``Mathematics and Physics of the Holographic Principle''

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Excited States
and
Holographic Entanglement Entropy

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Based on

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① Introduction

What is the entanglement entropy (EE) ?

A measure how much a given quantum state is quantum mechanically entangled (or complicated).

~ The amount of `active' degrees of freedom (or its information)

Why interesting and useful ?

At present, it seems still difficult to observe EE in real experiments (→ a developing subject).

But, recently it is very common to calculate EE in `**numerical experiments**' of cond-mat systems.

e.g. computing central charges, detecting spin liquids

Advantages of EE

- EE = A quantum order parameter (~a generalization of `Wilson loops') ➡ Classify quantum phases.

- The entanglement entropy (EE) is a helpful bridge between gravity (string) and cond-mat physics.

Gravity ⟺ Entanglement ⟺ Cond-mat.

$g_{\mu\nu}$ AdS/CFT (Holography) $S_A \approx \text{Area}$ systems $|\Psi\rangle$

- A universal quantity which characterizes the properties of non-equilibrium states.

Information vs. Energy

1st law of thermodynamics: $T \cdot dS = dE$

Temp. · Information · Energy

⇒ Can we find an analogous relation in any quantum systems which are far from the equilibrium ?

Something like: $T_{ent} \cdot dS_A = dE_A$??

Information in A = EE · Energy in A

What ? (→ T_{ent})

Can we observe EE ??

➡ The main motivation of this talk.

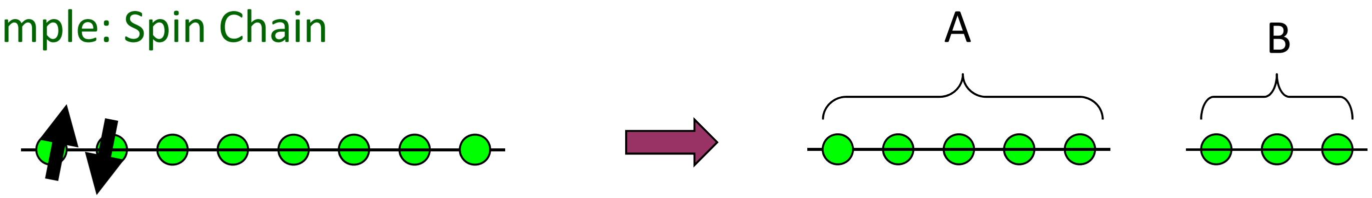
Contents

② Basic Facts about the Entanglement Entropy (EE)

(2-1) Definition of Entanglement Entropy

Divide a quantum system into two subsystems A and B. $H_{tot} = H_A \otimes H_B$.

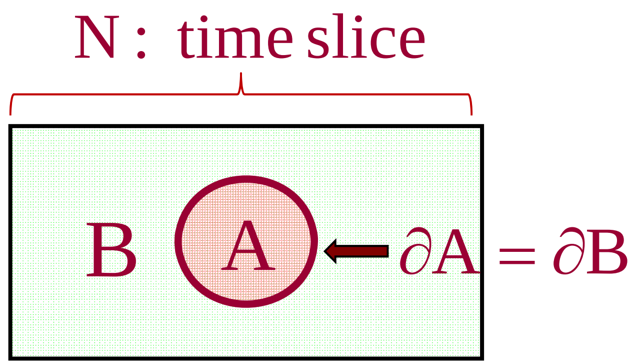
Example: Spin Chain



Define the reduced density matrix ρ_A for A by $\rho_A = \mathrm{Tr}_B \rho_{tot}$, taking a trace over the Hilbert space of B .

Now the entanglement entropy S_A is defined by the von-Neumann entropy:

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \, .$$



(2-2) Basic Properties of EE

(i) If ρ_{tot} is a pure state (i.e. $\rho_{tot} = |\Psi\rangle\langle\Psi|$) and $H_{tot} = H_A \otimes H_B$,

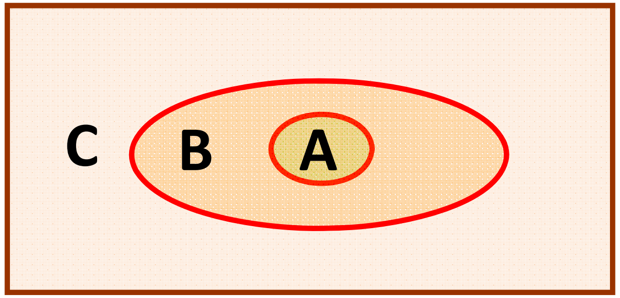
then $S_A = S_B$. $\Rightarrow$ EE is not extensive !

(ii) Strong Subadditivity (SSA) [Lieb-Ruskai 73]

When $H_{tot} = H_A \otimes H_B \otimes H_C$, for any ρ_{tot},

$$S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B,$$

$$S_{A+B} + S_{B+C} \geq S_A + S_C.$$



(Actually, these two inequalities are equivalent .)

The strong subadditivity can also be regarded as the concavity of von-Neumann entropy.

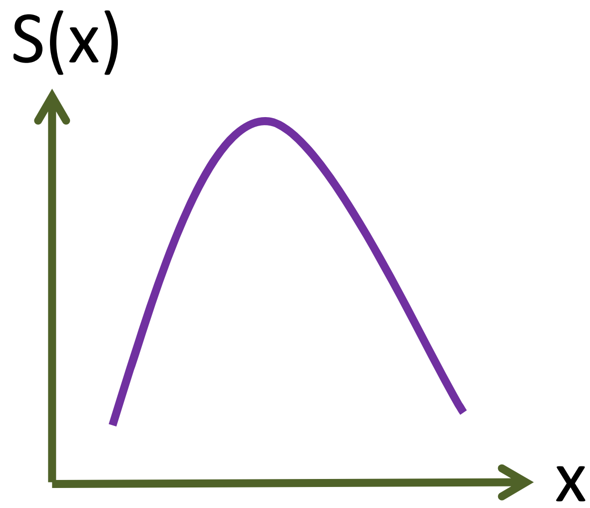
Indeed, if we assume A,B,C are numbers, then

$$S(A+B)+S(B+C)\geq S(A+B+C)+S(B),$$

$$\Rightarrow 2\cdot S\left(\frac{x+y}{2}\right)\geq S(x)+S(y),$$

$$\Rightarrow \frac{d^2}{dx^2}S(x)\leq 0.$$

$$(\text{i.e. concave function of } x)$$



(2-3) Holographic Entanglement Entropy (HEE)

[Ryu-TT 06]

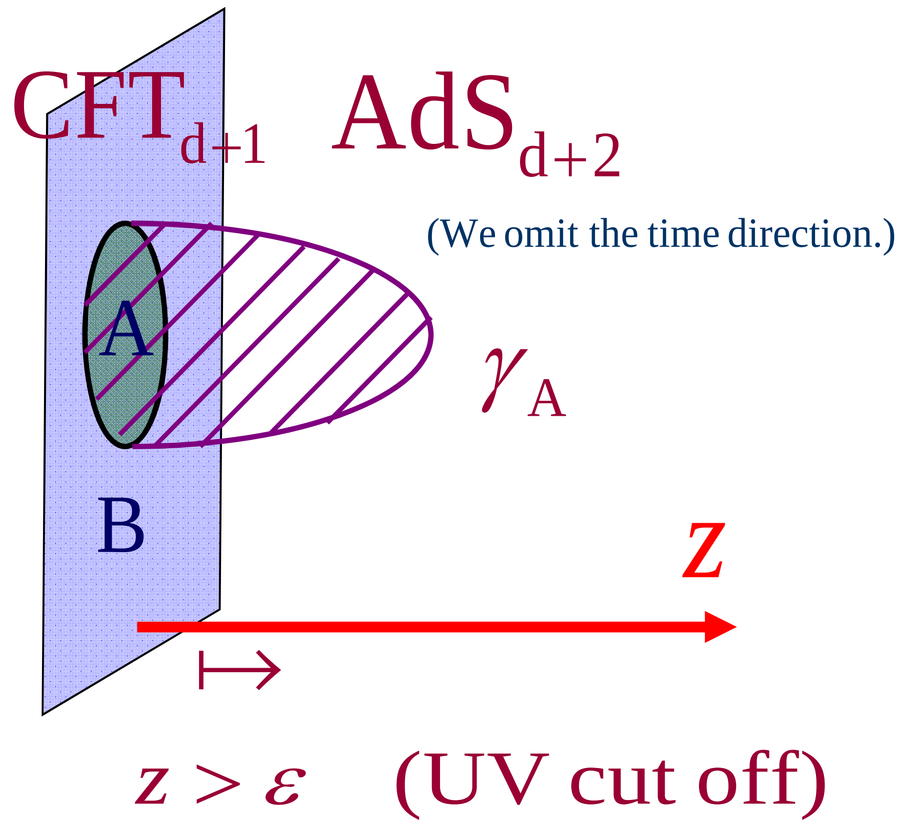
$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial\gamma_A \text{ and } A \sim \gamma_A .$$

homologous

$$ds^2_{AdS} = R^2_{AdS}\frac{-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2}{z^2}$$



Note: In time-dependent b.g., we need to employ the covariant version [Hubeny-Rangamani-TT 07].

