III A.1

Including non-linearities Dynamical RG:

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fluctuations in $\vec{V}_\perp$, $\delta\rho \to \infty$ as $q \to 0$ ($L \to \infty$)

(Justifies hydrodynamic approach)

But: Can you neglect NL's?

is $V_\perp \nabla V_\perp$ still "small"?

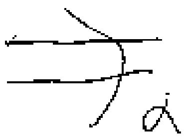
Solving NL problem:

Exact: hopeless: Different Fourier modes coupled

$$V(r)\nabla V(r) \to \sum_p V(q-p)(q-p)\,V(q-p)$$

Proceed perturbatively in NL's

Represent Graphically: (Feynmann Graphs)

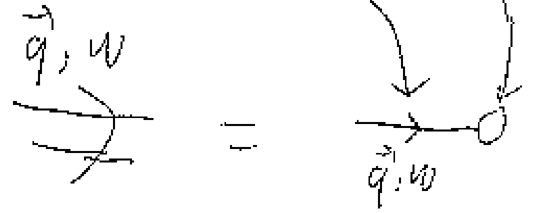


$= \Psi_\alpha \qquad (\Psi_\rho = \delta\rho$

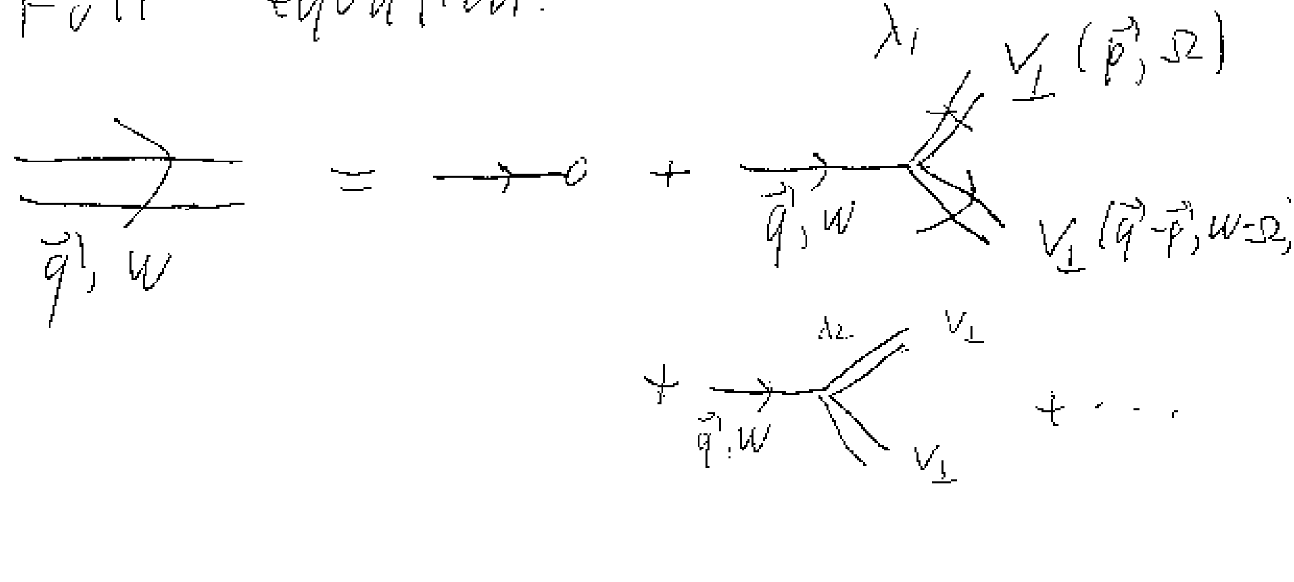
$\Psi_{V_{\perp i}} = V_{\perp i})$

Linear theory:

