replace the usual definitions of the double shuffle and Kashiwara-Vergne Lie algebra with versions in terms of polynomials in commutative variables, using properties defined by Ecalle.

Ecalle defines the analogue of the stuffle relations on *moulds*, which are families $\{m_r | r \geq 1\}$ of functions m_r of r variables. We will only consider polynomial moulds here.

Ecalle calls the "stuffle" property on moulds *alternility*. The alternility relations can be derived directly from the usual stuffle relations, as follows.

The alternility relations are defined from the stuffle relations by replacing every term containing a contraction in the standard stuffle relation with the following sum of two terms:

$$\frac{1}{u_i - u_j} m(\ldots, u_i, \ldots) + \frac{1}{u_j - u_i} m(\ldots, u_j, \ldots).$$

Thus for example, the standard stuffle relation in depth 2 is

$$st((u_1), (u_2)) = (u_1, u_2) + (u_2, u_1) + (u_1 + u_2),$$

and the corresponding alternility condition in depth 2 is given by

$$0 = m(u_1, u_2) + m(u_2, u_1) + \frac{1}{u_1 - u_2} m(u_1) + \frac{1}{u_2 - u_1} m(u_2).$$

The standard depth 3 stuffle relation is

$$st((u_1), (u_2, u_3)) = (u_1, u_2, u_3) + (u_2, u_1, u_3) + (u_2, u_3, u_1) + (u_1 + u_2, u_3) + (u_2, u_1 + u_3),$$

and the corresponding alternility condition in depth 3 is given by

$$\begin{aligned} 0 = m(u_1, u_2, u_3) + m(u_2, u_1, u_3) + m(u_2, u_3, u_1) \\ + \frac{1}{u_1 - u_2} m(u_1, u_3) + \frac{1}{u_2 - u_1} m(u_2, u_3) \\ + \frac{1}{u_1 - u_3} m(u_2, u_1) + \frac{1}{u_3 - u_1} m(u_2, u_3). \end{aligned}$$

A mould $m = \{m_r\}$ is *alternil* if it satisfies all the alternility conditions.

Definition. The *double shuffle Lie algebra* $\mathfrak{ds}$ is given by

$$\mathfrak{ds} = \{f \in \mathbb{Q}\langle x, y\rangle | ma(f) \text{ is alternal and } mi(f) \text{ is alternil}\}.$$

Recall that we also have

$$\mathfrak{lds} = \{f \in \mathbb{Q}\langle x, y\rangle | ma(f) \text{ is alternal and } mi(f) \text{ is alternal}\}$$

and

$$\mathfrak{plds} = \{f \in \mathbb{Q}\langle x, y\rangle | ma(f) \text{ is alternal and } mi(f) \text{ is pseudo-alternal}\}.$$

Kashiwara-Vergne

Let $m = \{m_r | r \geq 1\}$ be a *polynomial mould*, i.e. a family of polynomials m_r in r variables.

Define the following operators on polynomial moulds:

$$\begin{cases} push(m_r)(u_1,\ldots,u_r) = m_r(-u_1-\cdots-u_r,u_1,\ldots,u_r) \\ \\ cyc(m_r)(u_1,\ldots,u_r) = m_r(u_1,\ldots,u_r)+m_r(u_2,\ldots,u_r,u_1)+ \\ \quad \cdots+m_r(u_r,u_1,\ldots,u_{r-1}) \\ \\ mantar(m_r)(u_1,\ldots,u_r) = (-1)^{r-1}m_r(u_r,\ldots,u_1) \\ \\ teru(m_r)(u_1,\ldots,u_r) = m_r(u_1,\ldots,u_r)+ \\ \qquad \frac{1}{u_r}\big(m_{r-1}(u_1,\ldots,u_{r-2},u_{r-1}+u_r)-m_{r-1}(u_1,\ldots,u_{r-2},u_{r-1}\big). \end{cases}$$

If $mi(f)$ is *mantar-invariant*, i.e.

$$mi(f)(u_1,\ldots,u_r) = (-1)^{r-1}mi(f)(u_r,\ldots,u_1),$$

then setting $mi(h) = (u_1-u_2)\cdots(u_{r-1}-u_r)u_ru_1mi(f)$, we find that $mi(h)$ satisfies

$$mi(h)(u_1,\ldots,u_r)+mi(h)(u_r,\ldots,u_1) = 0.$$

We say that $mi(h)$ is *pseudo-mantar invariant.*

If $mi(f)$ is cyc-null, i.e. $cyc(mi(f)) = 0$, then we find that $mi(h)$ satisfies

$$\sum_{i=1}^{r} u_iu_{i+1}(u_{i+2}-u_{i+3})mi(h)(u_i,\ldots,u_r,u_1,\ldots,u_{i-1}) = 0.$$