Verification of HEE

- Confirmations of basic properties:

Area law, Strong subadditivity (SSA), Conformal anomaly,….

- Direct Derivation of HEE from AdS/CFT:

(i) Pure AdS, A = a round sphere [Casini-Huerta-Myers 11]

(ii) Euclidean AdS/CFT [Lewkowycz-Maldacena 13, cf. Fursaev 06]

(iii) Disjoint Subsystems [Headrick 10, Faulkner 13, Hartman 13]

(iv) General time-dependent AdS/CFT → Not yet.

[But, SSA: Allais-Tonni 11, Callan-He-Headrick 12, Wall 13]

- Corrections to HEE beyond the supergravity limit:

[Higher derivatives: Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11,…..]

[1/N effect: Barrella-Dong-Hartnoll-Martin 13, Faulkner-Lewkowycz-Maldacena 13]

[Higher spin gravity: de Boer-Jottar 13, Ammon-Castro-Iqbal 13]

③ `The First law' for EE of Excited States

[Bhattacharya-Nozaki-Ugajin-TT 12]

(3-1) Outline

Since the EE in a QFT is UV divergent, we would like to focus on the difference between the values of EE.

In other words, we will consider excited states and calculate:

$$\Delta S_A = S_A - S_A^{\mathrm{GroundState}}.$$

This is always finite and we will compare this entropy with the energy in A:

$$\Delta E_A = \int_A dx^d T_{tt}.$$

(3-2) Holographic Calculation

Consider an asymptotically AdSd+2 background

(= an excited state in CFTd+1):

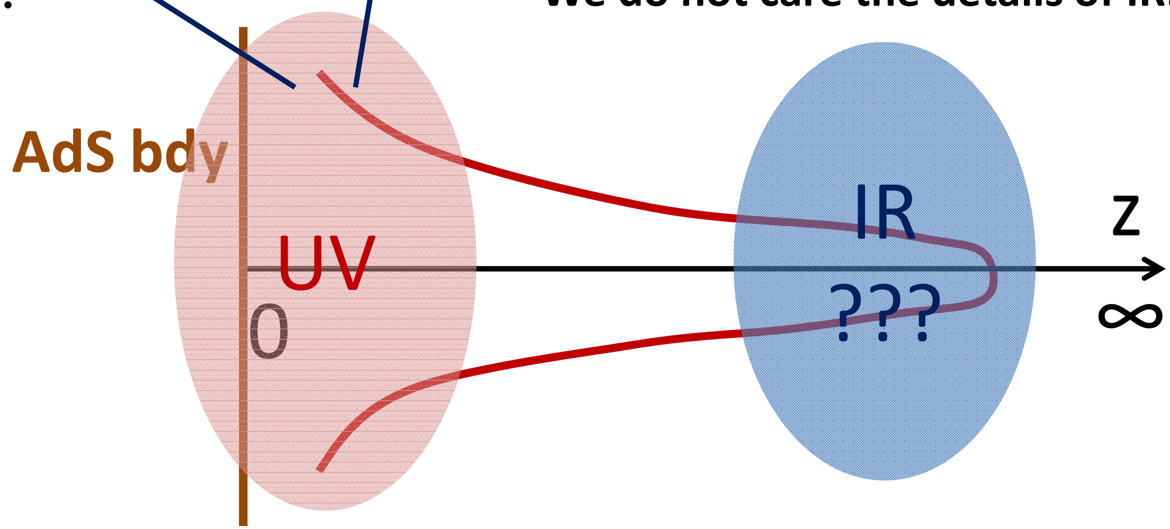
$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + \sum_{i=1}^{d} dx_i^2\right) ,$$

$$f(z) = 1 - mz^{d+1} + ..., \qquad g(z) = 1 + mz^{d+1} + ...$$

$$\Rightarrow \varepsilon = T_{tt} = \frac{dR^{d+1}m}{16\pi G_N}.$$

energy density

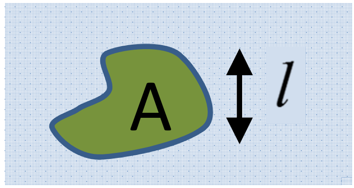
We do not care the details of IR.



Holographic Entanglement Entropy Analysis

If we assume a small subsystem A with the size l such that

$$ml^{d+1} << 1,$$



then we can show

$$T_{ent} \cdot \Delta S_A = \Delta E_A,$$

where $\Delta S_A = S_A - S_A^{\text{Pure AdS}}, \qquad \Delta E_A = \int_A dx^d T_{tt}.$

The `entanglement temperature' is given by

$$T_{ent} = \frac{c}{l}.$$

The constant c is universal in that it only depends on the shape of the subsystem A: $e.g. \quad c = \dfrac{d+2}{2\pi}$ when A = a round sphere.

Holographic Prediction

Consider an exited state in a CFT which has an approximate translational and rotational invariance.

If the size of the subsystem A (= l) is small enough such that

$$T_{tt} \cdot l^{d+1} << R^d / G_N \approx O(N^2),$$

then the following `1st law' like relation is satisfied:

$$T_{ent} \cdot \Delta S_A = \Delta E_A, \qquad T_{ent} \equiv \frac{c}{l} \ ,$$

Info. **Energy**

Note 1: The constant c depends only on the geometry of A.

Note 2: For more general critical points with the dynamical exponent **z**, we have $T_{ent} = c \cdot l^{-z}$.

More Progresses

If the rotational invariance is broken, ΔS_A is a linear combination Of not only Ttt but also other components of EM tensor.

[Pointed out in Guo-He-Tao 13, Allahbakhshi-Alishahiha-Naseh 13, Blanco-Casini-Hung-Myers 13;
This problems does not occur when A= a round ball]

However, we can generally show the following equivalence

$$\Delta S_A = \Delta H_A \, (= -\mathrm{Tr}[\delta\rho_A \cdot \log\rho_{\mathrm{A}}]),$$
$$(\Delta H_A \equiv -\log\rho_{\mathrm{A}}.)$$

when A=a round ball, as pointed out in

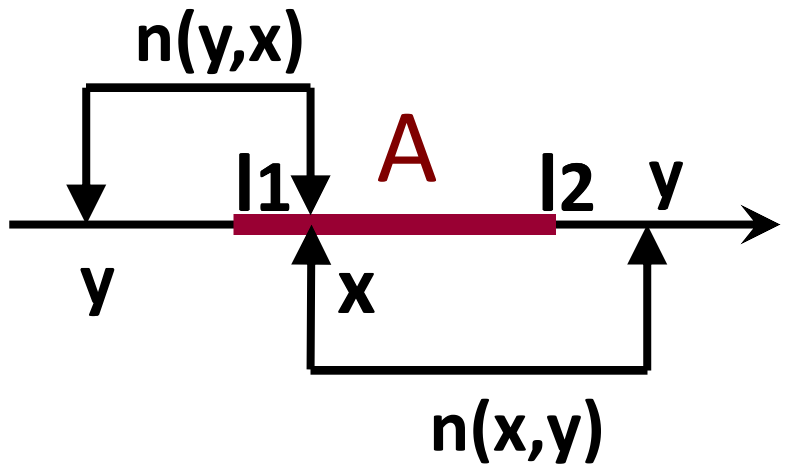
[Blanco-Casini-Hung-Myers 13, Wong-Klich-Pando Zayas-Vaman 13]

④ Entanglement Density [Nozaki-Numasawa-TT 13]

We focus on the EE for a pure state in 2d CFTs for simplicity.