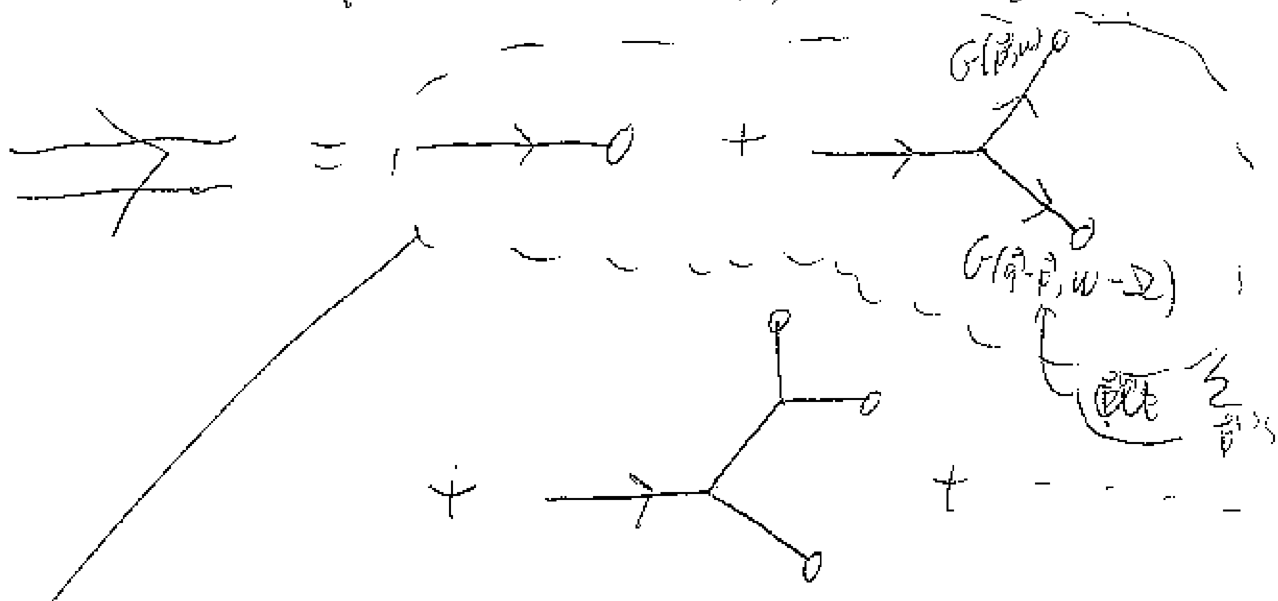
$\Psi_\alpha(\vec{q},w) = G_{\alpha B}(\vec{q},w) \, f_B(\vec{q})$ ← noise



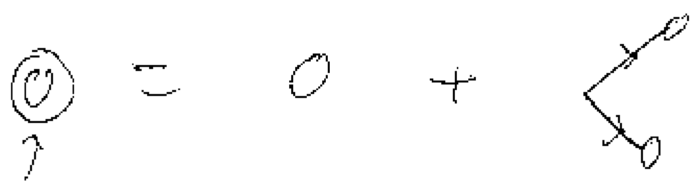
Full equation:



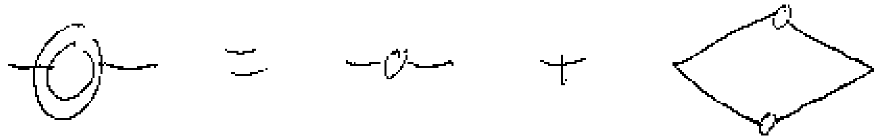
To solve perturbatively, iterate:



new, effective ~~noise~~: "renormalized" noise



⇒ renormalized Δ (noise correlation)



$\Delta_r = \Delta_0 +$ Graphical correction $(\Delta, D_B, \Lambda, g)$

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But: This perturbation theory may diverge $\left(\sum_{\vec{p}} \frac{1}{p^{\cdot}}\right)$ as $\vec{p} \to 0$, since fluctuations are big there.

How to ~~treat~~ deal with this?

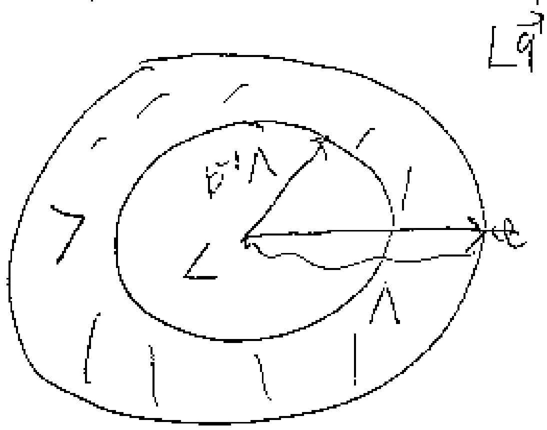
Dynamical RG: $\boxed{\text{Forster, Nelson, + Stephen Phys Rev A } \underline{16}, \text{ 732 (1977)}}$

