Zero-Error Classical Channel Capacity and Simulation Cost Assisted by Quantum Non-Signalling Correlations

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Motivation 1: General Channel Simulation Problem

Given two communication channels from Alice to Bob, when one channel can be used to exactly (or approximately) realize another one?

**Assumptions:**

1. Alice and Bob can access certain class of free resources $\Omega$: a. Local Operations (LO); b. Shared Randomness (SR); c. Shared Entanglement (E); d. Non-Signalling Correlations (NS); or e. Feedback (F).
2. Available resources themselves are not sufficient for the simulation.

**Zero-error (Exactly):**

$$\mathcal{M} \prec_\Omega \mathcal{N}$$

**Small-error (Approximately):**

$$\mathcal{M} \prec_\Omega^\epsilon \mathcal{N}, \ \epsilon > 0$$
Motivation 1: Channel Exchange Rate

Zero-Error Exchange Rate:

\[ R_{0,\Omega}(\mathcal{M},\mathcal{N}) = \inf \left\{ \frac{n}{m} : \mathcal{M}^\otimes m \lesssim_\Omega \mathcal{N}^\otimes n \right\} \]

Intuition: Optimal uses of one channel required to exactly realize another one.

Small-Error Exchange Rate:

\[ R_{\Omega}(\mathcal{M},\mathcal{N}) = \lim_{\epsilon \to 0^+} \inf \left\{ \frac{n}{m} : \mathcal{M}^\otimes m \lesssim_\Omega \mathcal{N}^\otimes n \right\} \]

The Channel Exchange Rate has embedded two fundamental quantities in information theory:

1. **Communication Capacity**: the number of noiseless bits (or qubits) can be transmitted via one use of the given channel,

\[ C_\Omega(\mathcal{N}) = R_\Omega(\mathcal{I}_2,\mathcal{N})^{-1} \]

2. **Simulation Cost**: the number of noiseless bits (or qubits) are required to realize one use of the given channel,

\[ G_\Omega(\mathcal{N}) = R_\Omega(\mathcal{N},\mathcal{I}_2) \]
Motivation 1: Small-Error Case

Completely solved for quantum channels with shared entanglement (or even more general non-signalling correlations) (Bennett, Shor, Smolin, Thapliyal 2002; Bennett, Devetak, Harrow, Shor, Winter 2009):

\[ R_E(\mathcal{M}, \mathcal{N}) = \frac{C_E(\mathcal{M})}{C_E(\mathcal{N})} \]

Remarks:

1. The case of classical channels holds for shared randomness, and essentially follows from the Shannon channel capacity theorem (Shannon 1948) and the reverse Shannon theorem (BSST2002).

2. The entanglement-assisted classical capacity uniquely determines the channel exchange rate.

3. The simulation process is reversible in the presence of entanglement (or non-signalling correlations):

\[ R_E(\mathcal{M}, \mathcal{N}) = R_E(\mathcal{N}, \mathcal{M})^{-1} \]
Motivation 2: Communication Channels as Graphs

(Shannon 1956)

Classical Channels → Bipartite Graphs → (Confusability) Graphs

Ordinary (small-error) communication
Zero-error communication assisted by feedback or non-signalling correlations
Zero-error communication (assisted by entanglement)

\[ C(N) = \max_{p(x)} I(X : Y) \]
\[ C_{0,F}(N) = C_{0,NS}(N) = \alpha^*(\Gamma) \]
\[ C_0(N) = C_0(G) = \sup_{k>1} \frac{\log_2 \alpha(G^k)}{k} \]
\[ C_0(G) \leq C_{0,E}(G) \leq \log_2 \theta(G) \]

1. Bipartite graphs and confusability graphs can be regarded as different level of abstractions of communication channels;
2. All information theoretic aspects of communication channels can be studied for graphs.
Motivation 2: Communication Channels as Graphs

Quantum Channels $\Rightarrow$ Non-commutative Bipartite Graphs $\Rightarrow$ Non-commutative Graphs

$$N = \sum_k E_k \cdot E_k^\dagger,$$

$$\sum_k E_k^\dagger E_k = I.$$  

$$K = \text{span}\{E_k\}. \
I \in K^\dagger K.$$  

$$S = \text{span}\{E_k^\dagger E_j\}.$$  

1) $I \in S$; 2) $S^\dagger = S$.

