Typical behaviour of extremes of chaotic dynamical systems for general observables

Tobias Kuna

University of Reading

Non-equilibrium Statistical Mechanics and the Theory of Extreme Events in Earth Science
28.10.2013,
Newton Institute

In collaboration with
V. Lucarini (Hamburg,Reading) J. Wouters (Hamburg),
D. Faranda (Hamburg)

Overview

- General theory of extremes
- Extremes for dynamical systems
- Distance vs generic observables
- Linear Response
- Open Problem

Extreme values, i.i.d. case

- Let X_i be i.i.d. RV.
- Extremes

$$M_n := \max_{i \leq n} X_i$$

in analogy to

$$S_n := \sum_{i=1}^{n} X_i$$

- One says that X fulfils an extreme value law iff there exists normaliziations

$$(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$$

such that

$$\frac{M_n - b_n}{a_n}$$

converges in distribution.

Extreme values, i.i.d. case

- $(a_n)_n$ and $(b_n)_n$ are essentially unique up to a scaling symmetry

$$x \mapsto ax + b, \qquad a > 0$$

- Classification of limit laws
- Pareto-Fréchet

$$P(Y \leq x) = e^{-y^{-\alpha}} \quad y > 0$$

- Weibull, maximal value

$$P(Y \leq y) = e^{-(x_{max}-y)^{\alpha}} \qquad y < x_{max}$$

- Gumbel

$$P(Y \leq y) = e^{-e^{-y}}$$

Domain of attraction of Extreme laws

- Let F be the cumulative distribution function
- Tippett-Fischer-Gnedenko theorem
- X has Pareto-Fréchet distributed Extreme law iff

$$1 - F(x) = x^{-\alpha} l_F(x)$$

  where l_F are log-factors (slowly varying function)
- X has Weibull distributed Extreme law iff
  there exists $x_{max} := \inf\{x | F(x) = 1\}$ and

$$1 - F(x_{max} - x) = (x_{max} - x)^{\alpha} l_F(1/(x_{max} - x))$$

Scheme of practical usage

- Statistical prediction of unseen extreme events
- Easiest Pareto-Fréchet case
- Log-log plot gives

  $$\ln(1 - F(e^{x+x_0})) \sim \alpha x + \ln(1 - F(e^{x_0}))$$

  for a large enough threshold x_0.
- In extremal regime one gets a line
- Extreme events can be predicted by linear interpolation

Extreme values for non i.i.d. case

- One need to check two properties
- Over-threshold: $P(X \geq a)$
- Some kind of mixing property
- If both hold; limit laws as in the i.i.d. case
- Recurrence of maxima is Poisson distributed

Extreme values for dynamical systems

- Let $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$, $t \in \mathbb{Z}$ be a dynamical system, with

$$\Phi_{t+s} = \Phi_s \circ \Phi_t$$

- Large time behaviour controlled by invariant measure
- Krylov-Bogolubov theorem

$$\mu := \lim_{T\to\infty} \frac{1}{T} \int_0^T (\Phi_T)_{\#}\mu_0$$

- SRB measure: if μ_0 Lebesgue measure
- equivalent definition: small noise limit
- Roughly: dynamical system with random initial condition large times distributed w.r.t. SRB

Extreme values for dynamical systems

- Series of works: P. Collet, A. Freitas, J. Freitas, M. Todd, C. Gupta, M. Holland, M. Nicol, G. Turchetti, and S. Vaienti
- Observable: $A(x) := d(x, x_0)^{\beta}$.
- x_0 in the compact attractor of Φ_t
- Distribution of maxima

  $$t_i := \inf\{t | A(t) \geq A(t_{i-1})\}$$

  What is the distribution of $(A(t_i))_i$.
- Weibull distributed if strong enough mixing.