Seperate $\Psi(\vec{q}, \omega)$ into "fast" and "slow" ~~p~~ spatial components:

slow: $\Psi_<(\vec{q}, \omega) = \Psi(\vec{q}, \omega)$ for $|\vec{q}| < b^{-1}\Lambda$

$\Psi_>(\vec{q}, \omega) = \Psi(\vec{q}, \omega)$ " $b^{-1}\Lambda < |\vec{q}| < \Lambda$

Brillouin zone



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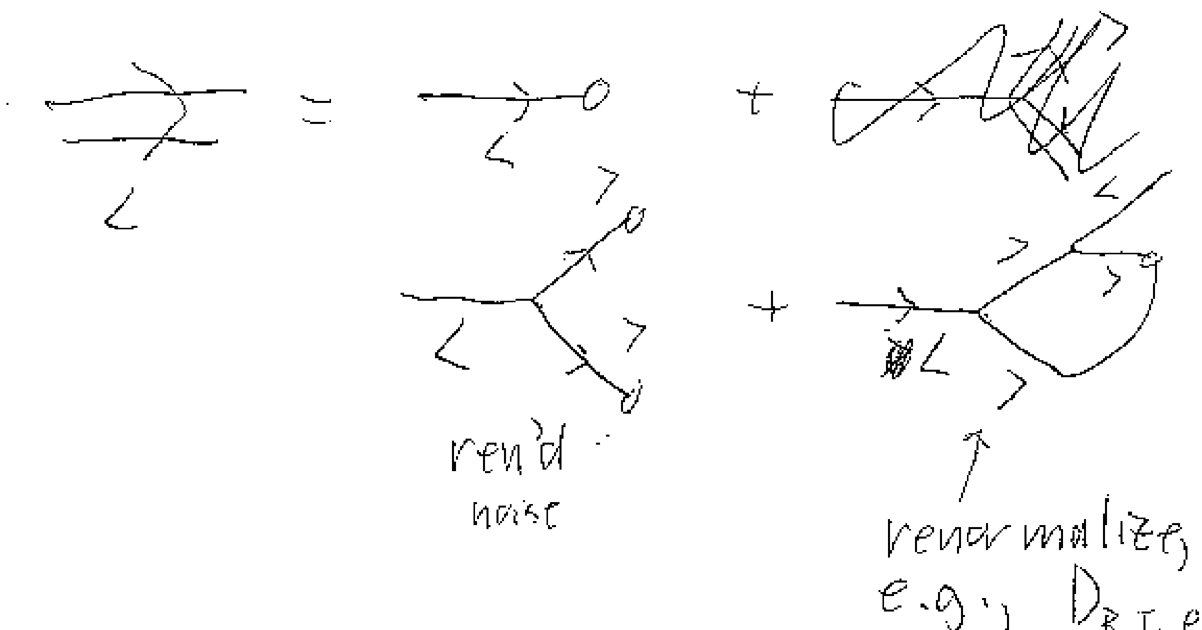
Average out all fields $\psi_>$, in "shell"

Do this in P.T.

No divergences, because we don't includ $\vec{q} \to 0$ in sums

Left with EOM's for $\psi_<$'s alon



$+ \ldots$

unrenormaliz

$$\{D, \Delta, \lambda, \Lambda\} \to \{D_I, \Delta_I, \lambda_I, b^{-1}\Lambda\}$$

Non-linearities

repeat.

Big question: Does problem get <u>easier</u> as we repeat, or harder

I.e., do perturbations ↑ or ↓ as we renormalize?

