Integrable Maps which Preserve Functions with Symmetries

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Outline

1. Basic Idea
   - A Coupled McMillan Map
   - A Symmetry Vector and its Invariants
   - Another Symmetry Vector and its Invariants
   - Commuting Maps
   - $4D$ Non-Invariant Poisson Brackets

2. The Main Ingredients
   - Another Choice of Basis
   - The Choice of Invariant Functions and Involutions
   - The Adler-Yamilov (Yang-Baxter) Map

3. Higher Dimensions

4. Conclusions
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A Coupled McMillan Map

Consider the functions

\[ h_1 = (1 - x_1 x_2)(1 - y_1 y_2) - 2 a x_1 y_1, \quad h_2 = x_1 y_1 - x_2 y_2. \]

Invariant under the two involutions

\[ \iota_{xy} : (x_i, y_i) \mapsto (y_i, x_i), \quad i = 1, 2, \quad \rho_x : h_k(\tilde{x}, y) = h_k(x, y), \quad k = 1, 2. \]

Under the map \( \varphi = \iota_{xy} \circ \rho_x \)

\[ \varphi : (x, y) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2) \]

\[ = \left( y_1, y_2, -\frac{x_2 y_2}{y_1} - \frac{2 a y_2}{1 - y_1 y_2}, -\frac{x_1 y_1}{y_2} - \frac{2 a y_1}{1 - y_1 y_2} \right), \]

the volume form \( \Omega_4 = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \) is anti-invariant.
A Coupled McMillan Map

Consider the functions

\[ h_1 = (1 - x_1 x_2)(1 - y_1 y_2) - 2ax_1 y_1, \quad h_2 = x_1 y_1 - x_2 y_2. \]

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Under the map \( \varphi = \iota_{xy} \circ \rho_x \),

\[ \varphi : (x, y) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2) \]

\[ = \left( y_1, y_2, -\frac{x_2 y_2}{y_1} - \frac{2ay_2}{1 - y_1 y_2}, -\frac{x_1 y_1}{y_2} - \frac{2ay_1}{1 - y_1 y_2} \right), \]

the volume form \( \Omega_4 = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \) is anti-invariant.
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A Symmetry Vector and its Invariants

The vector field $\mathbf{X} = (x_1, -x_2, -y_1, y_2)$, has invariants

$$\gamma_1 = x_1 y_1, \quad \gamma_2 = x_2 y_2, \quad \gamma_3 = x_1 x_2, \quad \gamma_4 = y_1 y_2,$$

satisfying $\gamma_3 \gamma_4 = \gamma_1 \gamma_2$. Under $\iota_{xy}$ we have $\gamma_3 \leftrightarrow \gamma_4$ and $\gamma_1, \gamma_2$ invariant.

We have $\mathbf{X}(h_i) = 0$. $h_i$ functions of invariants:

$$h_1 = (1 - x_1 x_2)(1 - y_1 y_2) - 2ax_1 y_1 = (1 - \gamma_3)(1 - \gamma_4) - 2a\gamma_1,$$

$$h_2 = x_1 y_1 - x_2 y_2 = \gamma_1 - \gamma_2,$$

and $\varphi^* : \mathbf{X} \mapsto -\tilde{\mathbf{X}}$.

In coordinates $u_1 = \gamma_3, \quad v_1 = \gamma_1, \quad u_2 = h_2, \quad v_2 = y_2,$

$$\Omega_4 = \frac{du_1 \wedge dv_1 \wedge du_2 \wedge dv_2}{u_1 v_2}, \quad \mathbf{X} = v_2 \partial_{v_2},$$

$$h_1 = (1 - u_1) \left(1 - \frac{v_1(v_1 - u_2)}{u_1}\right) - 2av_1.$$
A Symmetry Vector and its Invariants

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satisfying \( \gamma_3\gamma_4 = \gamma_1\gamma_2 \).

Under \( \iota_{xy} \) we have \( \gamma_3 \leftrightarrow \gamma_4 \) and \( \gamma_1, \gamma_2 \) invariant.

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Under \( \iota_{xy} \) we have \( \gamma_3 \leftrightarrow \gamma_4 \) and \( \gamma_1, \gamma_2 \) invariant.

