

# Globally regular instability of $AdS_3$

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# Outline

- Instability of  $\text{AdS}_{d+1}$  for  $d \geq 3$
- Why  $d = 2$  is different
- Analyticity strip method
- Evidence for weak turbulence
- Open questions

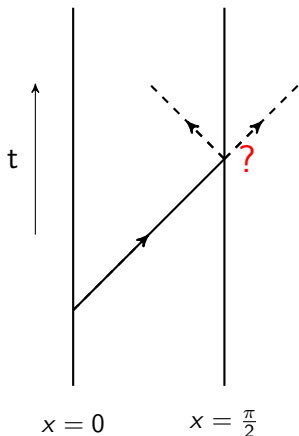
## Anti-de Sitter spacetime in $d + 1$ dimensions ( $\text{AdS}_{d+1}$ )

- $\text{AdS}_{d+1}$  spacetime is the maximally symmetric solution of the vacuum Einstein equations  $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$  with negative  $\Lambda$

$$g = \frac{\ell^2}{\cos^2 x} (-dt^2 + dx^2 + \sin^2 x d\omega_{S^{d-1}}^2), \quad \Lambda = -\frac{2}{d(d-1)\ell^2}$$

where  $0 \leq x < \pi/2$  and  $-\infty < t < \infty$ .

- Spatial infinity  $x = \pi/2$  is the timelike cylinder  $\mathcal{I} = \mathbb{R} \times S^{d-1}$  with the boundary metric  $ds_{\mathcal{I}}^2 = -dt^2 + d\omega_{S^{d-1}}^2$
- AdS is **not globally hyperbolic** -  
to make sense of evolution one needs to choose boundary conditions at  $\mathcal{I}$
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



## Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes)
- Basic question for any equilibrium solution: **do small perturbations of it at  $t = 0$  remain small for all future times?**
- Key difference between Minkowski and AdS: the main mechanism of stability of Minkowski - **dissipation of energy by dispersion** - is absent in AdS (for no flux boundary conditions  $\mathcal{I}$  acts as a mirror)
- The problem seems tractable only in spherical symmetry; we need matter to generate dynamics
- Simple matter model: massless scalar field

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left( \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial\phi)^2 \right)$$
$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0$$

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{\cos^2 x} \left( -Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x d\omega_{d-2}^2 \right)$$

where  $A$  and  $\delta$  are functions of  $(t, x)$ .

- Define mass function  $m(t, x) = \frac{\sin^{d-2} x}{\cos^d x} (1 - A)$
- Field equations (using  $8\pi G = d - 1$  and  $' = \partial_x$ ,  $\dot{\phantom{x}} = \partial_t$ )

$$\begin{aligned} (A^{-1} e^{\delta} \dot{\phi})' &= (\tan x)^{1-d} \left( \tan^{d-1} x A e^{-\delta} \phi' \right)' \\ m' &= (\tan x)^{d-1} A S, \quad \delta' = -\sin x \cos x S, \quad S := A^{-2} e^{2\delta} \dot{\phi}^2 + \phi'^2 \end{aligned}$$

- Initial-boundary problem is locally well-posed under the following boundary conditions near  $x = \pi/2$  (Holzegel-Smulevici 2011)

$$\phi \sim \left( \frac{\pi}{2} - x \right)^d, \quad \delta \sim \left( \frac{\pi}{2} - x \right)^{2d}, \quad 1 - A \sim \left( \frac{\pi}{2} - x \right)^d$$

- We are interested in the long time evolution of small smooth perturbations of AdS spacetime  $\phi = 0, m = 0, \delta = 0$ .

## Conjecture (B-Rostworowski 2011)

*AdS<sub>4</sub> is unstable against the formation of a black hole for a large class of arbitrarily small perturbations*

Evidence:

- **Perturbative:** due to the nondispersive character of the linear spectrum, resonant interactions between harmonics give rise to secular terms at higher orders of the formal perturbation expansion. This **shifts the energy spectrum to higher frequencies**. The same happens for vacuum Einstein equations (Dias-Horowitz-Santos 2011).
- **Heuristic:** the transfer of energy to higher frequencies (eo ipso, concentration of energy on finer and finer spatial scales) is expected to be eventually cut-off by horizon formation.
- **Numerical:** perturbations of size  $\varepsilon$  start growing after time  $\mathcal{O}(\varepsilon^{-2})$ . Subsequent nonlinear evolution leads to black hole formation (confirmed independently by Buchel-Lehner-Liebling 2012).

Generalization to  $d \geq 4$  is straightforward (Jałmużna-Rostworowski-B).