We say that $mi(h)$ is *pseudo-cyc-null.*

The definitions of the KV Lie algebras are:

$$\mathfrak{kv} = \{f \in \mathbb{Q}\langle x,y\rangle | ma(f) \text{ is alternal and for } r \geq 1,$$
$$mi(f^r) \text{ is mantar-invariant and}$$
$$teru(ma(f^r)) = push \circ mantar \circ teru \circ mantar(ma(f^r))\},$$

$$\mathfrak{lkv} = \{f \in \mathbb{Q}\langle x,y\rangle | ma(f) \text{ is alternal},$$
$$mi(f) \text{ is mantar-invariant and cyc-null}\},$$

$$\mathfrak{plkv} = \{f \in \mathbb{Q}\langle x,y\rangle | ma(f) \text{ is alternal},$$
$$mi(f)\text{is pseudo-mantar-invariant and pseudo-cyc-null}\}.$$

Conjectures: We conjecture that

$$\mathfrak{lkv} \simeq gr\mathfrak{kv} \simeq \mathfrak{lds} \simeq gr\mathfrak{ds},$$

and indeed that

$$\mathfrak{ds} \simeq \mathfrak{kv}.$$

We also conjecture that

$$\mathfrak{plds} = \mathfrak{plkv} = \mathcal{P} \simeq \mathcal{E}.$$

The linearized conjectures are known to hold in depths 2 and 3.

Grothendieck-Teichmüller

The definition of $\mathfrak{grt}$ is the well-known

$$\mathfrak{grt} = \{f \in \text{Lie}[x,y] | f(x_{12},x_{23}) + f(x_{23},x_{34}) + f(x_{34},x_{45}) +$$

$$f(x_{45},x_{51}) + f(x_{51},x_{12}) = 0\},$$

where the x_{ij} generate the 5-strand braid Lie algebra $Lie(P_5)$.

B. Enriquez defines the *elliptic Grothendieck-Teichmüller Lie algebra* $\mathfrak{grt}_{ell}$ as follows.

Define the genus 1 braid Lie algebra $\mathfrak{t}_{1,n}$ to be generated by $x_1^+, \ldots, x_n^+$ and $x_1^-, \ldots, x_n^-$ with relations:

$x_1^+ + \cdots + x_n^+ = x_1^- + \cdots + x_n^- = 0$
$[x_i^+, x_j^+] = [x_i^-, x_j^-] = 0$ if $i \neq j$
$[x_i^+, x_j^-] = [x_j^+, x_i^-]$ for $i \neq j$
$[x_i^+, [x_j^+, x_k^-]] = [x_i^-, [x_j^+, x_k^-]] = 0$ for i, j, k distinct.

Notation. For any map $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$, there exists a corresponding injective map $\phi : \mathbf{t}_{1,m} \to \mathbf{t}_{1,n}$ defined by

$$\phi(x_i^+) = \sum_{j \in I_i} x_j^+, \quad \phi(x_i^-) = \sum_{j \in \phi^{-1}(i)} x_j^-, \quad 1 \leq i \leq m,$$

where $I_i = \phi^{-1}(i)$, $i = 1, \ldots, n$. We write $\phi(\alpha) = \alpha^{I_1, \ldots, I_n}$ for every $\alpha \in \mathbf{t}_{1,m}$. (Note that the I_i are unordered sets; we have $\alpha^{2,13} = \alpha^{2,31}$ for example.)

Main examples: $\phi : \{1,2,3\} \to \{1,2,3\}$ defined by $\phi(i) = i$ gives the identity map $\mathbf{t}_{1,3} \to \mathbf{t}_{1,3}$: we write $\psi = \psi^{1,2,3}$.

$\phi : \{1,2,3\} \to \{2,1,3\}$ defined by $\phi(1) = 2$, $\phi(2) = 1$, $\phi(3) = 3$ gives the map $\phi : \mathbf{t}_{1,3} \to \mathbf{t}_{1,3}$ mapping $x_1^\pm \mapsto x_2^\pm$, $x_2^\pm \mapsto x_1^\pm$, $x_3^\pm \mapsto x_3^\pm$; we write $\phi(\psi) = \psi^{2,1,3}$.

$\phi : \{1,2,3\} \to \{1,2\}$ defined by $\phi(1) = 1$, $\phi(2) = \phi(3) = 2$ yields a map $\mathbf{t}_{1,2} \to \mathbf{t}_{1,3}$ given by

$$\begin{cases} x_1^+ \mapsto x_1^+ \\ x_1^- \mapsto x_1^- \\ x_2^+ \mapsto x_2^+ + x_3^+ \\ x_2^- \mapsto x_2^- + x_3^-, \end{cases}$$

and we write $\phi(\alpha) = \alpha^{1,23}$.