Let us estimate the EE for the subsystem A (=an interval) by summing all of the EE between two infinitesimal regions:

$$S_A(l_1,l_2)=\int_{l_1}^{l_2}dx\left[\int_{-\infty}^{l_1}dy\ n(y,x)+\int_{l_2}^{\infty}dy\ n(x,y)\right].$$



$$S_A = \int_{l_1}^{l_2} dx \left[\int_{-\infty}^{l_1} dy \;\; n(y,x) + \int_{l_2}^{\infty} dy \;\; n(x,y) \right].$$

$$\frac{\partial S_A}{\partial l_2} = \int_{-\infty}^{l_1} dy \;\; n(y, l_2) + \int_{l_2}^{\infty} dy \;\; n(l_2, y) - \int_{l_1}^{l_2} dx n(x, l_2).$$

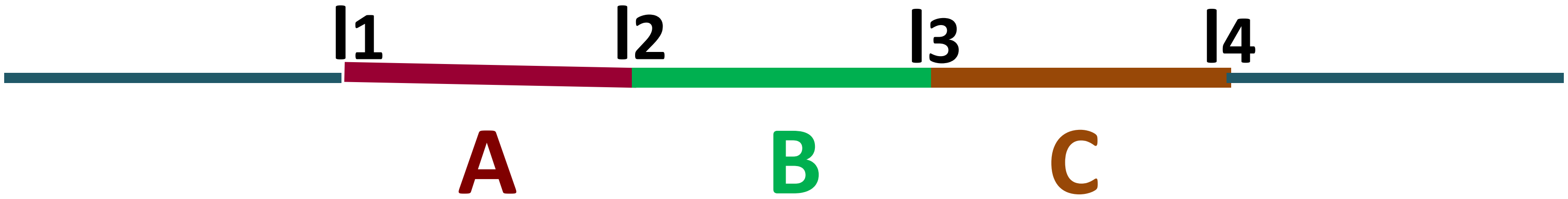
Therefore we find

$$\frac{\partial^2 S_A}{\partial l_1 \partial l_2} = 2n(l_1, l_2).$$

We will call $n(l_1, l_2)$ the *entanglement density*.

Clearly this quantity should be non-negative.
As we will see, this fact comes from the SSA.

Now, let us apply the SSA relation to the intervals:



$$S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B,$$

$$\Leftrightarrow S(l_1, l_3) + S(l_2, l_4) \geq S(l_1, l_4) + S(l_2, l_3),$$

$$\Leftrightarrow \frac{\partial^2 S(x, y)}{\partial x \partial y} \geq 0,$$

$$\Leftrightarrow n(x, y) \geq 0.$$

Note: This property is true for any excited states.

The 1st law and entanglement density

$$l = l_2 - l_1,$$
$$\xi = (l_1 + l_2)/2.$$

In 2d CFTs, we find the following property from the 1st law:

$$\Delta S_A(l,\xi,t) = \frac{\pi l^2}{3} T_{tt}(\xi,t) + O(l^3),$$

$$\Rightarrow \lim_{l\to 0} \int_{-\infty}^{\infty} d\xi \, \Delta S_A(l,\xi,t) = \text{conserved.}$$

In terms of the entanglement density, we obtain:

$$\lim_{l\to 0} \Delta n(l,\xi,t) = -\frac{\pi}{3} T_{tt}(\xi,t).$$

Entanglement ⇔ Energy

$$\Rightarrow \int_{-\infty}^{\infty} d\xi \, \Delta n(0,\xi,t) = \text{conserved.}$$

Full conservation law of entanglement density

Claim: $\int dl_1 dl_2 \ \Delta n(l_1, l_2, t) = \int dl d\xi \ \Delta n(l_1, l_2, t) = 0.$

[Proof] We compactify the space on a circle with periodicity L.

Then we find $S_A = S_B \ \ \Rightarrow \ \ \Delta n(l, \xi, t) = \Delta n(L-l, L-\xi, t).$

$$\int_0^L d\xi \int_0^{L/2} dl \ \Delta n(l, \xi, t) = \frac{1}{2}\int_0^L dl_1 \int_{l_2}^{l_1+L/2} dl_2 \ \frac{\partial^2 \Delta S_A(l_1, l_2)}{\partial l_1 \partial l_2}$$

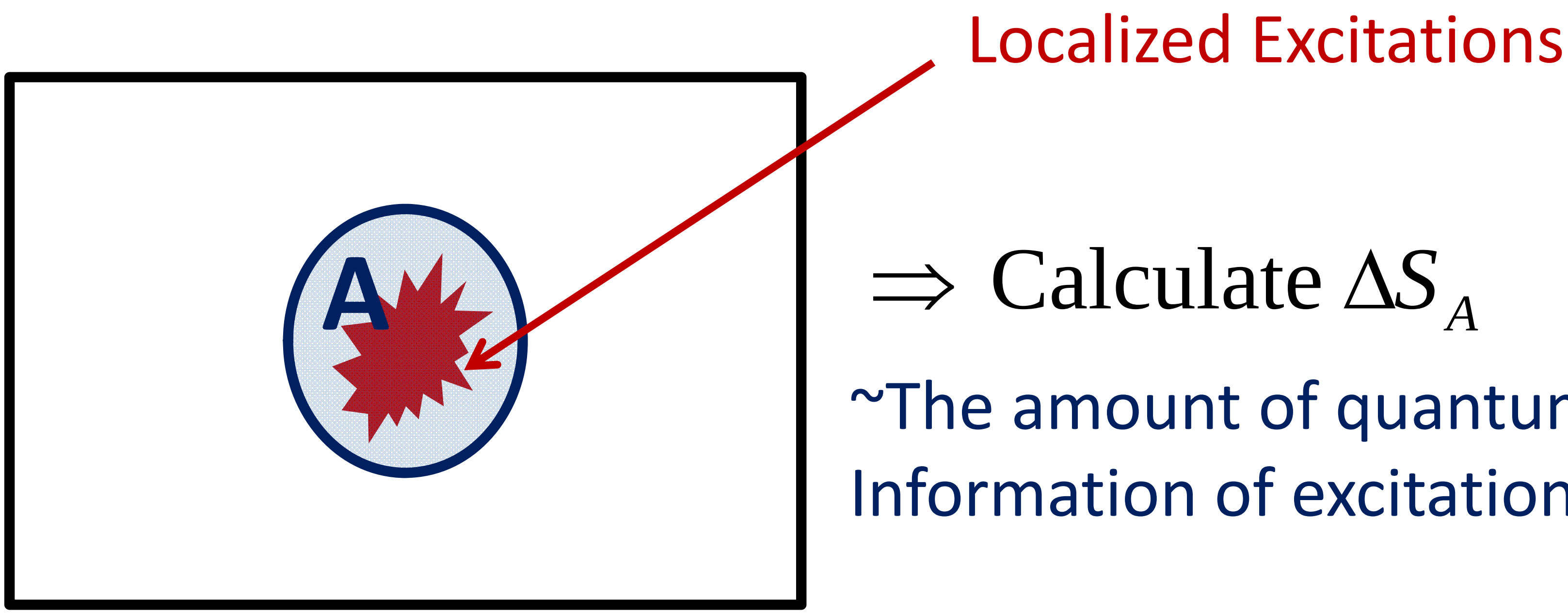
$$= \frac{1}{2}\int_0^L dl_1 \left[\underbrace{\frac{\partial}{\partial x}\Delta S_A(x, l_1 + L/2)}_{=0 \text{ because SA=SB.}} - \underbrace{\frac{\partial}{\partial x}\Delta S_A(x, l_1)}_{=0 \text{ because } 1^{st} \text{ law.}} \right]_{x=l_1} = 0.$$

⑤ Holographic Local Quenches and EE

[Nozaki-Numasawa-TT 13]

(5-1) Outline

We would like to study **the relation between the EE and energy** when we excite the system **locally**.