## AdS gravity in $d = 2$

- Spectral properties and nonlinear perturbation analysis are qualitatively the same in all dimensions  $d \geq 3$
- Dimensionless measure of gravity's strength is  $GM/L^{d-2}$  so for  $d = 2$  the *total* mass matters (not its concentration) ( $d = 2$  is the critical dimension for Einstein's equations)
- AdS-Schwarzschild family in  $d = 2$  (using  $r = \ell \tan x$ )

$$g = -N dt^2 + N^{-1} dr^2 + r^2 d\varphi^2, \quad N = 1 - M + r^2/\ell^2$$

There is a mass gap between AdS<sub>3</sub> and the lightest black hole:

- ▶  $M = 0$  AdS<sub>3</sub>
  - ▶  $0 < M < 1$  conical (naked) singularities (Staruszkiewicz 1963)
  - ▶  $M > 1$  BTZ black holes (Bañados-Teitelboim-Zanelli 1992)
- Small perturbations of AdS<sub>3</sub> cannot evolve into BTZ black holes
  - Numerical studies of the threshold for black hole formation has been pioneered by Pretorius and Choptuik 2000

## Possible scenarios of evolution

- Field equations (using  $e^\beta := Ae^{-\delta}$ )

$$\begin{aligned} (e^{-\beta} \dot{\phi})' &= \frac{1}{\tan x} (\tan x e^\beta \phi')' \\ m' &= \tan x A (e^{-2\beta} \dot{\phi}^2 + \phi'^2), \quad \beta' = 2 \sin x \cos x \frac{m}{A} \end{aligned}$$

- **Black hole formation is excluded for  $M = \lim_{x \rightarrow \pi/2} m(t, x) < 1$ .**

Proof:  $g^{\alpha\beta} \partial_\alpha r \partial_\beta r = A = 1 - m \cos^2 x > 0$ .

- There remains a dichotomy:
  - (a) Global-in-time regularity
  - (b) Naked singularity formation
- We will give evidence against (b) using the **analyticity strip method** (Sulem-Sulem-Frisch 1983).



## Analyticity strip method

- Let  $u(t, x)$  be a solution of an evolution equation starting from real-analytic initial data and let  $u(t, z)$  be its analytic extension to the complex  $z$ -plane.
- Typically  $u(t, z)$  will have complex singularities. Let  $z = x + i\rho$  be the location of the singularity closest to the real axis (hence  $\rho$  measures the width of the analyticity strip around the real axis).
- If  $\rho(t)$  vanishes at some  $t = T < \infty$ , then the solution "blows up"; otherwise it is globally regular in time.
- Fourier coefficients of  $u(t, x)$  behave for large  $k$  as

$$\hat{u}_k(t) \sim k^{-\alpha} \exp(-\rho k)$$

- Method: compute  $\rho(t)$  by fitting an exponential decay to the tail of the numerically computed Fourier spectrum

## Example

$$u_t = xu_x + \alpha u^2, \quad u(0, x) = \frac{\varepsilon}{1+x^2}$$

$$u(t, x) = \frac{\varepsilon}{1 + e^{2t}x^2 - \alpha\varepsilon t}$$

$$\hat{u}(t, k) = \frac{\varepsilon\pi e^{-t}}{\sqrt{1 - \varepsilon\alpha t}} H(k) \exp(-k \underbrace{e^{-t}\sqrt{1 - \varepsilon\alpha t}}_{\rho(t)}) + (k \leftrightarrow -k)$$

- $\alpha > 0$ : blowup at  $x = 0$  in time  $T = 1/\varepsilon\alpha$
- $\alpha = 0$ : global regularity but

$$\|u\|_{H^s}^2 := \int_{-\infty}^{\infty} (\partial_x^s u)^2 dx = c_s e^{(2s-1)t}$$

$L^2$ -asymptotic stability ( $s = 0$ ) and instability for  $s > 1/2$ .

## Spectral properties

- Linearized equation (Breitenlohner-Freedman 1982, Ishibashi-Wald 2004)

$$\ddot{\phi} + L\phi = 0, \quad L = -\frac{1}{\tan x} \partial_x (\tan x \partial_x)$$

$L$  is essentially self-adjoint on  $L^2([0, \pi/2], \tan x dx)$ .

- Eigenvalues and eigenvectors of  $L$  are ( $k = 0, 1, \dots$ )

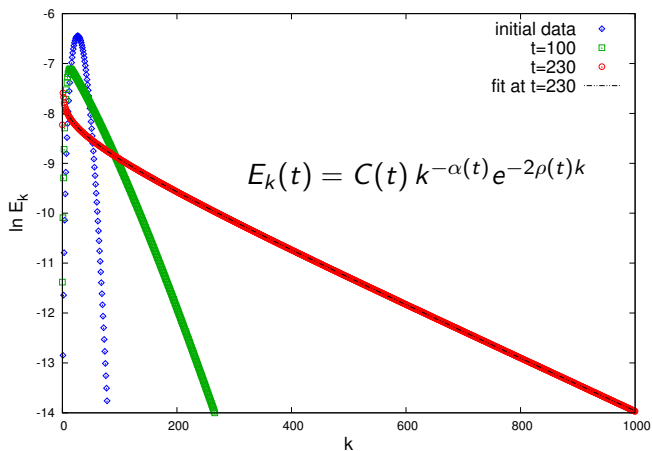
$$\omega_k^2 = (2 + 2k)^2, \quad e_k(x) = 2\sqrt{k+1} \cos^2 x P_k^{0,1}(\cos 2x)$$