Easy way to tell:

Add extra, rescaling step:

$\vec{q}_\perp = b^{-1}\vec{q}_\perp{}'$ ← new wavevector coordinates

$\Rightarrow \vec{r}_\perp = b\,\vec{r}_\perp{}'$ ← " position co-ordinates

$\vec{\nabla}_\perp = b^{-1}\vec{\nabla}_\perp{}'$ ← " gradient

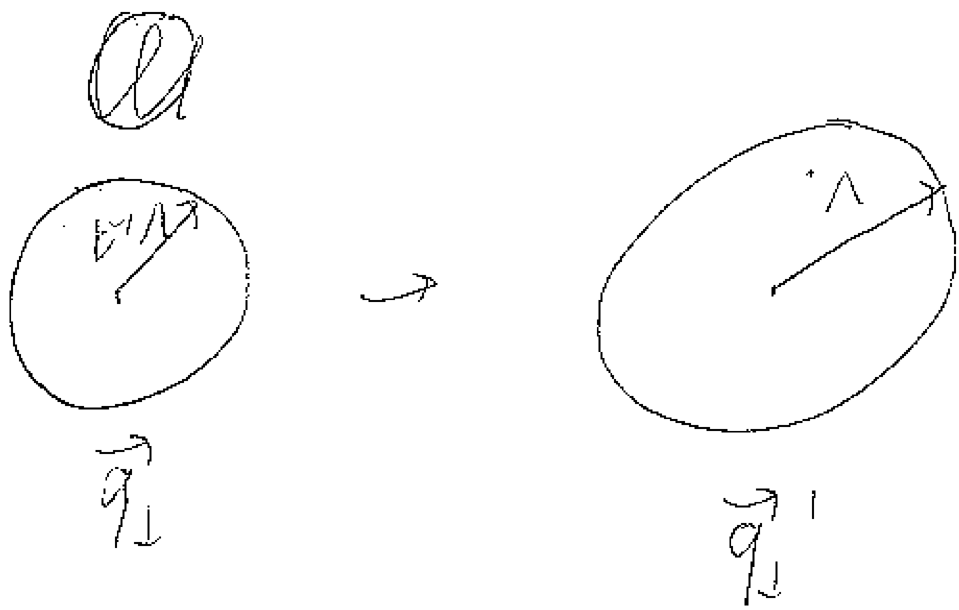
$r_\parallel \equiv b^\zeta r_\parallel{}'$ ← anisotropic scaling

$t \equiv b^z t'$

$\zeta \equiv$ anisotropy exponent

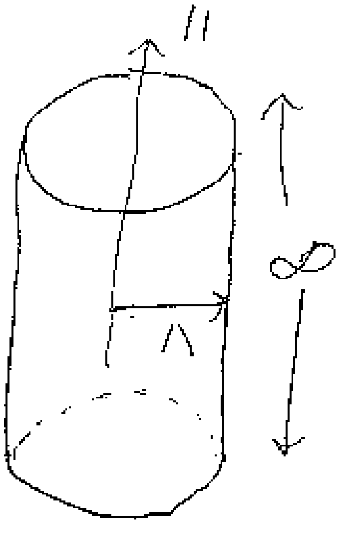
$z \equiv$ dynamical exponent

Technical aside: To make anisotropic



rescaling $\vec{q}_\perp$ restores UV cutoff to Λ

∞ Hypercylindrical BZ:



Also, rescale fields:

$$\psi_\alpha(\vec{r},t) \equiv b^{\chi_\alpha}\, \psi_\alpha'(\vec{r}',t')$$

$$\Rightarrow \quad g_\alpha^R = b^{\lambda_\alpha(\chi_\alpha)}\, g_\alpha^I = b^{\lambda_\alpha}\left(g_\alpha^0 + \text{Graphs}\right)$$

parameter in EOM after both RG steps

Choose χ_α's to keep fluctuation controlling parameters fixed ($D_{B,T,\rho}$, Δ in our problem)

$\Rightarrow$ On next step of RG, fluctuations will be same size

(Λ also same)

$\Rightarrow$ Perturbations will <u>only</u> grow if Non-linear coefficients $\{\lambda_1, \lambda_2, \sigma_2, \ldots\}$ grow