Approximate communication $\Rightarrow$ zero-error communication with feedback or non-signalling assistance $\Rightarrow$ zero-error communication without or with assistance of entanglement

1. Quantum channels, non-commutative bipartite graphs and non-commutative graphs represent different level of abstractions for communication channels.

2. All information theoretic aspects of communication channels can be studied for these graphs.

$K$ and $S$ could also be regarded as sets of quantum channels with the same Kraus operator spaces and non-commutative graphs, respectively.
Review of Zero-Error Case

Very difficult, and only special cases of communication and simulation have been studied...

1. Shannon formulated the assisted case as a problem to determine the independence number of a graph (NP-hard) (Shannon 1956).

2. Lovasz introduced a celebrated function name after him to upper bound the capacity of a graph (Lovasz 1979).

3. Entanglement can increase the classical zero-error classical capacity of classical channels (Cubitt, Leung, Matthews, Winter 2009; Leung et al 2010).

4. A quantum version of Lovasz theta function for non-commutative graphs, and establishes the connection to the entanglement-assisted zero-error capacity (Duan, Severini, Winter 2010; see also Beigi 2010 for the classical case).

5. Cubitt et al solved the case of classical channels assisted with classical non-signalling correlations both for the communication and simulation problems (Cubitt, Leung, Matthews, and Winter 2011).
Overview of Main Results

We imitate the approach of Cubitt et al to study zero-error classical communication/simulation cost of a quantum channel assisted by a new class of resources: *quantum non-signalling correlations*.

1. One-shot zero-error communication and simulation problems both can be formulated into rather simple semi-definite programmings (SDPs), and only depend on the non-commutative bipartite graph (Kraus operator space of the channel) *(Theorems 1, 2, and 3)*

2. The (asymptotic) zero-error classical simulation cost of classical-quantum (cq) channels is proven to be additive and equivalent to the one-shot simulation cost *(Theorem 4)*.

3. The (asymptotic) zero-error classical capacity of cq-channels can be obtained as the solution to a rather simple SDP proposed by Aram Harrow *(Theorem 5)*. “A semi-definite fractional packing number”

4. An operational interpretation of the celebrated Lovasz theta function as the zero-error classical capacity of the graph assisted by quantum non-signalling correlations *(Theorem 6)*.
Quantum Non-Signalling Correlations

Definition: A two-input and two-output quantum channel shared between Alice and Bob such that A and B cannot use the channel to communicate classical information. Assume the map is given by

$$\Pi : \mathcal{L}(A_i) \otimes \mathcal{L}(B_i) \rightarrow \mathcal{L}(A_o) \otimes \mathcal{L}(B_o)$$

The Choi matrix is

$$\Omega_{A_i^i A_o B_i^i B_o} = (\text{id}_{A_i^i} \otimes \text{id}_{B_i^i} \otimes \Pi)(\Phi_{A_i A_i^i} \otimes \Phi_{B_i B_i^i})$$

where $$\Phi_{A_i A_i^i} = |\Phi_{A_i A_i^i} \rangle \langle \Phi_{A_i A_i^i}|$$ and $$|\Phi_{A_i A_i^i} \rangle = \sum_k |k_{A_i} \rangle |k_{A_i^i} \rangle$$