We have \( \mathbf{X}(h_i) = 0. \) \( h_i \) functions of invariants:
\[
\begin{align*}
h_1 &= (1 - x_1 x_2)(1 - y_1 y_2) - 2ax_1 y_1 = (1 - \gamma_3)(1 - \gamma_4) - 2a\gamma_1, \\
h_2 &= x_1 y_1 - x_2 y_2 = \gamma_1 - \gamma_2,
\end{align*}
\]
and \( \varphi^* : \mathbf{X} \mapsto -\tilde{\mathbf{X}}. \)

In coordinates \( u_1 = \gamma_3, \ v_1 = \gamma_1, \ u_2 = h_2, \ v_2 = y_2, \)
\[
\Omega_4 = \frac{du_1 \wedge dv_1 \wedge du_2 \wedge dv_2}{u_1 v_2}, \quad \mathbf{X} = v_2 \partial_{v_2},
\]
\[
h_1 = (1 - u_1) \left( 1 - \frac{v_1(v_1 - u_2)}{u_1} \right) - 2av_1.
\]
The Map $\varphi$ in these Coordinates

The subspace $(u_1, v_1, u_2)$ is invariant under $\varphi$ (and $u_2 = k$) with

$$\Omega_3 = X \downarrow \Omega_4 = -\frac{du_1 \wedge dv_1 \wedge du_2}{u_1} \quad \text{invariant.}$$

The 2–dimensional map

$$\tilde{u}_1 = \frac{v_1(v_1 - k)}{u_1}, \quad \tilde{v}_1 = \frac{(k - v_1)(u_1 + (2a + k - v_1)v_1)}{u_1 + (k - v_1)v_1},$$

has invariant function $h_1$ and invariant symplectic form $\omega$, given by

$$h_1 = (1 - u_1) \left(1 - \frac{v_1(v_1 - k)}{u_1}\right) - 2av_1, \quad \text{and} \quad \omega = \frac{du_1 \wedge dv_1}{u_1},$$

and $\tilde{v}_2 = \alpha v_2^{-1}$, where $\alpha$ is a function of $u_1, v_1, k$.

$(u_1, v_1) \mapsto (\tilde{u}_1, \tilde{v}_1)$ is the QRT map from $h_1$.
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Another Symmetry Vector and its Invariants

Now take \( \mathbf{X} = (x_1, x_2, -y_1, -y_2) \), with invariants

\[
\gamma_1 = x_1 y_1, \quad \gamma_2 = x_2 y_2, \quad \gamma_3 = x_1 y_2, \quad \gamma_4 = x_2 y_1,
\]

also satisfying \( \gamma_3 \gamma_4 = \gamma_1 \gamma_2 \) and (under \( \iota_{xy} \)) \( \gamma_3 \leftrightarrow \gamma_4 \) and \( \gamma_1, \gamma_2 \) invariant.

In terms of the invariants, we choose the same \( h_i \):

\[
\begin{align*}
    h_1 &= (1 - \gamma_3)(1 - \gamma_4) - 2a \gamma_1 = (1 - x_1 y_2)(1 - x_2 y_1) - 2ax_1 y_1, \\
    h_2 &= \gamma_1 - \gamma_2 = x_1 y_1 - x_2 y_2.
\end{align*}
\]

In terms of \( x_i, y_i \), we get a different map:

\[
\hat{\phi} : (\mathbf{x}, \mathbf{y}) \mapsto \left( y_1, y_2, \frac{2a}{y_1} + \frac{1}{y_2} + \frac{y_2(1 - x_2 y_1)}{y_1^2}, \frac{2a}{y_2} + \frac{1}{y_1} + \frac{y_1(1 - x_1 y_2)}{y_2^2} \right).
\]
Another Symmetry Vector and its Invariants

Now take $X = (x_1, x_2, -y_1, -y_2)$, with invariants

\[ \gamma_1 = x_1 y_1, \quad \gamma_2 = x_2 y_2, \quad \gamma_3 = x_1 y_2, \quad \gamma_4 = x_2 y_1, \]

also satisfying $\gamma_3 \gamma_4 = \gamma_1 \gamma_2$ and (under $\iota_{xy}$) $\gamma_3 \leftrightarrow \gamma_4$ and $\gamma_1, \gamma_2$ invariant.

In terms of the invariants, we choose the same $h_i$:

\[ h_1 = (1 - \gamma_3)(1 - \gamma_4) - 2a \gamma_1 = (1 - x_1 y_2)(1 - x_2 y_1) - 2ax_1 y_1, \]
\[ h_2 = \gamma_1 - \gamma_2 = x_1 y_1 - x_2 y_2. \]

In terms of $x_i, y_i$, we get a different map:

\[ \hat{\phi} : (x, y) \mapsto \left( y_1, y_2, \frac{2a}{y_1} + \frac{1}{y_2} + \frac{y_2 (1 - x_2 y_1)}{y_1^2}, \frac{2a}{y_2} + \frac{1}{y_1} + \frac{y_1 (1 - x_1 y_2)}{y_2^2} \right). \]
The Map $\hat{\phi}$

Again the volume form $\Omega_4 = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$ is anti-invariant.