Full EOMs:

$$\partial_t \rho + \bar{\rho}_0 \vec{\nabla}_\perp \cdot \vec{v}_\perp + \lambda_\rho \vec{\nabla}_\perp \cdot (\delta\rho\, \vec{v}_\perp) + D_\rho \partial_\parallel^2 \delta\rho = 0$$

$$\partial_t \vec{v}_\perp + \lambda_1 (\vec{v}_\perp \cdot \vec{\nabla}) \vec{v}_\perp + \lambda_2 \vec{v}_\perp (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + \gamma \partial_\parallel \vec{v}_\perp + \gamma' \delta\rho\, \partial_\parallel \vec{v}_\perp + \gamma_2 \vec{v}_\perp \partial_\parallel \delta\rho$$

$$= -\vec{\nabla}_\perp \left(\sum_{n=1}^{\infty} \sigma_n (\delta\rho)^n \right) + D_B \vec{\nabla}_\perp (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + D_T \vec{\nabla}_\perp^2 \vec{v}_\perp + D_\parallel \partial_\parallel^2 \vec{v}_\perp + \vec{f}_\perp$$

Expansion of

Do rescaling: $\delta\rho' \equiv b^{\chi_\rho} \delta\rho'$ (I'll choose $\chi_\rho = \chi_v = \chi$, since $\delta\rho \sim v_\perp$)

$\vec{v}_\perp \equiv b^{\chi_a} \vec{v}_\perp{}'$

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$$b^{-z+\chi_\rho}\,\partial_{t'}\delta\rho' + b^{\chi_v-1}\,\vec{\nabla}_\perp'\cdot\vec{v}_\perp' + b^{2\chi_\rho-1}\lambda_\rho\,\vec{\nabla}_\perp'\cdot(\delta\rho'\,\vec{v}_\perp')$$

$$+\, b^{\chi_\rho-2\zeta}\,D_\rho\,\partial_\parallel'^2\,\delta\rho' = 0$$

$$b^{-z+\chi_v}\,\partial_{t'}\vec{v}_\perp' + b^{2\chi_v-1}\lambda_1(\vec{v}_\perp'\cdot\vec{\nabla}_\perp')\vec{v}_\perp' + b^{2\chi_v-1}\lambda_2\,\vec{v}_\perp'(\vec{\nabla}_\perp'\cdot\vec{v}_\perp')$$

$$+\, b^{\chi_v-\zeta}\gamma\,\partial_\parallel'\vec{v}_\perp' + b^{2\chi_\rho-\zeta}\gamma'\,\delta\rho'\,\partial_\parallel\vec{v}_\perp'$$

$$+\, b^{2\chi_\rho-\zeta}\,\vec{v}_\perp'\,\partial_\parallel'\delta\rho'$$

$$= -\vec{v}_\perp'\left(\sum_{n=1}^{\infty} b^{n\chi_\rho-1}\sigma_n(\delta\rho')^n\right) + D_B\, b^{\chi_v-2}\,\vec{\nabla}_\perp'(\vec{\nabla}_\perp'\cdot\vec{v}_\perp')$$

$$+\, b^{\chi_v-2}D_T\,\nabla_\perp'^2\,\vec{v}_\perp' + b^{\chi_v-2\zeta}\,D_\parallel\,\partial_\parallel'^2\,\vec{v}_\perp'$$

$$+\,\vec{f}_\perp$$

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$$\Rightarrow \quad D_\rho' = b^{z-2\zeta}\left(D_\rho + \text{Graphs}(\{\lambda_i, D_i, \Delta\})\right)$$

$$D_{B,T}' = b^{z-2}\left(D_{B,T} + \text{Graphs}\right)$$

$$D_{11}' = b^{z-2\zeta}\left(D_{11} + \text{Graphs}\right)$$

$$\vec{f}_\perp' = \vec{f}_\perp\, b^{z-\chi}$$

$$\Rightarrow \quad \langle f_\perp'(r_1,t_1)\, f'(r_2,t_2)\rangle = b^{2(z-\chi)}\Delta\, \delta^d(r_1-r_2)\,\delta(t_1-t_2)$$

$$\delta^{d-1}(\vec{r}_1-\vec{r}_2) = \delta^d\left(b(\vec{r}_1{}'-\vec{r}_2{}')\right) = b^{-(d-1)}\delta^d(\vec{r}_1{}'-\vec{r}_2{}')$$

$$\delta(t_1-t_2) = \delta\left(b^z(t_1'-t_2')\right) = b^{-z}\delta(t_1'-t_2')$$

$$\delta(r_{\parallel_1}-r_{\parallel_2}) = \delta\left(b^{\zeta}(r_{\parallel_1}'-r_{\parallel_2}')\right) = b^{-\zeta}\delta(r_{\parallel_1}'-r_{\parallel_2}')$$

property of δ fn: $\delta(kx) = \frac{1}{k}\delta(x)$

$$\text{So} \quad \langle f'f'\rangle = b^{2(z-\chi)-d-z+1-\zeta}\,\Delta\,\delta^d(r_1'-r_2')\,\delta(t_1'-t_2')$$

$$\Rightarrow \quad \Delta' = b^{z-2\chi-d+1-\zeta}\left(\Delta + \text{Graphs}\right)$$

