Linear fields in AdS spacetimes: boundary conditions and quasinormal modes

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Introduction

- AdS spacetimes hugely important, esp. in context of holographic correspondence
- Many classical properties of AdS spacetimes still not fully understood
  - Stability / instability of global AdS
    - Is collapse generic?
    - Are there islands of stability?
  - Stability / instability of Kerr-AdS
    - Does very slow linear decay translate into non-linear instability?

  [Dafermos; Holzegel; Smulevici; Anderson; Bizon–Rostworowski; Buchel–Lehner–Liebling; Dias–Horowitz–Marolf–Santos; ...]
- Nature of gravity in AdS very different from Minkowski / de Sitter
  - Initial-boundary problem vs. Cauchy problem
  - Requires different mathematical techniques to analyse

Introduction

- Most analytical work on AdS spacetimes in symmetrical backgrounds
  - Fourier transform reduces hyperbolic PDE to elliptic PDE
  - Equations of motion separate to give ODEs
- Goal: Prove results about AdS spacetimes under minimal assumptions of symmetry
  - No assumptions of separability
  - No explicit solutions to equations available

Introduction

Unless otherwise stated, consider:

- Asymptotically AdS spacetimes
- Classical bulk physics
- Linear fields (stick to Klein-Gordon Equation)
- No symmetry assumptions on background

What is an asymptotically AdS Spacetime?

- X an $(n+1)-$dimensional manifold with boundary ∂X, and g a smooth Lorentzian metric on $\mathring{X}$.

Definition

∂X is an *asymptotically anti-de Sitter end of* (X, g) *with radius* l if:

I) There exists a smooth r such that $r^{-1} = 0$, $\nabla r^{-1} \neq 0$ on $\partial X = \mathscr{I}$

II) There exist x^α, coordinates on the slices $r = \mathrm{const.}$, such that locally

$$g_{rr} = \frac{l^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \qquad g_{r\alpha} = \mathcal{O}\left(\frac{1}{r^2}\right), \qquad g_{\alpha\beta} = r^2 \mathfrak{g}_{\alpha\beta} + \mathcal{O}\left(1\right),$$

where $\mathfrak{g}_{\alpha\beta} dx^\alpha dx^\beta$ is a Lorentzian metric on $\mathscr{I}$.

III) $r^{-2} g$ extends as a smooth metric on a neighbourhood of $\mathscr{I}$.

- Examples include: AdS, AdS-Schwarzschild, AdS-Kerr, ...

Klein-Gordon Equation

- Simplest equation to consider is the Klein-Gordon equation

$$\Box_g u - \frac{\mu}{l^2} u = 0, \qquad \mu > -\frac{n^2}{4} \tag{1}$$

- On AdS^{n+1}, find that near infinity

$$u \sim a_+ r^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\mu}} + a_- r^{-\frac{n}{2}-\sqrt{\frac{n^2}{4}+\mu}}$$

- Can solve (1) with initial conditions $u = u_0, \partial_t u = u_1$ at $t = 0$ and choice of boundary conditions

$a_+ = 0$	$\mu > -\frac{n^2}{4}$	Dirichlet
$a_- = 0$	$-\frac{n^2}{4} + 1 > \mu > -\frac{n^2}{4}$	Neumann
$a_- - \beta a_+ = 0$	$-\frac{n^2}{4} + 1 > \mu > -\frac{n^2}{4}$	Robin

(Breitenlohner-Freedman; Ishibashi-Wald)

Theorem (CMW 2012)

Let Σ be a spacelike, n-dimensional surface in (X^{n+1}, g), an asymptotically AdS spacetime. Choose Dirichlet, Neumann or Robin boundary conditions on $\mathscr{I}$, appropriate to μ. Then for suitable $u_0, u_1 : \Sigma \to \mathbb{R}$ there exists a unique u solving

$$\Box_g u - \frac{\mu}{l^2} u = 0 \quad in \quad D^+[\Sigma \cup (\mathscr{I} \cap I^+(\Sigma))] := \mathcal{D}^+[\Sigma]$$

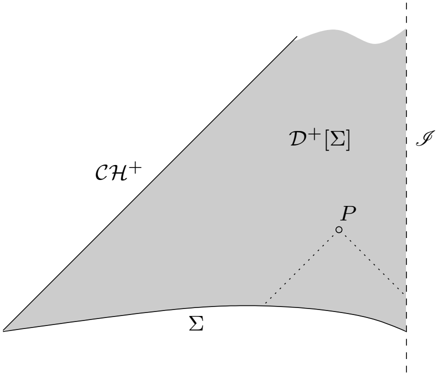
with

$$u|_\Sigma = u_0, \quad n_\Sigma u|_\Sigma = u_1.$$

Furthermore u has an expansion near infinity as in the AdS case.