SDP constraints for quantum non-signalling:

$$\Omega \geq 0, \quad (CP)$$

$$\text{Tr}_{A_o B_o} \Omega = \mathbf{1}_{A_o B_o}, \quad (TP)$$

$$\text{Tr}_{A_i A_i^i} \Omega X_{A_i}^T = 0, \quad \forall \text{Tr} X = 0, \quad (A \not\leftrightarrow B)$$

$$\text{Tr}_{B_o B_i^i} \Omega Y_{B_i}^T = 0, \quad \forall \text{Tr} Y = 0. \quad (B \not\leftrightarrow A)$$
Composing Quantum Channels and Quantum Non-Signalling Correlations

\[ \mathcal{M}^{A_i \rightarrow B_o} = \text{Tr}_{A_o, B_i} \Pi^{A_i \otimes B_i \rightarrow A_o \otimes B_o} \circ \mathcal{N}^{A_o \rightarrow B_i} \]

\[ \text{FIG. 2. Two different partitions of } \Pi \]

**Partition (a)**  
\[ A_i \rightarrow A_o | B_i \rightarrow B_o : \mathcal{M}^{A_i \rightarrow B_o} = \sum_k \mathcal{B}_k^{B_i \rightarrow B_o} \circ \mathcal{N}^{A_o \rightarrow B_i} \circ \mathcal{A}_k^{A_i \rightarrow A_o}, \Pi = \sum_k \mathcal{A}_k \otimes \mathcal{B}_k \]

**Partition (b)**  
\[ A_i \rightarrow B_o | B_i \rightarrow A_o : \mathcal{M}^{A_i \rightarrow B_o} = \sum_k \text{Tr}(\mathcal{F}_k^{B_i \rightarrow A_o} \circ \mathcal{N}^{A_o \rightarrow B_i}) \mathcal{E}_k^{A_i \rightarrow B_o}, \Pi = \sum_k \mathcal{E}_k \otimes \mathcal{F}_k \]

Partition (b) is useful when the communication from A to B is under consideration.
Structures of Quantum Non-Signalling Correlations

**Proposition 1** (bipartition $A_i A_o : B_i B_o$): A bipartite CPTP map $\Pi : \mathcal{L}(A_i) \otimes \mathcal{L}(B_i) \to \mathcal{L}(A_o) \otimes \mathcal{L}(B_o)$ is a non-signalling correlation between A and B if and only if $\Pi = \sum_k \lambda_k A_k \otimes B_k$, $\sum_k \lambda_k = 1$, where $A_k : \mathcal{L}(A_i) \to \mathcal{L}(A_o)$ and $B_k : \mathcal{L}(B_i) \to \mathcal{L}(B_o)$ are CPTP maps, and $\lambda_k$ are real numbers. Intuitively, any non-signalling correlation $\Pi$ between A and B can be written as a real affine combination of product CPTP maps $A_k$ and $B_k$.

**Proposition 2** (bipartition $A_i B_o : B_i A_o$): $\Pi$ is totally non-signalling (in the sense that none of $A_i$ and $B_i$ can send information to any of $A_o$ and $B_o$), if and only if $\Pi = \sum_k \mu_k E_k \otimes F_k$, where 1) $E_k : \mathcal{L}(A_i) \to \mathcal{L}(B_o)$ and $F_k : \mathcal{L}(B_i) \to \mathcal{L}(A_o)$ are CPTP maps; 2) $\mu_k$ are real numbers such that $\sum_k \mu_k = 1$; and 3) both $\sum_k \mu_k E_k$ and $\sum_k \mu_k F_k$ are constant maps.

**Proposition 3** (Eggeling, Schlingermann, and Werner 2001; Piani and M.P.R. Horodecki 2006): $\Pi$ is $B \leftrightarrow A$ iff $\Pi = G \circ F$ for two CPTP maps $F : \mathcal{L}(A_i) \to \mathcal{L}(A_o \otimes R)$ and $G : \mathcal{L}(B_i \otimes R) \to \mathcal{L}(B_o)$, where $R$ is an internal memory.
One-shot Zero-Error Classical Capacity Assisted by QNS

Goal: The maximum number of messages one can send without error using the given channel $\mathcal{N}$ assisted by QNS.