Suppose λ_i, $\sigma_{n>1}$, ~~are~~ γ_1', γ_2 all small initially (before RG)

$\Rightarrow$ can neglect graphs (at <u>this</u> point in RG)

$\Rightarrow$ To keep flucs fixed (keep D's, Δ fixed) choose

$D_{g,\parallel}$ fixed $\Rightarrow z = 2\zeta$

$D_{B,T}$ fixed $\Rightarrow \boxed{z = 2}$

$\Rightarrow \boxed{\zeta = 1}$

Δ fixed $\Rightarrow z = 2\chi + d + \zeta - 1 \Rightarrow \boxed{\chi = 1 - \frac{d}{2}}$ $\leftarrow$ Note: $\chi > 0$ for $d < 2$ (Mermin-Wagner)

$$\vec{v}_\perp = b^{\chi} \vec{v}_\perp{}'$$

These flucs fixed

$\Rightarrow$ These flucs growing

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Now, as we keep running RG, can be continue to neglect graphs?

Since flucs fixed with this choice of (z, ζ, χ), we can iff λ_i's, $\sigma_{n>2}$ all $\to 0$ as $b \uparrow$.

$$\lambda'_{1,2} = b^{\chi+z-1}(\lambda_{1,2} + \text{Graphs}) = b^{\overbrace{2-\frac{d}{2}}^{\phi_\lambda}}(\lambda_{1,2} + \text{Graphs})$$

$$\sigma'_n = b^{(n-1)\chi-1+z}(\sigma_n + \text{graphs}) = b^{\underbrace{n - \left(\frac{n-1}{2}\right)d}_{\phi_n}}(\sigma_n + \text{Graphs})$$

$\Rightarrow$ RG eigenvalues:

$$\phi_\lambda = 2 - \frac{d}{2} > 0 \text{ if } d < 4$$

$\Rightarrow \lambda_{1,2}$ grow for $d<4$

$$\phi_n = n - \left(\frac{n-1}{2}\right)d > 0 \text{ for } d < \frac{2n}{n-1} = d(n)$$

$\Rightarrow$ biggest is $n=2$

$\Rightarrow$ for $d<4$, σ_2, λ_i's grow upon RG, if initially small

$\Rightarrow$ $\boxed{\text{"relevant"}}$

$\Rightarrow$ can ~~not~~ neglect them for b big enough

$\Rightarrow$ can not neglect them for large distance L

$$\gamma' = b^{\alpha+z-3}(\gamma + \text{Graphs})$$

$$\gamma_2 = b^{\alpha+z-3}(\gamma + \text{Graphs})$$

$$\phi_{\gamma\text{'s}} = \alpha+z-3 = 1-\frac{d}{2}+2-1 = 2-\frac{d}{2} \Rightarrow \text{These}$$

also relevant for $d<4$

Eventually, must do graphs

Aside: What would have happened if we assumed finite range spatio-temporal correlations of $\vec{f}$?

$$\langle f_i(\vec{r},t) f_j(\vec{r}',t') \rangle = \delta_{ij}\, g(\vec{r}-\vec{r}', t-t')$$

$$\Rightarrow \langle f_i(\vec{q},\omega) f_j(-\vec{q},-\omega) \rangle = \delta_{ij}\, \tilde{g}(\vec{q},\omega)$$

$$g(\vec{q},\omega) \equiv \int d^d\Delta r\, dt\; g(\Delta \vec{r}, \Delta t)\, e^{i(\vec{q}\cdot\vec{r}-\omega t)}$$