Dirichlet case due to [Vasy; Holzegel]

Klein-Gordon Equation



Klein-Gordon Equation

- No assumption of symmetry for (X, g)
- Do not require that the Klein-Gordon equation separates
- Note initial conditions necessary: boundary data alone does *not* determine field everywhere
- Theorem applies for *inhomogeneous* Dirichlet, Neumann, Robin conditions also (for suitable μ), and for more general systems of PDEs.
- Proof uses energy space methods based on a *renormalized* energy
- In general, energy on a spacelike surface Σ' to the future of Σ will be larger than initial energy, but still finite
- Need more structure to say more about behaviour in time

Theorem (Holzegel, CMW 2013)

The coupled Einstein–Klein-Gordon system is well posed as a characteristic initial-boundary value problem within spherical symmetry, with inhomogeneous Dirichlet, Neumann or Robin boundary conditions imposed on the Klein-Gordon field at $\mathscr{I}$.

Case of homogeneous Dirichlet boundary conditions due to [Holzegel–Smulevici]

Stationary Black Holes

Definition

$(\mathcal{R}, g)$ is the exterior region of a stationary, asymptotically anti-de Sitter, black hole space time if the following holds

I) $(\mathcal{R}$, g) is asymptotically AdS with null infinity $\mathscr{I}$

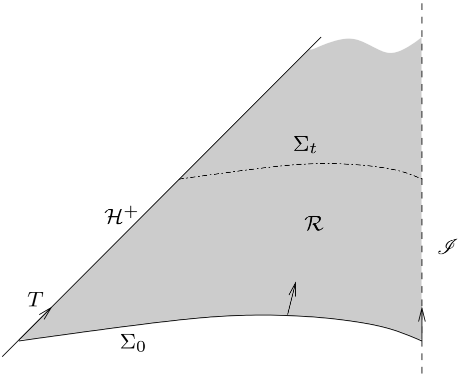
II) $\mathcal{R} = \mathcal{D}^+[\Sigma]$ for some spacelike Σ

III) g admits a Killing field T such that:

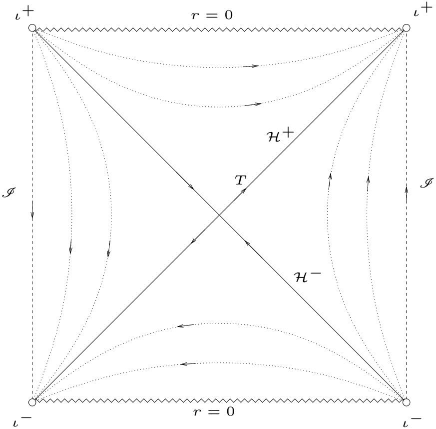
1. T is timelike in the interior of $\mathcal{R}$
2. T is tangent to $\mathscr{I}$ and normal to $\mathcal{H}^+ = \mathcal{CH}^+$
3. $\mathcal{H}^+$ is a Killing horizon of T, with surface gravity $\kappa > 0$.
4. $r^{-1}T$ is uniformly bounded on $\mathcal{R}$ (w.r.t g)

IV) Σ, $\mathcal{H}^+ \cap \Sigma$ and $\mathscr{I} \cap \Sigma$ are compact

Stationary Black Holes



Example: AdS Schwarzschild



$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{r^2}{l^2}} + r^2 d\Omega^2$$

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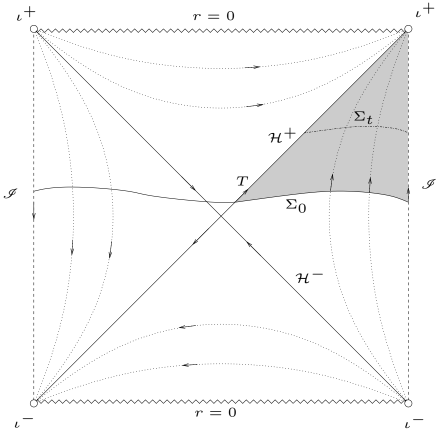

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- AdS-Schwarzschild with hyperbolic / flat symmetry orbits
- AdS-Kerr below Hawking-Reall bound
- AdS-Reissner-Nordström
- various 'hairy' black holes, supported by non-trivial fields
- Higher dimensional generalisations