$$\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger, \quad K = \text{span}\{E_k\}, \quad J_{AB} = (\text{id}_A \otimes \mathcal{N}) \Phi_{AA'}$$

Theorem 1. The one-shot zero-error classical capacity assisted by QNS is given by the integer part of the following SDP:

$$\Upsilon(K) = \max \text{Tr} S_A \text{ s.t. } 0 \leq U_{AB} \leq S_A \otimes 1_B,$$

$$\text{Tr}_A U_{AB} = 1_B,$$

$$\text{Tr} P_{AB} (S_A \otimes 1_B - U_{AB}) = 0.$$

Here $P_{AB}$ is the projection on the support of $J_{AB}$.

Remarks:

1. The capacity only depends on the Kraus operator space or the support of Choi matrix. This justifies the use of terminology “non-commutative bipartite graph” for $K$.

2. The SDP formulation indicates the one-shot zero-error capacity assisted by QNS is efficiently computable. This is a remarkable fact comparing to the entanglement-assisted case.
Proof outline of Theorem 1

Task: Determine the maximum integer $M$ such that $\mathcal{I}_M \prec_{QNS} \mathcal{N}$, or

$$\mathcal{I}_{M_i \rightarrow M_o}^M = \text{Tr}_{AB} \Pi_{M_i \otimes B \rightarrow A \otimes M_o} \circ \mathcal{N}^{A \rightarrow B}$$

Crucial property: the classical nature and permutation symmetry of $\mathcal{I}_M$.

Key Steps:

1. Dephasing and taking group average over classical registers $M_{i/o}$, we can choose the optimal QNS as the following special form (refer to Partition (b) in Fig. 2)

$$\Pi = \frac{1}{M} \mathcal{I}_M \otimes \mathcal{E}^{B \rightarrow A} + (1 - \frac{1}{M}) \hat{\mathcal{I}}_M \otimes \mathcal{F}^{B \rightarrow A}$$

2. The use of the given channel is to cancel the second (undesired) part, so that the resulting channel is noiseless:

$$\text{Tr} \mathcal{N} \circ \mathcal{F} = 0$$

3. Translating the non-signalling constraints and replacing the variable channel by Choi matrix, we obtain the desired SDP.
Zero-Error Classical Simulation Cost Assisted by QNS

Goal: Determine the minimum integer $M$ such that $\mathcal{N} <_{QNS} \mathcal{I}_M$.

**Theorem 2.** The one-shot simulation cost is $\left[2^{-H_{\min}(A|B)}\right]$, where $H_{\min}(A|B)$ is the conditional min-entropy defined by (Koenig, Renner, Schaffner 2009)

$$2^{-H_{\min}(A|B)} = \min \text{Tr} \Gamma_B, \text{s.t.}, J_{AB} \leq 1_A \otimes \Gamma_B.$$ 

Corollary. The asymptotic zero-error classical simulation cost of channel $\mathcal{N}$ is $-H_{\min}(A|B)$ bits per realization.

The more important problem is to find the simulation cost of the “cheapest” channel supporting on $K$.

**Theorem 3.** The one-shot simulation cost of non-commutative bipartite graph $K$ is the integer part of the following SDP ($P$ is the projection over the Choi-matrix)

$$\Sigma(K) = \min \text{Tr} T_B \text{ s.t. } 0 \leq V_{AB} \leq 1_A \otimes T_B,$$

$$\text{Tr}_B V_{AB} = 1_A,$$

$$\text{Tr}(1 - P)V = 0.$$
Zero-Error Simulation Cost of cq-channels Assisted by QNS

\[ N : \ k \rightarrow \rho_k, \ 1 \leq k \leq n. \ P_{AB} = \sum_{k=1}^{n} |k\rangle\langle k|_A \otimes P_k. \]

Simplified one-shot simulation cost for non-commutative bipartite graph:

\[ \Sigma(K) = \min \text{Tr} \ T, \ \text{s.t.} \ T \geq V_k, \ \text{Tr} \ V_k = 1, 0 \leq V_k \leq P_k. \]