If $g(\Delta\vec{r}, \Delta t)$ decays sufficiently rapidly as $\Delta\vec{r}, \Delta t \to \infty$,

$$g(\vec{q}\to\vec{0}, \omega\to 0) \to \int d^d\Delta r\, dt\; g(\Delta\vec{r}, \Delta t)$$

finite $\equiv \Delta$

$$\Rightarrow \langle f_i(\vec{q},\omega) f_j(-\vec{q},-\omega) \rangle \underset{q,\omega\to 0}{\to} \Delta$$

$\Rightarrow$ small $\vec{q}, \omega$ behavior exactly the same

$$\lambda_g' = b^{\eta+z-1}\lambda_g = b^{2-\frac{d}{2}}\lambda_g$$

$\Rightarrow$ λ_g relevant for $d<2$

$\Rightarrow$ $\lambda_{1,2}$, σ_2, λ_g, γ', γ_2 all relevant NL's for $d<4$.

How to deal with them?

1. Do graphs.

Example: Subset of graphs for λ_2 (only those linear in λ_2), drop γ's)

14 graphs, 200+ pages of algebra, 3 months of JT's life

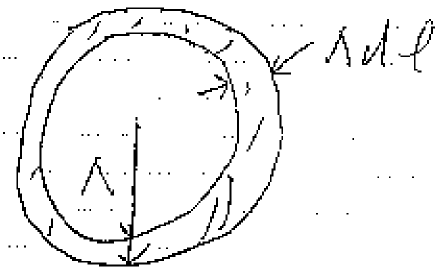
a Find $\delta\lambda_2\Big|_{\text{Graph}} = 0$

Not very useful anyway, because only valid near $d = 4$.

Why?

Easy way to see: Turn ~~into diff. eqn~~ recursion relation into differential equation:

$b = 1 + d\ell\,,\quad d\ell \ll 1$



$\Rightarrow$ Graphs $\propto d\ell$ as well

$\Rightarrow$ $\cancel{\lambda_1' = 1 + \left(\frac{4-d}{2}\right) d\ell}$

$$b^{\frac{4-d}{2}} \approx 1 + \left(\frac{4-d}{2}\right) d\ell \equiv 1 + \frac{\epsilon}{2} d\ell\,,\quad \epsilon \equiv 4 - d$$

EG $\Rightarrow \lambda_1' \cong (1+\frac{\epsilon}{2}dl)(\lambda_1 + f(\lambda)\,dl)$

$$= \lambda_1 + dl\left[\frac{\epsilon}{2}\lambda_1 + f\right]$$

$$\Rightarrow \frac{\lambda_1' - \lambda_1}{dl} = \left(\frac{\epsilon}{2}\lambda_1 + f\right)$$

$$\Rightarrow \frac{d\lambda_1}{dl} = \frac{\epsilon}{2}\lambda_1 + f$$

Likewize for σ_2, γ_0's, etc.

Since we can only calculate f in P.T., must hope that λ's etc stay small. $\Rightarrow$ hope $f < 0$.

~~Typically, f~~

$f = -C\lambda^2$ (lowest order in λ

$$\Rightarrow \frac{d\lambda_1}{d\ell} = \frac{\epsilon}{2}\lambda_1 - c\lambda_1^2 \Rightarrow \lambda_1 = \frac{\epsilon}{2c}$$: small if $\epsilon \ll 1$

Basis of ϵ-expansion

Only valid near $d=4$.

Often works well in $d=3$, but ~~need~~ need to go to $O(\epsilon^2)$ to decide delicate competition between λ, γ, etc.

Nothing one can do near $d=2$.

However: Let's assume that

Graphs are such that γ's $\to 0$.

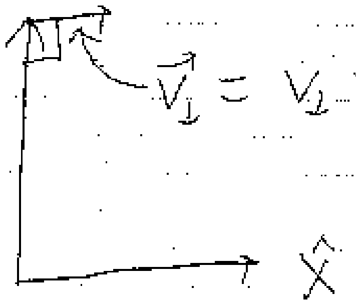
$\Rightarrow$ In $d=2$, only NL's left are