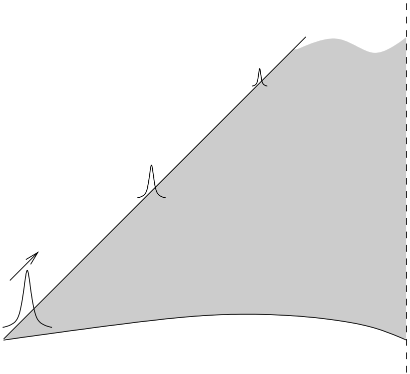
Quasinormal Modes

- Want to understand behaviour of solutions, ψ of Klein-Gordon equation at late times t
- Expect to see

$$\psi \sim \sum e^{s_i t} u_i(x)$$

  for discrete frequencies s_i, and functions u_i on Σ, characteristic of the spacetime.
- Obstruction to decay at late times: outgoing wavepackets at the horizon

Wavepackets at the horizon



Wavepackets at the horizon

- Rate of decay determined by how sharply localised the wave packet is
- Measure localisation using Sobolev norms
- For a function $u(x)$ defined on $\mathbb{R}^n$, with Fourier transform $\tilde{u}(\xi)$, define

$$||u(x)||^2_{H^k} = \int d^n\xi (1 + |\xi|^2)^k \, |\tilde{u}(\xi)|^2$$

- Can extend definition to curved manifolds
- The larger k is, the smoother a function with $||u(x)||_{H^k} < \infty$ is.
- Crudely, an outgoing wavepacket localised at the horizon, with $||\psi(x,t)||_{H^k} < \infty$ will decay like

$$|\psi(x,t)| \sim e^{-\varkappa(k-\frac{1}{2})}$$

where $\varkappa$ is the surface gravity.

The wave semigroup

- Solutions of the Klein-Gordon equation satisfy

$$||\psi||^2_{H^k(\Sigma_t)} + ||T\psi||^2_{H^{k-1}(\Sigma_t)} \leq C \left(||\psi||^2_{H^k(\Sigma_0)} + ||T\psi||^2_{H^{k-1}(\Sigma_0)} \right)$$

Key ingredient: Redshift effect [Dafermos–Rodnianski]

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- Time evolution gives an operator semigroup on the Hilbert space $H^k \times H^{k-1}$:

$$\mathcal{S}(t) = e^{\mathcal{A}t}$$

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- The quasinormal frequencies are the eigenvalues, s, of $\mathcal{A}$ satisfying

$$\Re(s) > \left(\frac{1}{2} - k\right)\varkappa$$

The main theorem

Theorem (Discreteness of QNF [CMW, 2013])

The spectrum of $\mathcal{A}$ in the region $\Re(s) > \left(\frac{1}{2} - k\right) \varkappa$ consists solely of isolated eigenvalues of finite multiplicity. The eigenfunctions u are smooth at the horizon and if $\psi = e^{st}u$, we have

$$L\psi = 0.$$

Related work: [Bachelot; Gannot; Melrose–Sá Baretto–Vasy; Dyatlov; Sá Baretto–Zworski; Bony–Häfner; ...]

Corollary

Let $\psi(x,t)$ be a smooth solution of the Klein-Gordon equation on an asymptotically AdS black hole. Then the Laplace transform

$$\hat{\psi}(x,s) = \int_0^\infty e^{-st}\psi(x,t)dt$$

extends meromorphically to $\mathbb{C}$, and the location of its poles belong to a countable set Λ_{QNF} which is independent of ψ.

The main theorem

- No separability of the equations is assumed
- Regularity as a boundary condition is very natural
- Can extend to any other of the usual linear fields (Dirac, Maxwell, etc.)
- Can extend to locally stationary black holes
- Unlike the usual definition using 'ingoing' boundary conditions, QNM are honest eigenfunctions of an operator on a Hilbert space
- Do not need to restrict to perturbations supported away from the horizon
- Can show that 'ingoing' QNF are a subset of these QNF, and they typically agree

Completeness of the spectrum

- Since QNF spectrum is countable, is it true by analogy with Fourier series that if ψ is a solution of KGE, then

$$\psi(x,t) = \sum_{i=0}^{\infty} a_i e^{s_i t} u_i(x)?$$

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NO!

- In fact, can arrange that $\sum_{i=0}^{\infty} e^{s_i t} u_i(x)$ converges,

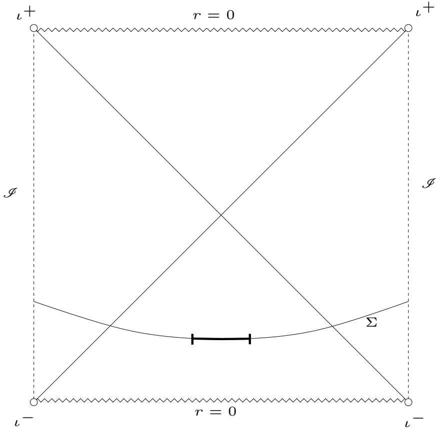
$$\psi(x,t) \sim \sum_{i=0}^{\infty} a_i e^{s_i t} u_i(x) \qquad \text{as } t \to \infty$$

but nevertheless

$$\psi(x,t) \neq \sum_{i=0}^{\infty} e^{s_i t} u_i(x)$$

for *any* finite t.