$\phi : \{1,2,3\} \to \{1,2\}$ defined by $\phi(2) = 1$, $\phi(1) = \phi(3) = 2$ yields a map $\mathbf{t}_{1,2} \to \mathbf{t}_{1,3}$ given by

$$\begin{cases} x_1^+ \mapsto x_2^+ \\ x_1^- \mapsto x_2^- \\ x_2^+ \mapsto x_1^+ + x_3^+ \\ x_2^- \mapsto x_1^- + x_3^-, \end{cases}$$

and we write $\phi(\alpha) = \alpha^{2,13}$.

The Lie algebra $\mathfrak{grt}_{ell}$ is the set of triples $\Psi = (\psi, \alpha_+, \alpha_-)$ such that setting

$$\begin{cases} \Psi(x_1^+) = \alpha_+^{1,23} + [x_1^+, \psi^{1,2,3}] \\ \Psi(x_1^-) = \alpha_-^{1,23} + [x_1^-, \psi^{1,2,3}] \\ \Psi(x_2^+) = \alpha_+^{2,31} + [x_2^+, \psi^{2,1,3}] \\ \Psi(x_2^-) = \alpha_-^{2,31} + [x_2^-, \psi^{2,1,3}] \\ \Psi(x_3^+) = \alpha_+^{3,12} \\ \Psi(x_3^-) = \alpha_-^{3,12}, \end{cases}$$

this extends to a derivation of $\mathfrak{t}_{1,3}$.

Enriquez showed the following facts concerning $\mathfrak{grt}_{ell}$.

(1) There is a surjection $\mathfrak{grt}_{ell} \to \mathfrak{grt}$ given by $(\psi, \alpha_+, \alpha_-) \mapsto \psi$. Let $\mathfrak{r}_{ell}$ denote the kernel.

(2) There is a section $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ obtained as follows.

The relation between the genus 0 and the genus 1 Lie algebras is given by the following inclusion of $\mathrm{Lie}[x, y] \subset \mathrm{Lie}[a, b]$:

$$\begin{cases} x = t_{01} = \frac{ad(a)}{1-exp(ad(a))}(b) \\ y = t_{02} = \frac{ad(-a)}{1-exp(ad(-a))}(-b) \\ z = t_{12} = -[a, b]. \end{cases}$$

The standard action of $f \in \mathfrak{grt}$ on $\mathrm{Lie}[x, y]$ given by

$$D_f(x) = [x, f(z, x)], \quad D_f(y) = [y, f(z, y)]$$

can be interpreted as a derivation action of $\mathfrak{grt}$ on the subspace $\mathrm{Lie}[x, y] \subset \mathrm{Lie}[a, b]$. Enriquez extends it to a derivation action on the whole of $\mathrm{Lie}[a, b]$ by setting

$$D_f(e^a) = \psi^{0,2,1} e^a - e^a \psi^{0,1,2}, \quad D_f([a, b]) = 0.$$

(3) He shows that $\mathcal{E} \subset \mathfrak{r}_{ell}$ and conjectures that they are equal.

Elliptic double shuffle and Kashiwara-Vergne Lie algebras

We identify an element $h \in \mathfrak{plds}$ or $\mathfrak{plkv}$ with a derivation D_h of $\mathrm{Lie}[a,b]$ by taking $D_h(a) = h$, $D_h([a,b]) = 0$.

We consider $\mathfrak{ds}$ and $\mathfrak{kv}$ as subspaces of $\mathrm{Der}\,\mathrm{Lie}[a,b]$ by following Enriquez' idea for $\mathfrak{grt}$.

Indeed, it is possible to show (using a result of Ecalle) that for $f \in \mathfrak{ds}$, setting $F(x,y) = f(z,-y)$, there exists a unique G such that $[x,G]+[y,F] = 0$. This also holds, by definition, for elements of $\mathfrak{kv}$.

Following Enriquez, we extend this standard action of $\mathfrak{ds}$ or $\mathfrak{kv}$ on $\mathrm{Lie}[x,y]$ to all of $\mathrm{Lie}[a,b]$ by associating the derivation D_f to $f \in \mathfrak{ds}$ defined by

$$D_f(e^a) = Fe^a - e^a G, \quad D_f([a,b]) = 0.$$

Conjecture. *Viewed inside* $\mathrm{Der}\,\mathrm{Lie}[a,b]$*, the subspace* $\mathfrak{ds}$ *(resp.* $\mathfrak{kv}$*) normalizes* $\mathfrak{plds}$ *(resp.* $\mathfrak{plkv}$*).*

Keeping in mind the conjecture $\mathfrak{plds} \simeq \mathfrak{plkv} = \mathcal{P} \simeq \mathcal{E}$, the elliptic double shuffle and Kashiwara-Vergne Lie algebras would be the semi-direct products

$$\mathfrak{ds}_{ell} = \mathfrak{plds} \rtimes \mathfrak{ds}, \quad \mathfrak{kv}_{ell} = \mathfrak{plkv} \rtimes \mathfrak{kv}.$$