λ_g, $\lambda_{1,2}$, σ_2

Look at $\lambda_{1,2}$ in $d=2$

$\vec{v}_\perp, \vec{\nabla}_\perp$

$\langle \vec{v} \rangle = v_0 \hat{y}$



$\vec{v}_\perp = v_\perp \hat{x}$: Only one component

$$\Rightarrow (\vec{v}_\perp \cdot \vec{\nabla}_\perp)\vec{v}_\perp = \hat{x}\, v_\perp \partial_x v_\perp$$
$$= \frac{1}{2}\hat{x}\,\partial_x (v_\perp^2)$$

$$\vec{v}_\perp(\vec{\nabla}_\perp \cdot \vec{v}_\perp) = \hat{x}\, v_\perp \partial_x v_\perp$$
$$= \frac{1}{2}\hat{x}\,\partial_x(\vec{v}_\perp^{\,2}) \text{ : Same.}$$

Both λ_1 and λ_2 terms are the same in $d=2$!

Furthermore, both are total
x ~~decision~~ (i.e., $\perp$) derivatives

$\Rightarrow$ 1) Renormalization of noise correlations $\Delta = O(q^2) \ll \Delta_0$

$\Rightarrow$ Neglect
(No noise renorm)

$\Rightarrow$ $\Delta' = b^{z-2\chi-d}\Delta$

exact in $d=2$

$\Rightarrow$ To fix flucs, must choose

$$\cancel{\boxed{z = 2\chi + d} = 2\chi + 2} \quad (1)$$

$$\boxed{z = 2\chi + d + \zeta - 1 = 2\chi + \zeta + 1} \quad (1)$$

since $d=2$

2) ~~$D_{\parallel}$ graph~~

No $D_{\parallel}$ graphs $\quad D_{\parallel}\partial_{\parallel}^{2}$, λ's $\propto \partial_{\perp}$

$$\Rightarrow D_{\parallel}' = b^{z-23} D_{\parallel} \quad \text{Exactly}$$

$$\Rightarrow \boxed{z = 23} \quad (2)$$

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3) If $\lambda_{\rho} = \lambda_1 + \lambda_2 \equiv \lambda$, EOM's have exact Pseudo-Gallilean invariance:

$$\vec{V}_{\perp}(\vec{r}, t) \to \vec{V}_{\perp}(\vec{r} - \lambda \vec{v}_{\perp}^{0} t, t) + \vec{v}_{\perp}^{0}$$

$$\delta\rho(\vec{r}, t) \to \delta\rho(\vec{r} - \lambda \vec{v}_{\perp}^{0} t, t)$$

Must be preserved with same value of λ

$\Rightarrow$ No graphs for λ in this case.

Furthermore, we know there are no graphs for λ_ξ (~~[illegible]~~ # conservation is exact)

$\Rightarrow$ $G_{\lambda_\xi} = 0$ always

$\Rightarrow$ if bare λ $< \lambda_\xi$, renormalized λ will always be $< \lambda_\xi$ (if equal, λ gets no graphs $\Downarrow$ has same RR as λ_ξ $\Rightarrow$ remains equal)

$\Rightarrow$ If F.P. exists, since same λ (λ or λ_ξ) must be $\neq 0$ at it, λ_ξ must be non-zero at it.

RP for λ_ϱ

$$\lambda_\varrho' = b^{z+\gamma-1}\lambda_\varrho$$

$\Rightarrow$ To get FP, must have

$$\boxed{z+\gamma=1} \quad (3)$$

3 eqns (1, 2, 3) for 3 unknowns (z, γ, ζ).

Solve: (2), (3) into (1) $\Rightarrow$ $z = 2(1-z) + 1 + \frac{z}{2}$

$$\Rightarrow \frac{5}{2}z = 3 \Rightarrow \boxed{z = \frac{6}{5}}$$

$$\Rightarrow \zeta = \frac{3}{5}, \quad \gamma = 1 - z = -\frac{1}{5} < 0 \Rightarrow LRO!$$

If you believe this works in all d,

$$\frac{5}{2}z = d - 1 + 2 = d + 1 \Rightarrow z = \frac{2(d+1)}{5} = \begin{cases} 2, & d=4 \\ 6/5, & d=2 \end{cases}$$

$$\zeta = \frac{d+1}{5} = \begin{cases} 1, & d=4 \\ 3/5, & d=2 \end{cases}$$

$$\gamma = \frac{3-2d}{5} = \begin{cases} -1, & d=4 \; (=1-\frac{d}{2}) \\ -\frac{1}{5}, & d=2 \end{cases}$$

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