$\Rightarrow$ Calculate ΔS_A

~The amount of quantum Information of excitations

(5-2) Local Quantum Quenches

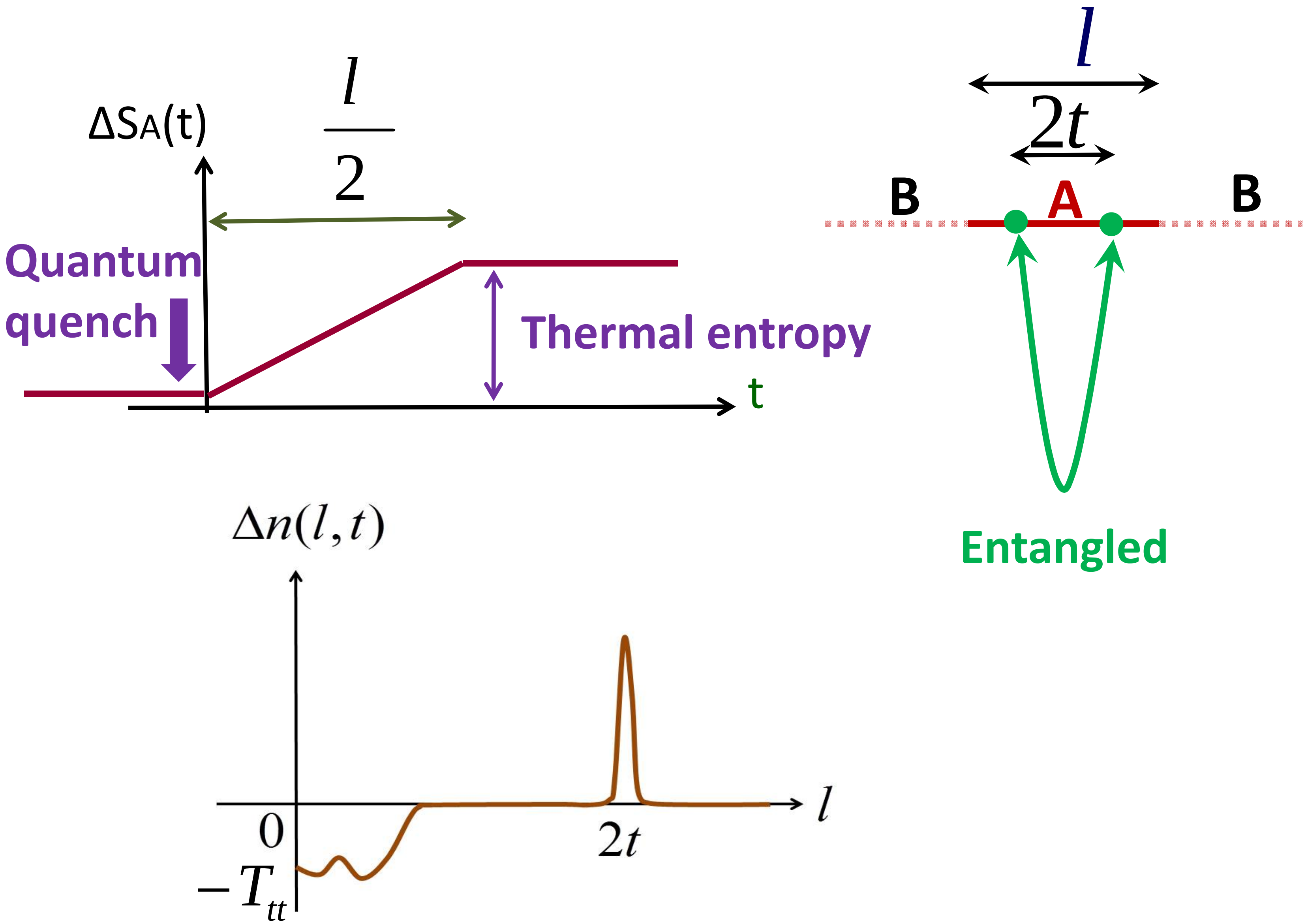
<u>Global Quench</u> (Global Excitations)

The thermalization under a sudden change of Hamiltonian is called **quantum quench** and has been intensively studied in condensed matter physics. [Calabrese-Cardy 05-10]

This corresponds to the time-dependent setup of AdS/CFT:

Black hole formation in AdS ⇔ Thermalization in CFT

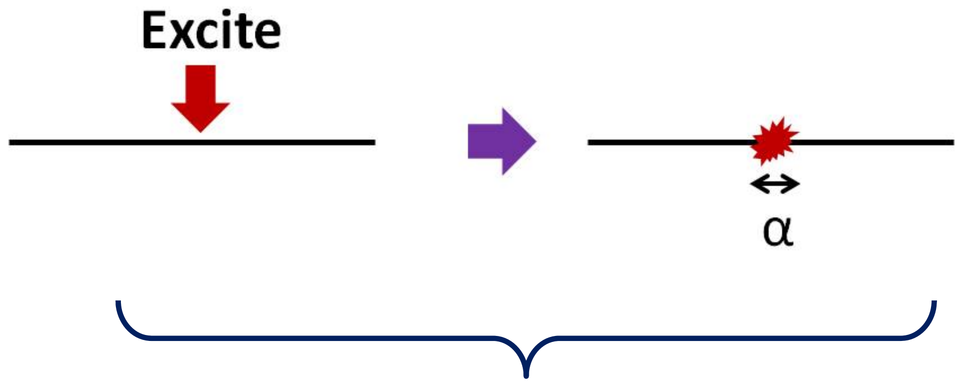
EE in Global Quenches and Entanglement Density in 2d CFTs



Local Quenches (Local excitations)

Original Local Quench:

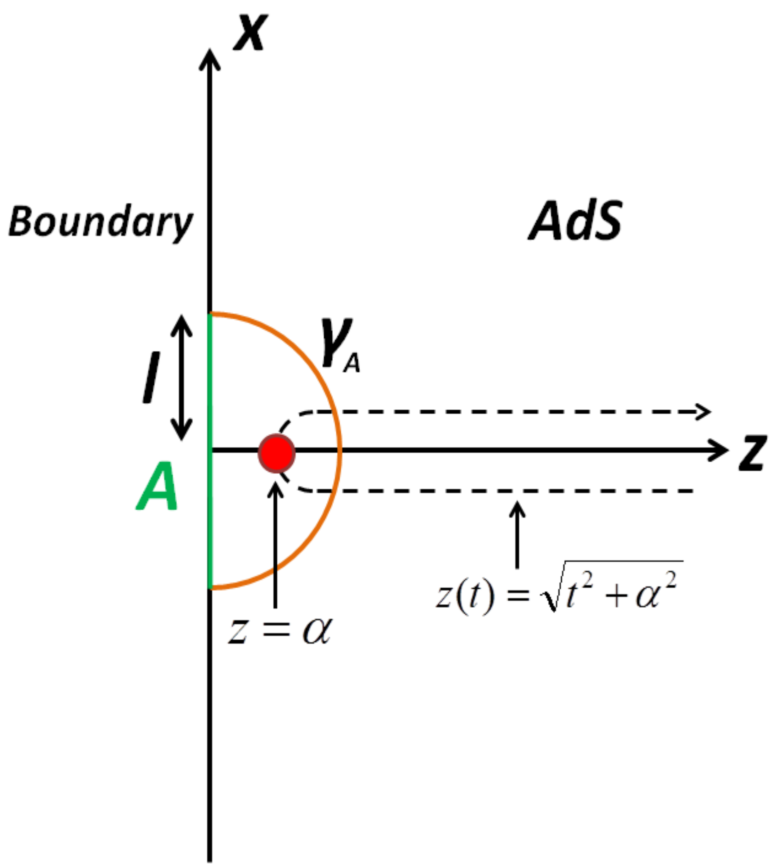
Generalization
(Local Excitations):



We consider this kind of example using the AdS/CFT.

(5-3) Holographic Local Quenches

We argue that a simple model of holographic local quench is given by a free falling particle (mass **m**) in AdSd+2.



$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}$$

Trajectory: $z(t) = \sqrt{t^2 + \alpha^2}$.