Again, $\hat{\phi}_*: X \mapsto -\tilde{X}$

Again the constraint $\gamma_3 \gamma_4 = \gamma_1 \gamma_2$, and the reduction to $h_2 = k$, gives us

$$h_1 = (1 - \gamma_3) \left(1 - \frac{\gamma_1 (\gamma_1 - k)}{\gamma_3}\right) - 2a\gamma_1,$$

written in terms of two variables $\gamma_1, \gamma_3$.

Using coordinates:

$$u_1 = \gamma_3 = x_1y_2, \quad v_1 = \gamma_1 = x_1y_1, \quad u_2 = h_2 = x_1y_1 - x_2y_2, \quad v_2 = y_2,$$

the reduced volume form

$$\Omega_3 = X \downarrow \Omega_4 = \frac{du_1 \wedge dv_1 \wedge du_2}{u_1}$$

is invariant.
The Map $\hat{\phi}$

Again the volume form $\Omega_4 = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$ is anti-invariant.

Again, $\hat{\phi}^* : X \mapsto -\tilde{X}$

Again the constraint $\gamma_3 \gamma_4 = \gamma_1 \gamma_2$, and the reduction to $h_2 = k$, gives us

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Commuting Maps

We can again reduce the map to the \((u_1, v_1)\) space, with \(u_2 = k\) and 
\[
\tilde{v}_2 = \alpha v^{-1}_2,
\]
where \(\alpha\) is a function of \(u_1, v_1, k\).

On the level surface \(u_2 = k\):

\[
\hat{u}_1 = \frac{u^2_1 + v^2_1 + u_1 v_1 (2a - v_1)}{u^2_1}, \quad \hat{v}_1 = 2a + k - v_1 + \frac{u_1}{v_1} + \frac{v_1}{u_1},
\]

having the same invariant function \(h_1\) and invariant symplectic form \(\omega\),
given by

\[
h_1 = (1 - u_1) \left(1 - \frac{v_1 (v_1 - k)}{u_1}\right) - 2a v_1, \quad \text{and} \quad \omega = \frac{d u_1 \wedge d v_1}{u_1}.
\]

On this 2–dimensional space, \(\varphi\) and \(\hat{\varphi}\) commute.
Commuting Maps

We can again reduce the map to the \((u_1, v_1)\) space, with \(u_2 = k\) and \(\tilde{v}_2 = \alpha v_2^{-1}\), where \(\alpha\) is a function of \(u_1, v_1, k\).

On the level surface \(u_2 = k\):

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\hat{u}_1 = \frac{u_1^2 + v_1^2 + u_1 v_1 (2a - v_1)}{u_1^2}, \quad \hat{v}_1 = 2a + k - v_1 + \frac{u_1}{v_1} + \frac{v_1}{u_1},
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having the same invariant function \(h_1\) and invariant symplectic form \(\omega\), given by

\[
h_1 = (1 - u_1) \left( 1 - \frac{v_1 (v_1 - k)}{u_1} \right) - 2av_1, \quad \text{and} \quad \omega = \frac{du_1 \wedge dv_1}{u_1}.
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Some 4D Poisson Brackets

Functions $h_1 = (1 - x_1 x_2)(1 - y_1 y_2) - 2ax_1 y_1$, $h_2 = x_1 y_1 - x_2 y_2$.

$\{h_1, h_2\} = 0$ with respect to canonical bracket $\{x_i, y_j\} = \delta_{ij}$, etc.

Poisson tensor $P_0$ with $P_0 \nabla h_2 = X$.

Invariants of $X$, commute with $h_2$: $\{\gamma_i, h_2\} = 0$.

$P_0$ has a simple block structure in coordinates $\gamma_3, \gamma_1, h_2, y_2$.

---

Bi-rational map $\varphi$ with invariant functions $h_1, h_2$.

Poisson tensor transforms: $P \mapsto d\varphi P (d\varphi)^T$ and $P_0 \mapsto P_{\pm 1} \mapsto \cdots$.

$\{h_1, h_2\} = 0$ with respect to each bracket.