- Inner product:  $(f, g) = \int_0^{\pi/2} f(x)g(x) \tan x dx$
- Let us define  $\phi_k := (\sqrt{A} \phi', e'_k)$  and  $p_k := (\sqrt{A} e^{-\beta} \dot{\phi}, e_k)$ . Then

$$M = \int_0^{\pi/2} A \left( e^{-2\beta} \dot{\phi}^2 + \phi'^2 \right) \tan x dx = \sum_{k=0}^{\infty} E_k(t)$$

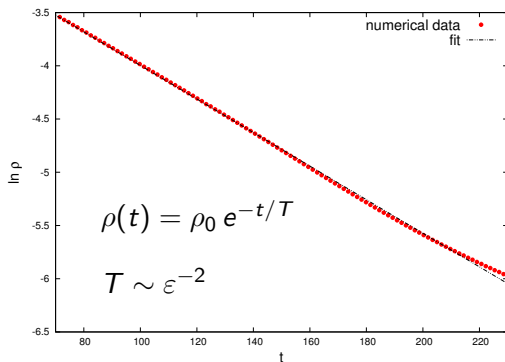
where  $E_k = p_k^2 + \omega_k^{-2} \phi_k^2$  is the energy of the  $k$ -th mode.

# Computation of $\rho(t)$ from the energy spectrum



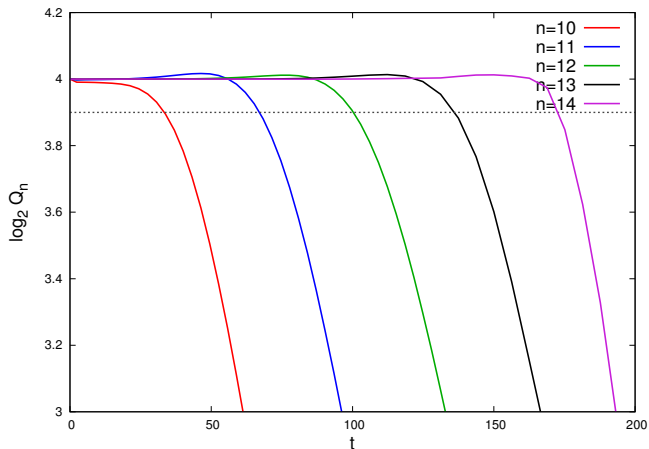
Initial data  $\phi(0, x) = \varepsilon \exp(-\tan^2 x / \sigma^2)$ ,  $\dot{\phi}(0, x) = 0$

## Evidence for global regularity



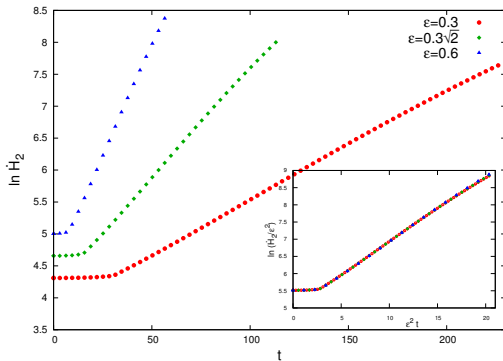
- Conjecture: as  $t \rightarrow \infty$ , solutions develop progressively finer spatial scales without ever losing smoothness (weak turbulence).
- Similar weakly turbulent loss of regularity occurs for the incompressible Euler equation in two dimensions (Yudovich 1974).

## Convergence test



Convergence factor for the solution  $\phi_n$  computed on the  $2^n$ -grid is defined by  $Q_n = \frac{\|\phi_n - \phi_{n+1}\|}{\|\phi_{n+1} - \phi_{n+2}\|}$ , where  $\|\cdot\|$  is the spatial  $\ell_2$ -norm.

# $\dot{H}^s$ -instability for $s > 1$



Time evolution of the upper envelope of  $\dot{H}^2 = \|\phi''(t, x)\|_2$

- $\text{AdS}_3$  is  $\dot{H}^s$ -unstable for arbitrarily small perturbations (for  $s > 1$ ).
- The turbulent instability is not active for some perturbations - time-periodic solutions (see Andrzej's talk).

## Final remarks

- Weak turbulence is expected to be common for nonlinear wave equations in bounded domains.
- In the case of Einstein's equations, **the weakly turbulent dynamics can proceed forever only in  $d = 2$** , whereas in higher dimensions it is unavoidably cut off in finite time by the black hole formation.
- The nature of the threshold at  $M = 1$  is not well understood. Does every solution with  $M > 1$  evolve into a black hole?
- **Finite energy threshold for blowup is typical for nonlinear wave equations in critical dimensions** (for instance, wave maps in  $d = 2$  or Yang-Mills equations in  $d = 4$ ).
- Are there any interesting holographic implications of weak turbulence?