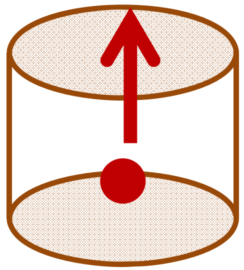
α~ the size of localized excitations

Construction of Back-reacted Solutions

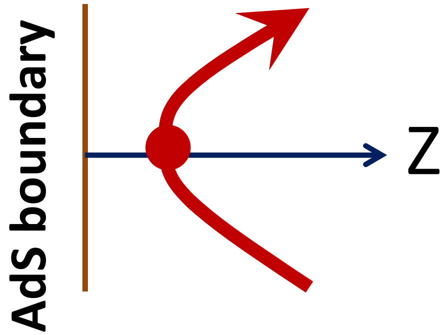
We use the method first noticed by [Horowitz-Itzhaki 99].

Start with the global AdS BH:

$$ds^2 = -f(r)d\tau^2 + \frac{R^2}{f(r)}dr^2 + r^2 d\Omega_d^2, \qquad f(r) = r^2 + R^2 - M/r^{d-2}.$$



Coordinate transformation



$$\sqrt{R^2 + r^2}\cdot\cos\tau = \frac{R^2 e^{\beta} + e^{-\beta}(z^2 + x^2 - t^2)}{2z},$$

$$\sqrt{R^2 + r^2}\cdot\sin\tau = \frac{Rt}{z},$$

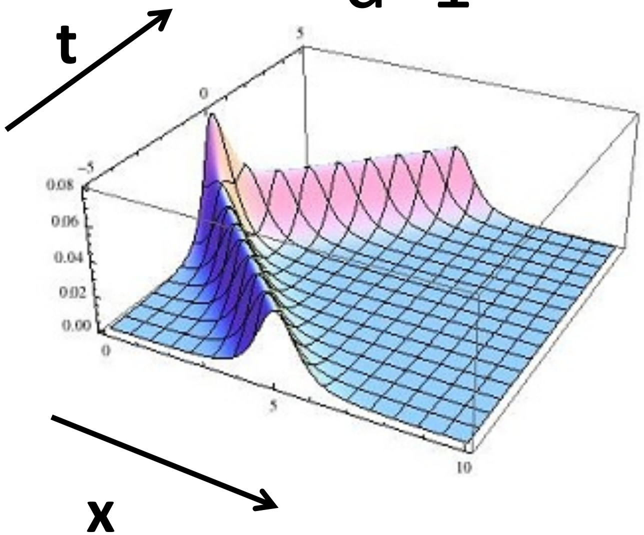
$$r\Omega_i = \frac{Rx_i}{z} \quad (i = 1,2,\ldots,d),$$

$$r\Omega_{d+1} = \frac{-R^2 e^{\beta} + e^{-\beta}(z^2 + x^2 - t^2)}{2z}.$$

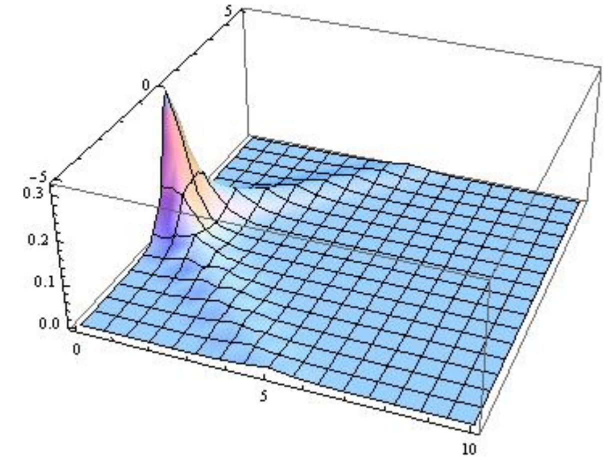
Asymptotic AdS space (M=0 → Pure AdS)

Energy density via AdS/CFT

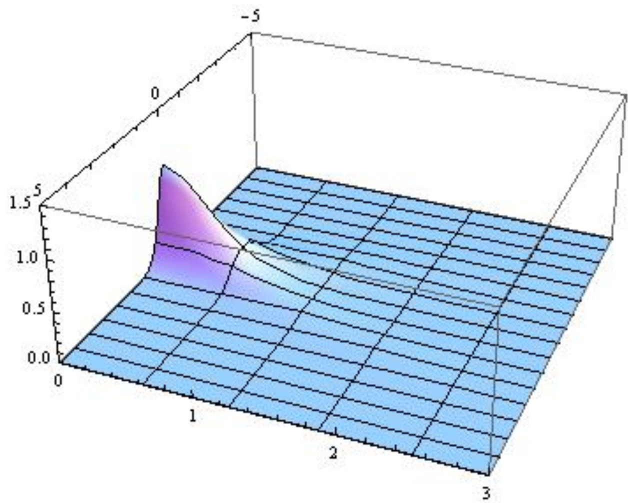
d=1



d=2



d=3



Analysis of HEE

(i) Perturbation theory w.r.t M → Any dim. d.

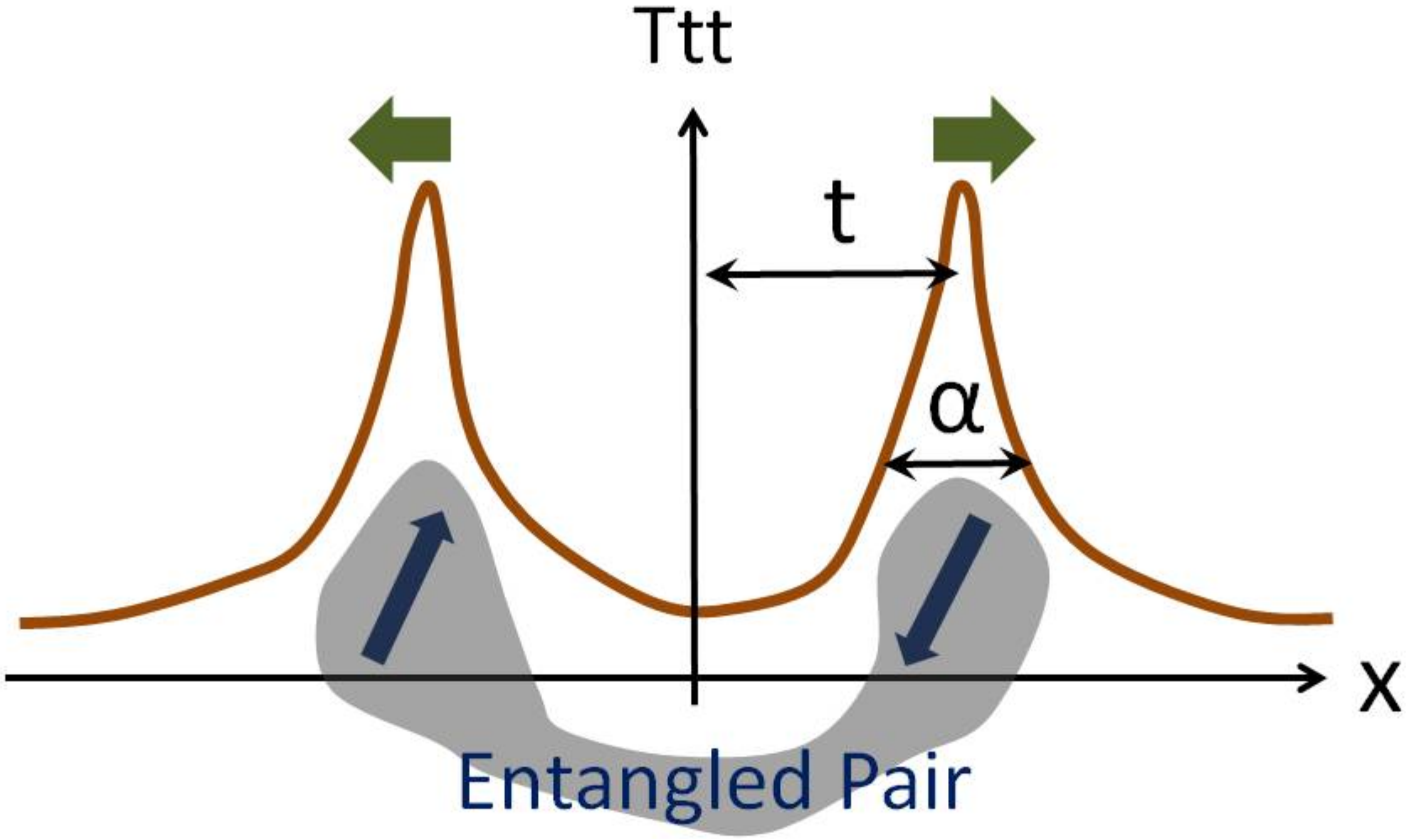
→ We can confirm the `1st law' in the small size limit.