$P_{\pm 1}$ also have simple block structures in coordinates $\gamma_3, \gamma_1, h_2, y_2$.

Invariant $2 \times 2$ block.
Some 4D Poisson Brackets

Functions \( h_1 = (1 - x_1 x_2)(1 - y_1 y_2) - 2ax_1 y_1, \ h_2 = x_1 y_1 - x_2 y_2 \).

\( \{ h_1, h_2 \} = 0 \) with respect to canonical bracket \( \{ x_i, y_j \} = \delta_{ij} \), etc.

Poisson tensor \( P_0 \) with \( P_0 \nabla h_2 = X \).

Invariants of \( X \), commute with \( h_2 \): \( \{ \gamma_i, h_2 \} = 0 \).

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Bi-rational map \( \phi \) with invariant functions \( h_1, h_2 \).

Poisson tensor transforms: \( P \mapsto d\phi P (d\phi)^T \) and \( P_0 \mapsto P_{\pm 1} \mapsto \cdots \).

\( \{ h_1, h_2 \} = 0 \) with respect to each bracket.

\( P_{\pm 1} \) also have simple block structures in coordinates \( \gamma_3, \gamma_1, h_2, y_2 \).

Invariant 2 \times 2 block.
Three 4D Poisson Brackets in \( u - v \) Coordinates

\[
P_{-1} = \begin{pmatrix}
0 & u_1 & 0 & \frac{u_1 v_2}{v_1} \\
-u_1 & 0 & 0 & v_2 \\
0 & 0 & 0 & v_2 \\
-\frac{u_1 v_2}{v_1} & -v_2 & -v_2 & 0
\end{pmatrix},
\]

\[
P_0 = \begin{pmatrix}
0 & u_1 & 0 & -\frac{u_1 v_2}{u_2-v_1} \\
-u_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -v_2 \\
\frac{u_1 v_2}{u_2-v_1} & 0 & v_2 & 0
\end{pmatrix},
\]

\[
P_1 = \begin{pmatrix}
0 & u_1 & 0 & -\frac{u_1 v_2}{u_2-v_1} \\
-u_1 & 0 & 0 & v_2 \\
0 & 0 & 0 & v_2 \\
\frac{u_1 v_2}{u_2-v_1} & -v_2 & -v_2 & 0
\end{pmatrix}.
\]

The fact that \( \{ h_1, h_2 \} = \{ h_1, u_2 \} = 0 \) for each Poisson bracket is obvious in these coordinates.
The Main Ingredients

**Vector:** \( \mathbf{X} = (x_1, -x_2, -y_1, y_2) \), with **invariants** \( \gamma_i \), satisfying

\[
\iota_{xy} : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\gamma_1, \gamma_2, \gamma_4, \gamma_3) \quad \text{and} \quad \gamma_3 \gamma_4 = \gamma_1 \gamma_2.
\]

The choice of invariant functions: \( h_i(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \).

*Reduction to 2–dimensions depends upon the above constraint.*

**Commuting maps** as reductions associated with different vectors \( \mathbf{X} \).

---

The choice of the basis \( \gamma_i \): Clearly

\[
\hat{\gamma}_1 = \gamma_1, \quad \hat{\gamma}_2 = \gamma_2, \quad \hat{\gamma}_3 = \frac{\gamma_3}{\gamma_1}, \quad \hat{\gamma}_4 = \frac{\gamma_4}{\gamma_1}
\]

also satisfy

\[
\iota_{xy} : (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4) \mapsto (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_4, \hat{\gamma}_3), \quad \text{but now} \quad \hat{\gamma}_3 \hat{\gamma}_4 = \hat{\gamma}_2 / \hat{\gamma}_1.
\]

The same functions can be used: \( h_i(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4) \),

but the reduction is different.
The Main Ingredients

Vector: \( \mathbf{X} = (x_1, -x_2, -y_1, y_2) \), with invariants \( \gamma_i \), satisfying

\[ \iota_{xy} : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\gamma_1, \gamma_2, \gamma_4, \gamma_3) \quad \text{and} \quad \gamma_3 \gamma_4 = \gamma_1 \gamma_2. \]

The choice of invariant functions: \( h_i(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \).

Reduction to 2–dimensions depends upon the above constraint.

Commuting maps as reductions associated with different vectors \( \mathbf{X} \).