Extreme values for dynamical systems

- Idea of proof:
- Chaotic (hyperbolic systems)
- locally invariant split into subspace of expanding and contracting directions
- d_u dimension of expanding directions
- attractor splits locally along this split of subspaces
- attractor in stable direction like Cantor set
- d_s Hausdorff dimension of Cantor set
- local scaling of volume

$$\mu\left(d(x, x_0) < C\right) \sim C^{d_s + d_u}$$

Extreme values for dynamical systems

- Scaling behaviour of over thresholds

$$\frac{\mu\left(d(x,x_0)^\alpha > B + A_0\right)}{\mu\left(d(x,x_0)^\alpha > A_0\right)}$$

gives rise to Weibull index $(d_s + d_u)/\alpha$

- Mixing property implies proper extreme value distribution
- Restrictions
  1. x_0 has to be in the attractor
  2. low dimensional systems
  3. hyperbolic systems
  4. Geometric link between observable and attractor

Meteorological application

- High dimensional dynamical system
- Complicated attractor structure
- Observables are often additive like energy, momentum, density etc.
- Model uncertainty

Extreme values for generic observables

- Let A be a generic observable.
- As the attractor is very thin it has typically not the maximum on the attractor
- Locally around the maximum

$$A(x) = A(x_0) + \nabla A(x_0) \cdot x + \text{quadratic terms}$$

- Scaling of measure

$$\frac{\mu(A(x) \geq A(x_0) - T - B_0)}{\mu\left(A(x) \geq A(x_0) - B_0\right)} = C\left(1 - \frac{T}{(A(x_0) - B_0)}\right)^{\delta}$$

- $\delta = d_s + \frac{1}{2} d_u$.
- M. Holland, R. Vitolo, P. Rabassa, A. Sterk

Extreme values for generic observables

- Geometrical picture
- Denote the attractor by Ω
- locally A is linear
- locally Ω is a paraboloid
- $$\mu(A(x) \geq A(x_0) - T)$$

  paraboloid cut by plane
- unstable directions are normal to $\nabla A(x_0)$
- If $\nabla A(x_0)$ is not parallel to one of the stable directions
- then volumes scale like d_s.

Geometry Generic?

- Problem: hyperbolic theory works for generic points in attractor
- x_0 is by construction on surface
- Surface is of measure zero.
- Question: "$\nabla A(x_0)$ is not parallel to one of the stable directions" is **generic?**
- True for product of horse shoe
- False for generic differential deformation of horseshoe
- True for generic continuous deformation of horseshoe?
- Conclusion: Property is not stable in the the usual categories

Response theory for Extreme values

- Linear Response theory (rigorous D. Ruelle)
- Small perturbations of dynamics

$$\Phi_t \mapsto \Phi_t^{(\varepsilon)}$$

- Perturbative expansion
- for ergodic means of observables

$$\frac{d}{d\varepsilon} \lim_{T\to\infty} \frac{1}{T} \int_0^T \int A(\Phi_t^{(\varepsilon)}(x))dx$$
$$= \int_0^\infty \int \nabla A(\Phi_t(x)) D_x \Phi_t X(x) \; \mu(dx)$$

Response theory for Extreme values

- Shape parameters can be expressed via moments

$$\xi^{(\varepsilon)} = \frac{1}{2}\left(1 - \frac{1}{\frac{m_2 - m_1^2}{m_1^2}}\right)$$

where m_i are the moments of the conditional distribution over threshold

$$\mu\left(A(x) \geq \cdot + A_0 \,|A(x) \geq A_0\,\right)$$

Response theory for Extreme values

- By response theory we get that

$$\frac{d}{d\varepsilon}\xi^{(\varepsilon)} = \frac{1}{d_s + d_u/2}\frac{d}{d\varepsilon}d_s^{(\varepsilon)}$$

- If we follow the conjecture for chaotic system that local scaling is the Kaplan-Yorke dimension

$$d_{KY} - d_u = d_s = \sum_{k=1}^{n}\frac{\lambda_k}{|\lambda_{n+1}|}$$

- For practical purposes $d_{KY}^{(\varepsilon)}$ should be smooth.

Open problem

- These are only hypothesis
- Numerical investigation.
- For low dimension depends on fine structure of system
- No universality, what is proper generalisation
- Influence of scales?
- Symmetry, indistinguishable particles
- Which category of conjugation of dynamical systems is appropriate
- Study of surface of attractor

Thanks

- THANK YOU FOR YOU ATTENTION