(ii) Exact analysis → So far only for AdS3 (d=1).

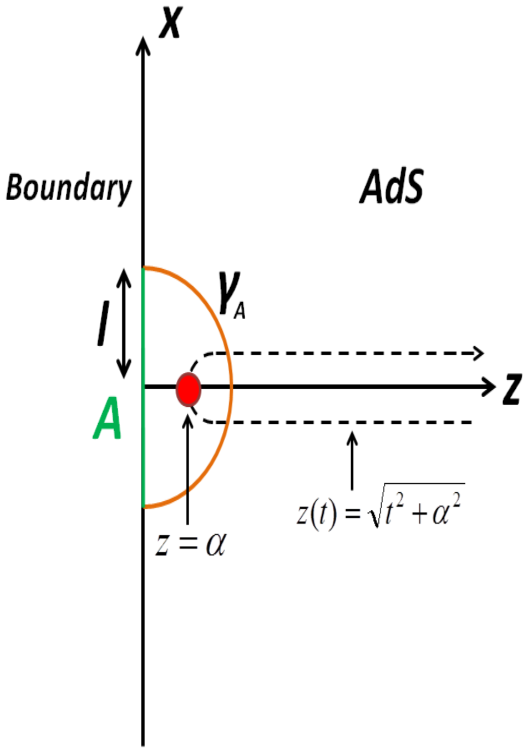
$$E_{particle} = \frac{R \cdot m}{\alpha},$$

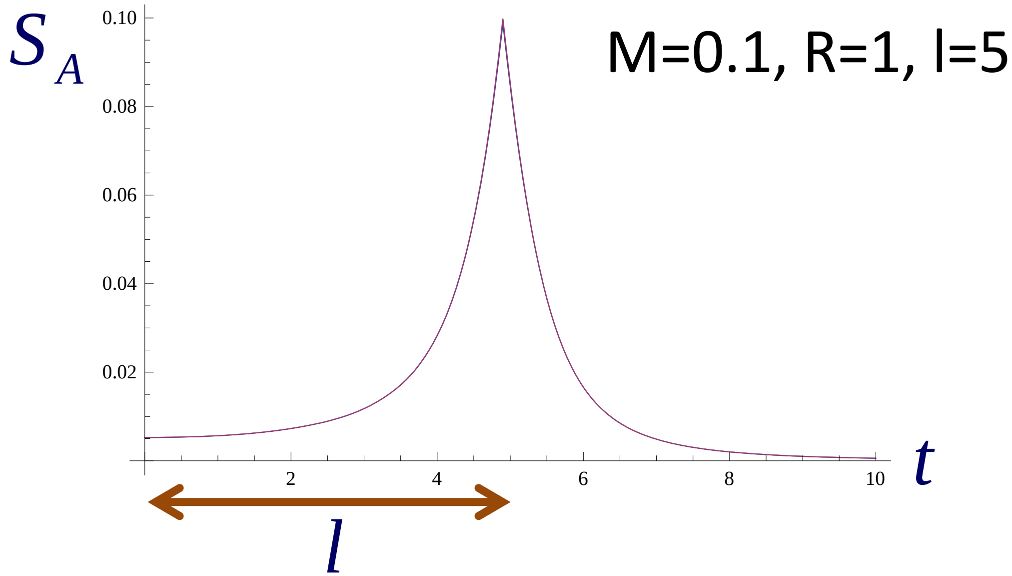
$$M = mG_N R^2 \cdot \frac{8\Gamma(d/2)}{(d-1)\pi^{d/2-1}}.$$

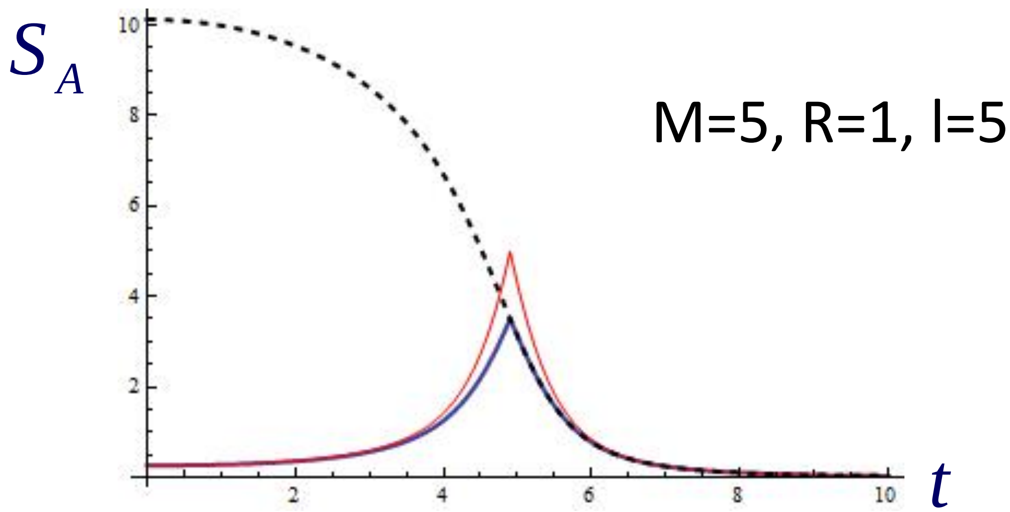
Basic picture



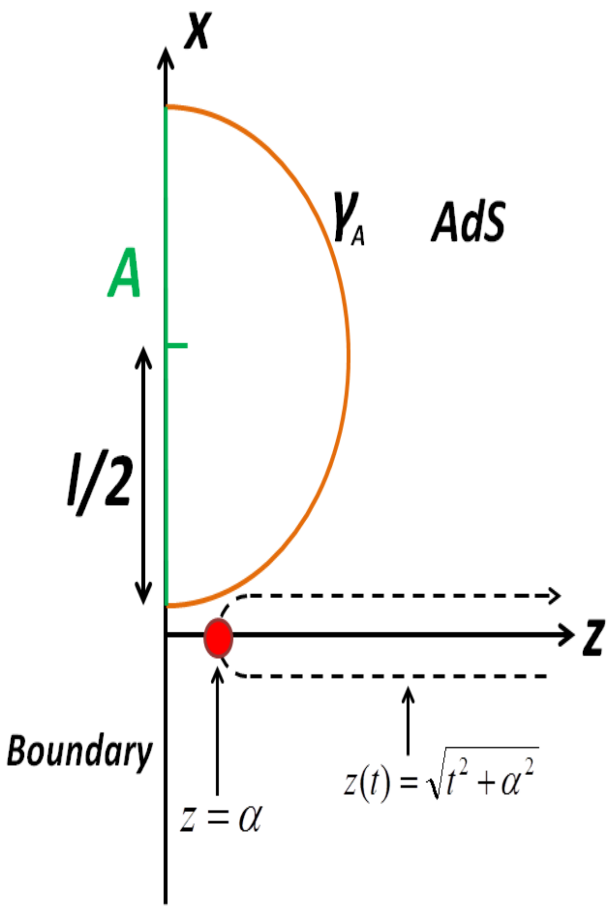
Time Evolution of HEE in AdS3/CFT2 (Case 1)