The choice of the basis \( \gamma_i \): Clearly

\[ \hat{\gamma}_1 = \gamma_1, \quad \hat{\gamma}_2 = \gamma_2, \quad \hat{\gamma}_3 = \frac{\gamma_3}{\gamma_1}, \quad \hat{\gamma}_4 = \frac{\gamma_4}{\gamma_1} \]

also satisfy

\[ \iota_{xy} : (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4) \mapsto (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_4, \hat{\gamma}_3), \quad \text{but now} \quad \hat{\gamma}_3 \hat{\gamma}_4 = \hat{\gamma}_2/\hat{\gamma}_1. \]

The same functions can be used: \( h_i(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4) \), but the reduction is different.
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Another Choice of Basis

For the vector field \( \mathbf{X} = (x_1, -x_2, -y_1, y_2) \), choose invariants

\[
\gamma_1 = x_1 y_1, \quad \gamma_2 = x_2 y_2, \quad \gamma_3 = \frac{x_2}{y_1}, \quad \gamma_4 = \frac{y_2}{x_1}, \quad \text{satisfying} \quad \gamma_3 \gamma_4 = \frac{\gamma_2}{\gamma_1}.
\]

Under \( \iota_{xy} \) we have \( \gamma_3 \leftrightarrow \gamma_4 \) and \( \gamma_1, \gamma_2 \) invariant.

Choose functions

\[
h_1 = (1 - \gamma_3)(1 - \gamma_4) - 2a\gamma_1 = \left(1 - \frac{x_2}{y_1}\right) \left(1 - \frac{y_2}{x_1}\right) - 2ax_1y_1,
\]

\[
h_2 = \gamma_1 - \gamma_2 = x_1y_1 - x_2y_2,
\]

Construct \( \varphi \) in the same way and \( \varphi_* : \mathbf{X} \mapsto -\tilde{\mathbf{X}}. \)

In coordinates \( u_1 = \gamma_3, \ v_1 = \gamma_1, \ u_2 = h_2, \ v_2 = y_2, \ h_1 \) reduces to

\[
h_1 = \frac{k(1 - u_1)}{u_1 v_1} - \frac{(1 - u_1)^2}{u_1} - 2av_1, \quad \text{where} \quad u_2 = k.
\]
Another Choice of Basis

For the vector field $\mathbf{X} = (x_1, -x_2, -y_1, y_2)$, choose invariants

$$
\gamma_1 = x_1 y_1, \quad \gamma_2 = x_2 y_2, \quad \gamma_3 = \frac{x_2}{y_1}, \quad \gamma_4 = \frac{y_2}{x_1},
$$
satisfying $\gamma_3 \gamma_4 = \frac{\gamma_2}{\gamma_1}$.

Under $\iota_{xy}$ we have $\gamma_3 \leftrightarrow \gamma_4$ and $\gamma_1, \gamma_2$ invariant.

Choose functions

$$
h_1 = (1 - \gamma_3)(1 - \gamma_4) - 2a\gamma_1 = \left(1 - \frac{x_2}{y_1}\right)\left(1 - \frac{y_2}{x_1}\right) - 2ax_1y_1,
$$

$$
h_2 = \gamma_1 - \gamma_2 = x_1 y_1 - x_2 y_2,
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Construct $\varphi$ in the same way and $\varphi^* : \mathbf{X} \mapsto -\tilde{\mathbf{X}}$.

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Allan Fordy (Leeds)
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The QRT map built from two involutions of \( h_1 \) is now

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Three Commuting Maps

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   - A Coupled McMillan Map
   - A Symmetry Vector and its Invariants
   - Another Symmetry Vector and its Invariants
   - Commuting Maps
   - 4D Non-Invariant Poisson Brackets

2. The Main Ingredients
   - Another Choice of Basis
   - The Choice of Invariant Functions and Involutions
   - The Adler-Yamilov (Yang-Baxter) Map

3. Higher Dimensions

4. Conclusions
The Choice of Invariant Functions and Involution

Given the vector field $X$, we build $h_i$ from its invariants.

The choice of functions $h_1, h_2$ is initially only constrained by the invariance under the simple involution.

However, the forms of $h_i$ are further restricted by the requirement that involution $\rho_x$ is rational.

Whilst we have mainly used $\rho_x$, solving $h_i(\tilde{x}, y) = h_i(x, y)$, there are many other options.

For example, given a map with $X(h_i) = 0$ for some $X$, can we decompose it into a pair of involutions?