“Sub-multiplicative under tensor product”

The dual problem:

\[ \Sigma(K) = \max \sum_k s_k, \ \text{s.t.} \ s_k P_k \leq P_k U_k P_k, \ \sum_k U_k = I, \ U_k \geq 0. \]

“Super-multiplicative under tensor product”

Multiplicative in general: \[ \Sigma(K_0 \otimes K_1) = \Sigma(K_0) \times \Sigma(K_1). \]

Theorem 4. The asymptotic zero-error classical simulation cost of a cq-channel \( K \) is given by

\[ G_{0,\text{NS}}(K) = \log_2 \Sigma(K). \]

Corollary (Additivity): \[ G_{0,\text{NS}}(K_0 \otimes K_1) = G_{0,\text{NS}}(K_0) + G_{0,\text{NS}}(K_1). \]
Zero-Error Classical Capacity of cq-channels Assisted by QNS

\[ \mathcal{N} : i \to \rho_i \]

Simplified one-shot capacity:

\[ \Upsilon(K) = \max \sum_i s_i, \text{ s.t. } 0 \leq s_i, R_i \leq s_i(1 - P_i), \sum_i (s_i P_i + R_i) = 1. \]

Harrow’s semi-definite fractional packing number:

\[ A(K) = \max \sum_i s_i, \text{ s.t. } 0 \leq s_i, \sum_i s_i P_i \leq 1, \]

(Asymptotic) zero-error classical capacity:

\[ C_{0,NS}(K) = \sup_{n \geq 1} \frac{\log \Upsilon(K \otimes n)}{n} \]

Theorem 5. \[ C_{0,NS}(K) = \log A(K) \]

Corollary (Additivity): \[ C_{0,NS}(K_0 \otimes K_1) = C_{0,NS}(K_0) + C_{0,NS}(K_1) \]
Proof outline of Theorem 5

Suffice to establish the following inequality for large enough $n$

$$A(K^\otimes n) \geq \Upsilon(K^\otimes n) \geq \frac{1}{\text{poly}(n)} A(K^\otimes n)$$

The first inequality is obvious (by corresponding SDPs); The second inequality needs to combine type method (standard technique) and the special case of group symmetry.

**Key Lemma:** For a set of projectors $P_i$ on $B$ with a transitive group action by conjugation under $U^g$, let

$$B = \bigoplus_{\lambda} Q_\lambda \otimes R_\lambda$$

be the isotypical decomposition of $B$ into irreps $Q_\lambda$ of $U^g$, with multiplicity spaces $R_\lambda$. Denote the number of terms $\lambda$ by $L$, and the largest occurring multiplicity by $M = \max_\lambda |R_\lambda|$. Then, for the corresponding cq-channel,

$$\Upsilon(K) \geq \frac{1}{4L^2 M^{9/2}} A(K),$$

if $A(K) \geq 64L^6 M^{14}$.

For permutation group $S_n$, both $L$ and $M$ are upper bounded by polynomials in $n$. 

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Two Examples

1. Classical channels with bipartite graph $\Gamma$:

$$K = \text{span}\{|j\rangle\langle i| : i \rightarrow j \text{ edge in } \Gamma\}$$

It was shown (Cubitt et al. 2010):

$$\Upsilon(K) = \Sigma(K) = A(K)$$

Thus

$$C_{0,\text{NS}}(K) = G_{0,\text{NS}}(K) = \log A(K) = \log \alpha^*(\Gamma)$$

Zero-error communication and simulation are Reversible!