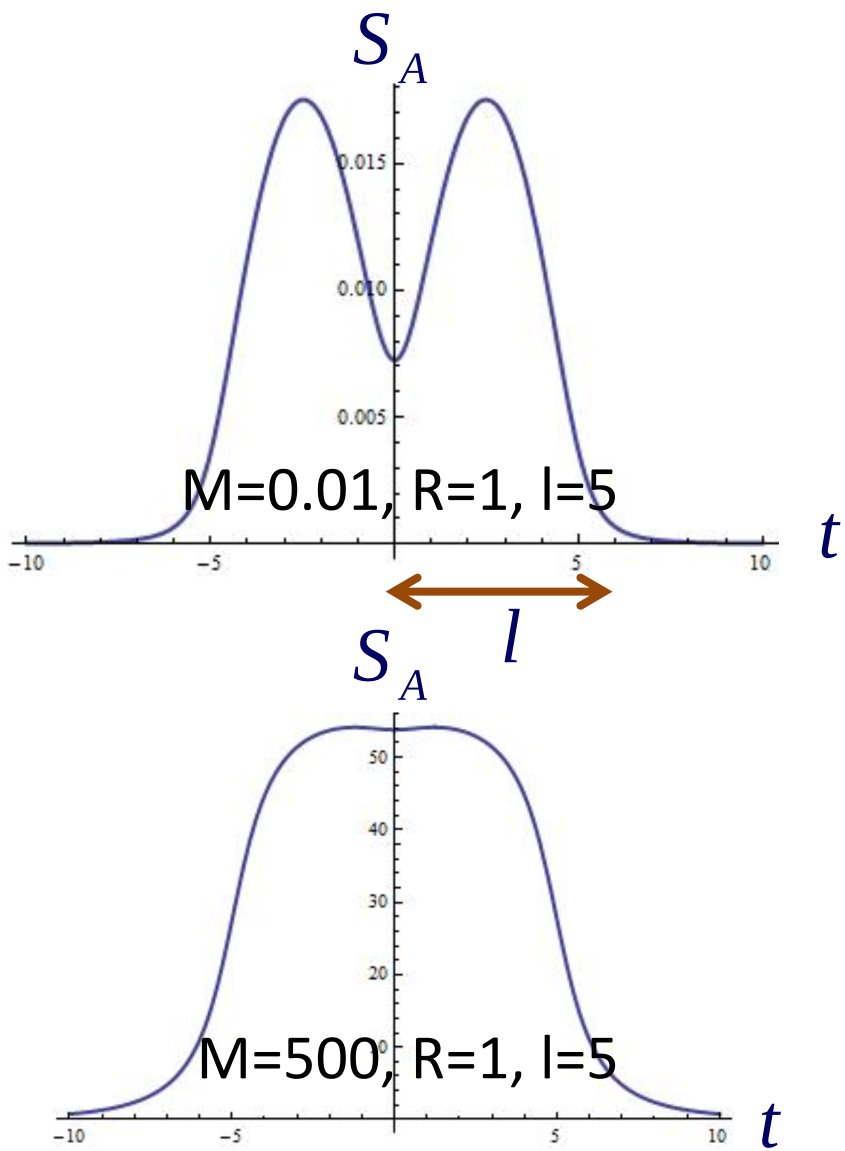






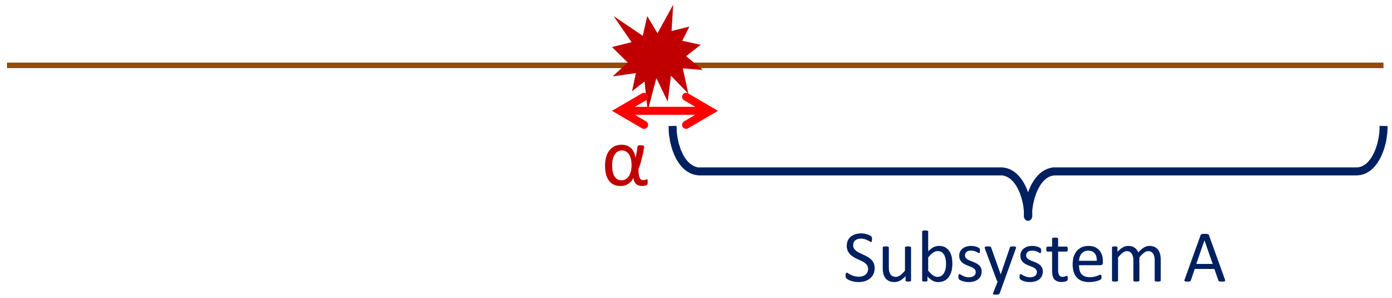
Time Evolution of HEE in AdS3/CFT2 (Case 2)





If we take the limit l>>t>>α in the case 2, we find

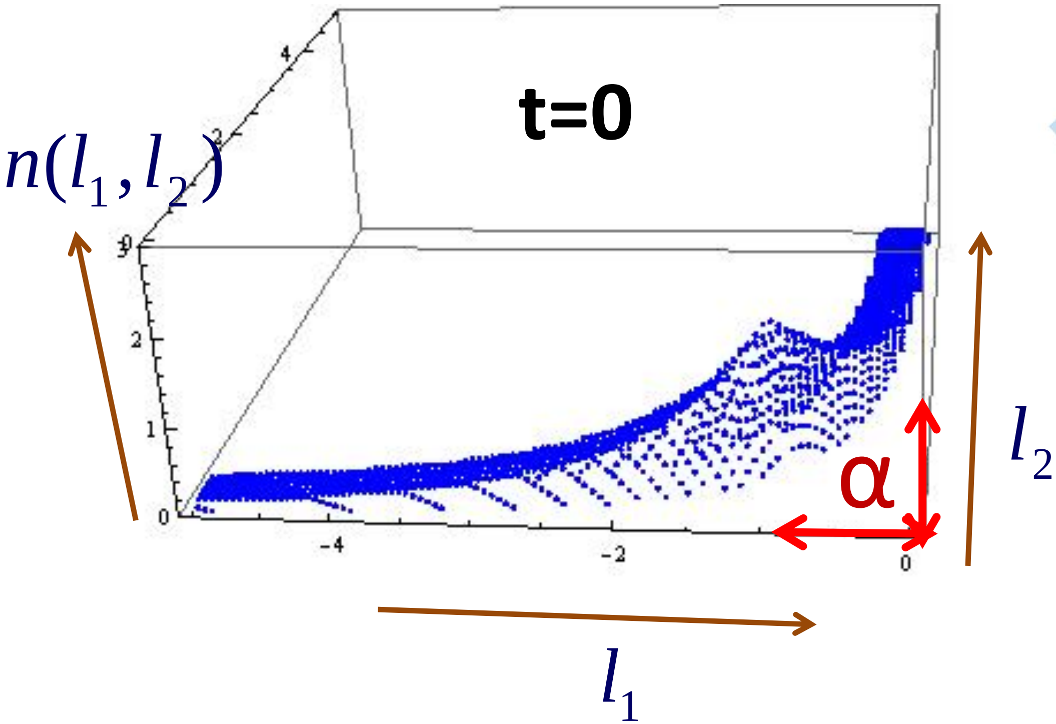
$$S_A = \frac{c}{6}\log\frac{t}{\alpha} + \frac{c}{3}\log\frac{l}{a} + \text{const.}$$

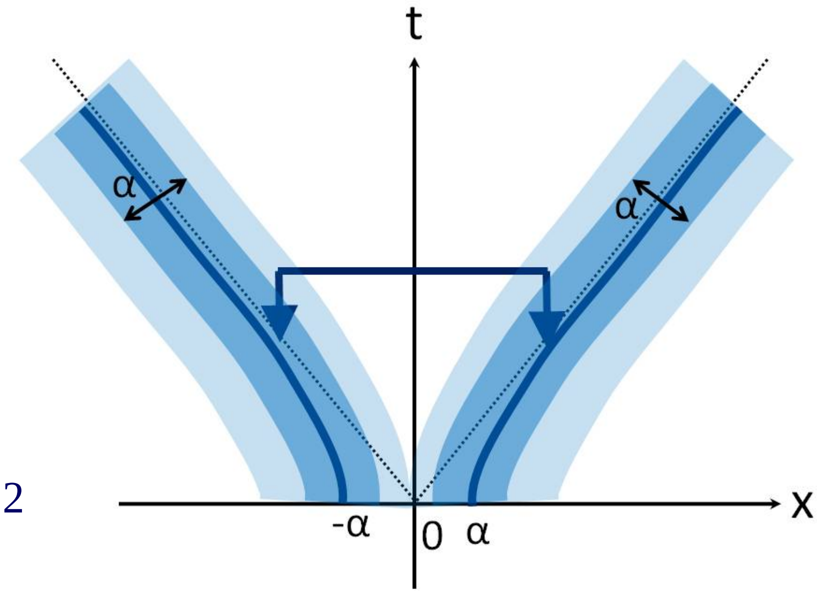


cf. 2d CFT result for local quenches by the joint procedure

$$S_A = \frac{c}{3}\log\frac{t}{a} + \frac{c}{6}\log\frac{l}{\alpha} + \text{const.}$$

Entanglement Density





(5-4) Quantum Information and Energy

We can estimate the maximal value of ΔS_A when we varies l and t. For a small M, we find for any dim.