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The Adler-Yamilov (Yang-Baxter) Map\textsuperscript{1,2}

Two invariant functions

\[ h_1 = x_1 x_2 + y_1 y_2, \quad h_2 = a y_1 y_2 + b x_1 x_2 + x_1 y_2 + x_2 y_1 + x_1 x_2 y_1 y_2, \]

with symmetry vector \( \mathbf{X} = (x_1, -x_2, y_1, -y_2) \) and involution \( \iota_{12} : 1 \leftrightarrow 2 \). When composed with

\[ \rho_{x_1 y_2} : (x_1, x_2, y_1, y_2) \mapsto \left( y_2 - \frac{(a - b)x_2}{1 + x_2 y_1}, y_1, x_2, x_1 + \frac{(a - b)y_1}{1 + x_2 y_1} \right), \]

derived from \( h_k(\tilde{x}_1, y_1, x_2, \tilde{y}_2) = h_k(x_1, x_2, y_1, y_2) \), we get the non-involutive YB map

\[ \varphi = \rho_{x_1 y_2} \circ \iota_{12} : (x_1, x_2, y_1, y_2) \mapsto \left( y_1 - \frac{(a - b)x_1}{1 + x_1 y_2}, y_2, x_1, x_2 + \frac{(a - b)y_2}{1 + x_1 y_2} \right). \]

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Common invariant function and symplectic form:

\[ h_2 = ak + v_1 + \frac{u_1(k - u_1)(1 + v_1)}{v_1} - (a - b)u_1, \quad \omega = \frac{du_1 \wedge dv_1}{v_1}. \]

A reduction of \( \omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2 \), written in \( u - v \) coordinates.

Using \( X = (x_1, -x_2, y_1, -y_2) \),

\[ \tilde{u}_1 = k - u_1 - \frac{a - b}{1 + v_1}, \quad \tilde{v}_1 = \frac{1}{v_1} \left( u_1 + \frac{a - b}{1 + v_1} \right) \left( u_1 - k + \frac{a - b}{1 + v_1} \right). \]

This is the reduced AY map and the QRT map of the above \( h_2 \).

Using \( X = (x_1, -x_2, -y_1, y_2) \),

\[ \hat{u}_1 = b - a + k - u_1 + \frac{k - u_1}{v_1} - \frac{v_1}{k - u_1}, \]

\[ \hat{v}_1 = (k - u_1)^2 \left( \frac{v_1 + 1}{v_1^2} \right) - \frac{(a - b)(k - u_1)}{v_1} - 1. \]
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The Six Dimensional Case

Starting with 3 invariant functions and 2 symmetry vectors, use 2 invariant functions and 2 symmetry vectors to reduce to 2–dimensions.

In 2–dimensions we obtain an invariant symplectic form and have 1 remaining invariant function.

Easy to write the general form of the symmetry vectors of the given 3 functions, but hard to satisfy \( \mathbf{X} \mapsto \pm \mathbf{X} \).

To guarantee \( d\Omega_3 = 0 \), we need that the Lie derivative \( L_X \Omega_4 = 0 \), since

\[
L_X \Omega_4 = d(\mathbf{X} \lrcorner \Omega_4) + \mathbf{X} \lrcorner d\Omega_4 = d(\mathbf{X} \lrcorner \Omega_4).
\]

Divergence condition on \( \mathbf{X} \): \( \Omega = \frac{dx_1 \wedge \ldots \wedge dx_n}{\sigma(x_1,\ldots,y_n)} \) gives

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d(\mathbf{X} \lrcorner \Omega) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\mathbf{X}^i}{\sigma} \right) + \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left( \frac{\mathbf{X}^{n+i}}{\sigma} \right) = 0.
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Conclusions

In 4D, with 2 invariant functions and 1 symmetry vector, we reduce to 2–dimensions, with invariant function $h_1$ and symplectic form $\omega$, with

$$X \downarrow \Omega_4 = \omega \wedge dh_2.$$

Different $X$ lead to different reductions with the same invariant function and symplectic form. The corresponding maps commute.

In 6D, with 3 invariant functions and 2 symmetry vectors, satisfying the divergence and transformation properties, reduce $\Omega_6$ to $\Omega_4$, with

$$X_1 \downarrow X \downarrow \Omega_6 = \Omega_4 = \omega \wedge dh_2 \wedge dh_3,$$

where $\omega$ is a symplectic form

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invariant under a 2D map $(r, s) \mapsto (\tilde{r}, \tilde{s})$.

We have no criteria for when a second good symmetry exists. Some maps will only reduce to 4D, with invariant 4–form.
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