Incompleteness for AdS Schwarzschild



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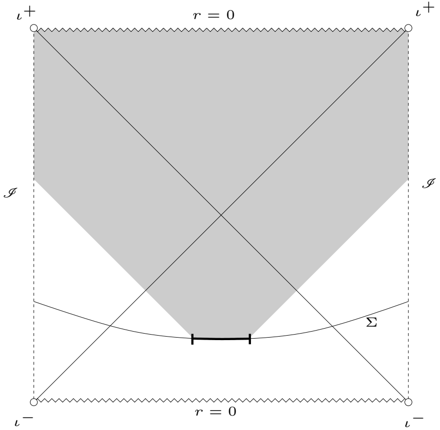

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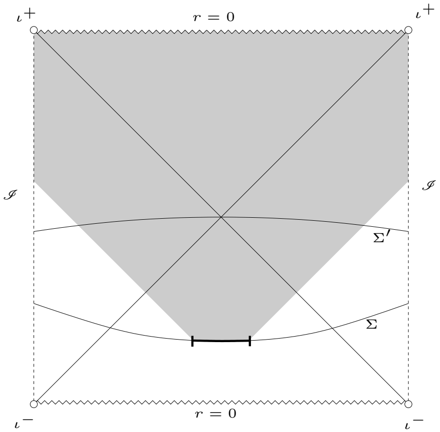

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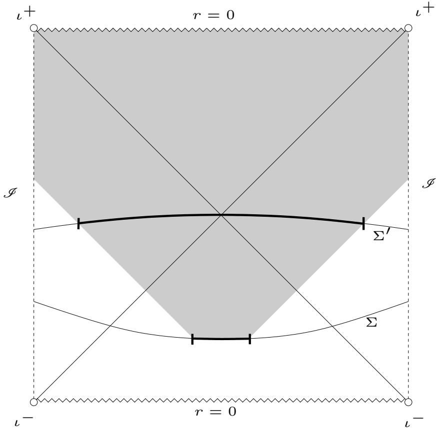

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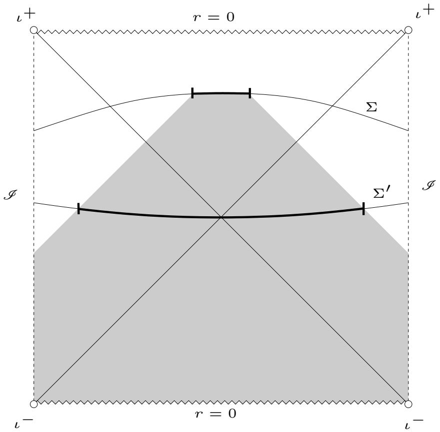

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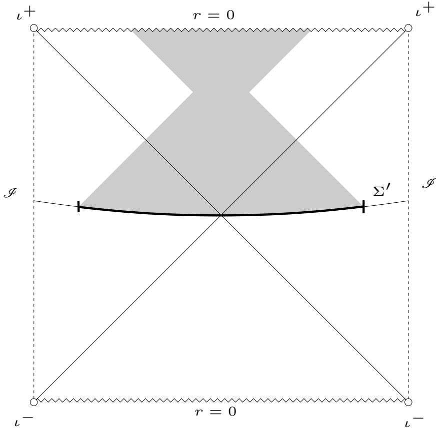

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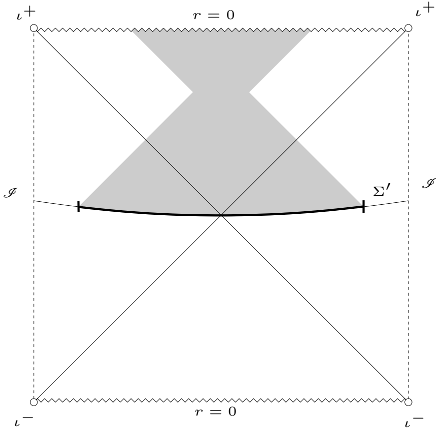

$$\psi \sim 0 \quad \text{as } t \to \infty, \qquad \text{but } \psi \not\equiv 0.$$

Conclusions

- Linear equations are well posed as initial-boundary problems in arbitrary asymptotically AdS spacetimes
- QNM should be thought of as eigenvalues of the infinitesimal generator of the solution operator on $H^k \times H^{k-1}$ for a regular slicing
- The QNF are a discrete, countable set of points in the complex plane
- The QNM do not form a complete basis for $H^k \times H^{k-1}$.