1<<Δ<<c

$$\Delta S^{\max} = C_d \cdot mR \approx C_d \cdot \Delta.$$

$$(C_1 = 2,\ C_2 = \pi / 2,\ C_3 = 4)$$

$$\Rightarrow \ \#\mathrm{Microstates\ \ of}$$

$$\mathrm{Localized\,Excitations} \sim \mathrm{e}^{\mathrm{C_s}\cdot\Delta}.$$

Exact result in AdS3

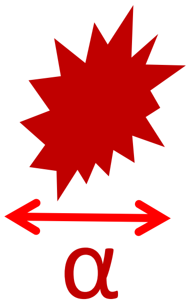
$$\Delta S^{\max} = \frac{c}{3}\log\left[\frac{R}{\sqrt{R^2 - M}}\sin\left(\frac{\pi\sqrt{R^2 - M}}{2R}\right)\right] \approx 2\Delta.$$

Summary:

The amount of quantum information

of Localized excitations **~ E ▪ α**

(`fire ball' of gluons) Energy Size



Cf. Bekenstein Bound

⑥ What is the Einstein equation for HEE ?

[Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharya-TT 13]

The `1st law-like' relation appears only when the size of A is small.

➡ What can we say if the size of subsystem is not small ?

This is related to a basic question in the AdS/CFT:

Gravity on AdS	⟺ AdS/CFT	**CFT**
$g_{\mu\nu}(t,\vec{x},z)$		$S_A(t,\vec{x},l)$
Einstein eq.	⟺	What ?

Below we study **a HEE counterpart of perturbative Einstein eq.** assuming small excitations of a CFT.

(see also Lashkari-McDermott-Raamsdonk 13 for a different approach)

How can we compute ΔS_A ?

We can calculate the area perturbation of the minimal surface from the minimal surface before the metric perturbation:

$$\Delta \text{Area} = \frac{1}{2} \int_{\gamma_A^{(0)}} d\xi^d \sqrt{G^{(0)}} \, \text{Tr}\left[G^{(1)} \left(G^{(0)} \right)^{-1} \right] ,$$

where

$\gamma_A^{(0)}$: minimal surface before the metric perturbati on.

γ_A : minimal surface after the metric perturbati on.

$G^{(0)}$ is the induced metric on $\gamma_A^{(0)}$.

$G^{(1)}$ is the pertabativ e contributi on to the induced metric on γ_A.

AdS4/CFT3

Let A be a round ball with radius l. Its center is situated at $(t, \vec{x})$.

The perturbative Einstein equation is rewritten as follows

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

Kinetic term **C.C.** **Matter field contributions** $\phi \leftrightarrow O$

$$\left(\partial_l^2 - \partial_l - \partial_{\vec{x}}^2 - \frac{3}{l^2} \right) \Delta S_A(t, \vec{x}, l) = \langle O \rangle \langle O \rangle$$

Note: There are **no time derivatives**.

$\Rightarrow$ This gives a **constraint** for HEE at a fixed time.

The time evolution is determined by IR bdy conditions.

AdS4 Schwarzschild BH

When the size of the subsystem is very large , we find

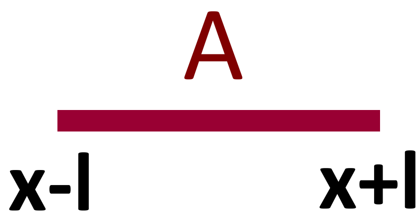
$$\left(\partial_l^2 - \partial_l - \partial_{\vec{x}}^2\right) \Delta S_A(t, \vec{x}, l) = \langle O \rangle \langle O \rangle.$$

This coincides with the holographic result for flat space.

AdS3/CFT2

In AdS3 gravity, we have two constraints:

$$\left(\partial_x^2 - \partial_t^2\right) \Delta S_A(t, x, l) = \langle O \rangle \langle O \rangle,$$
$$\left(\partial_l^2 - \partial_t^2 - \frac{2}{l^2}\right) \Delta S_A(t, x, l) = \langle O \rangle \langle O \rangle.$$

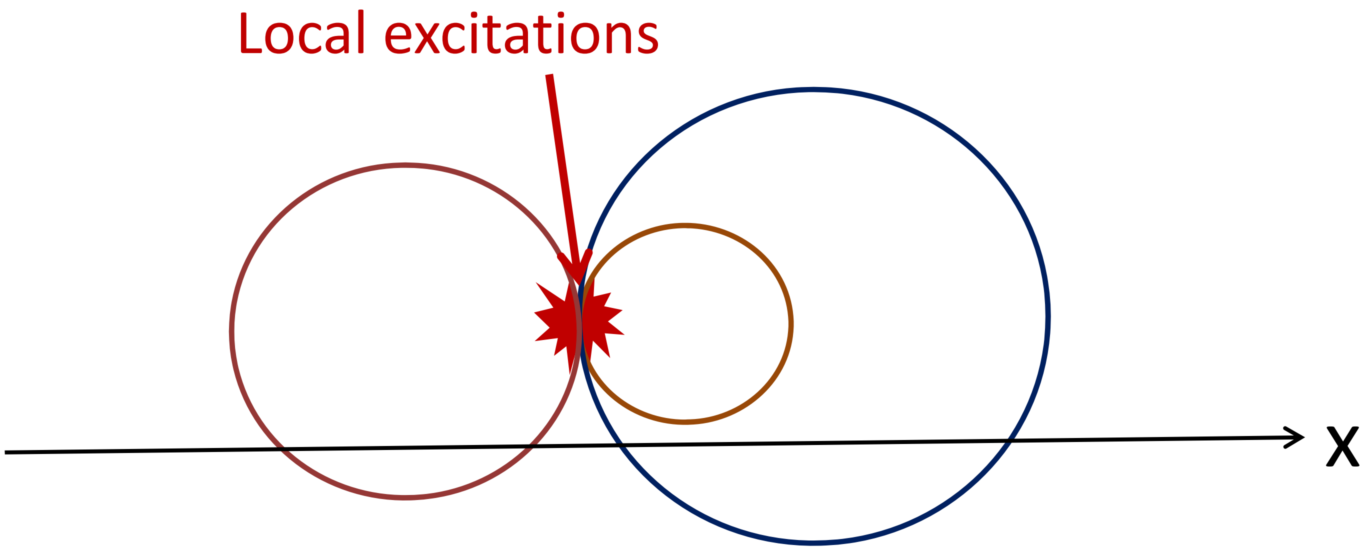


[Confirmed in CFT2 by Wong-Klich-Pando Zayas-Vaman 13]

Intuitive interpretation of these constraints

Hyperbolic PDE: $\left(\partial_l^2 - \partial_{\vec{x}}^2\right)\Delta S_A(t,\vec{x},l) \approx 0$

$$\Rightarrow \Delta S_A \propto f(l - |x|) + g(l + |x|).$$



ΔS_A becomes non-trivial only when ∂A intersects with the excited region $\Leftrightarrow\ l = \pm |x|$.

⑦ Conclusions

- The amount of quantum information ~ energy in CFTs.

  → Analogous to the thermodynamical first law.

- A simple holographic model of local quenches and its HEE.

- A counterpart of Einstein equation for entanglement entropy.

→ Can we employ these to study the holography
in other spacetimes (e.g. de-Sitter spaces) ?

→ Can we formulate quantum gravity in terms of HEE ?