2. Two-state cq-channels: $\mathcal{N}: i \rightarrow |\psi_i\rangle, |\psi_{0,1}\rangle = \alpha|0\rangle \pm \beta|1\rangle$, $\alpha > \beta > 0$.

$$K = \text{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$$

$$\Upsilon(K) = 1,$$

$$\Upsilon(K \otimes K) \geq \max \left\{1, \frac{1}{2\alpha^4}\right\},$$

$$\Upsilon(K^\otimes n) \geq \frac{1}{\alpha^{2n} + \beta^{2n}} \text{ for } n \gg 1.$$  

$$C_{0,\text{NS}}(K) = \log A(K) = -2\log \alpha$$

$$G_{0,\text{NS}}(K) = \log \Sigma(K) = \log(1 + 2\alpha\beta)$$

$$0 < C_{0,\text{NS}}(K) < G_{0,\text{NS}}(K)$$

Zero-error communication and simulation are Irreversible!
Operational Interpretation of the Lovasz theta function

For a graph $G=(V,E)$, the non-commutative graph corresponding to $G$ is

$$G = \text{span}\{|i\rangle\langle j| : (i,j) \in E \text{ or } i = j, i, j \in V\}$$

The zero-error classical capacity of $G$ assisted by non-signalling correlation is defined as

$$C_{0,\text{NS}}(G) = \min\{C_{0,\text{NS}}(K) : K^\dagger K = G\}$$

Theorem 6. For any classical graph $G$, $C_{0,\text{NS}}(G) = \log \vartheta(G)$.

Remarks:

1. Lovasz theta function of a graph $G$ can be operationally interpreted as the zero-error classical capacity of the graph assisted by quantum non-signalling correlations.

2. Previously it was only known that Lovasz theta function is an upper bound for the entanglement-assisted zero-error classical capacity (Beigi 2010; Duan, Severini, and Winter 2010).
Proof outline of Theorem 5

Orthogonal Representation (OR) \{ |\psi_i \rangle : i \in V \} for G=(V,E) (Lovasz 1979):
\[
\langle \psi_i | \psi_i \rangle = 1 \text{ and } \langle \psi_i | \psi_j \rangle = 0 \text{ for } \{i, j\} \not\in E
\]

Lovasz theta function for graph G:
\[
\vartheta(G) = \min_{\text{OR:} \{ |\psi_i \rangle \}} \min_{|c\rangle = 1} \max_{1 \leq i \leq n} |\langle c | \psi_i \rangle|^{-2}
\]

Unit vectors can be relaxed to projections

OR \{ P_i : i \in V \} for G=(V,E): \text{ Tr } P_i P_j = 0 \text{ for } \{i, j\} \not\in E
\[
\vartheta(G) = \min_{\text{OR:} \{ P_i \}} \min_{\rho \geq 0, \text{ Tr } \rho = 1} \max_{1 \leq i \leq n} (\text{ Tr } \rho P_i)^{-1}
\]

Harrow’s number for a classical graph G:
\[
A(G') = \min \{ A(K) : K^\dagger K = G \}
\]

Alternatively,
\[
A(G) = \min_{\text{OR:} \{ P_i \}} A(\{ P_i \})
\]

Then the problem is reduced to prove the following equality:
\[
A(G') = \vartheta(G')
\]

This can be shown by some algebraic manipulations and the Von Neumann’s minimax theorem.
Some Open Problems

1. Asymptotic communication capacity for more general quantum channels. For instance, amplitude damping channel:

   \[ K_r = \text{span}\{E_0, E_1\}, \ E_0 = |0\rangle\langle 0| + \sqrt{1-r}|1\rangle\langle 1|, \ E_1 = \sqrt{r}|0\rangle\langle 1|, \ 0 < r < 1 \]

2. Is Quantum Lovasz theta function is equal to the entanglement-assisted zero-error classical capacity of a graph?

3. Quantitative theory for quantum non-signalling correlations.

Finally, what is the zero-error channel exchange rate for general quantum channels assisted by quantum non-signalling correlations? That is,

\[ R_{0,QNS}(\mathcal{M}, \mathcal{N}) =? \]

Note that we have \[ R_{0,QNS}(\mathcal{N}, \mathcal{I}_2) = -H_{\text{min}}(A|B)_J. \]

However, even for general cq-channels this quantity is unknown. 😊
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