Interplay between Mathematics and Physics

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I. Principle of Dynamic Transitions

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I. Principle of Dynamic Transitions

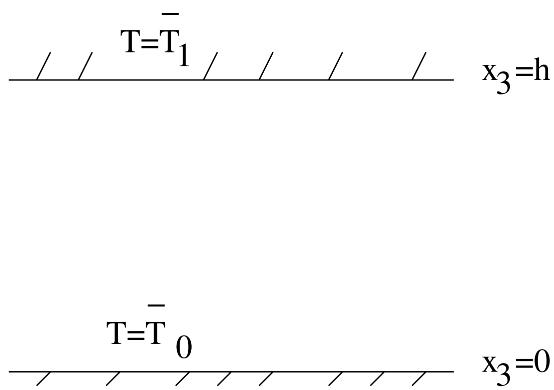
- A dynamic transition theory is established recently by Ma-Wang for dissipative systems, leading to the following:

  Principle of Dynamic Transitions: all dynamics transitions of dissipative systems are classified into three categories: 1) continuous, 2) catastrophic, and 3) random.

  Key philosophy of the theory is to search for the complete set of transition states, represented by a local attractor.

  For details on the theory and its applications, see (Ma-Wang, Phase Transition Dynamics, Springer-Verlag, 11/2013, 555 pp.)

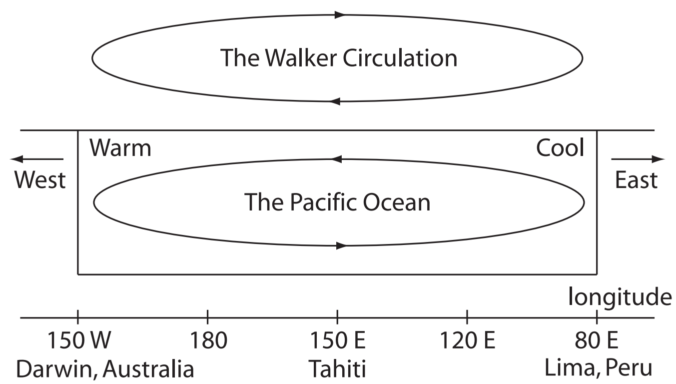
Consider the Bénard convection on a nondim. domain $\Omega = D \times (0,1)$ with a set of physically sound BCs. Let Rayleigh number be $\boxed{R = g\alpha(\bar{T}_0 - \bar{T}_1)h^3/(\kappa\nu)}$.



Thm (Ma & W., 04): As R crosses the first critical Rayleigh number R_c, the system always undergoes a continuous transition to an attractor $\Sigma_R \simeq S^{m-1}$, where m is the multiplicity of R_c.

In addition, Σ_R attracts every bounded set of $H \setminus \Gamma$, where Γ is the stable manifold of the basic solution.

II. Metastable State Oscillation Theory of ENSO

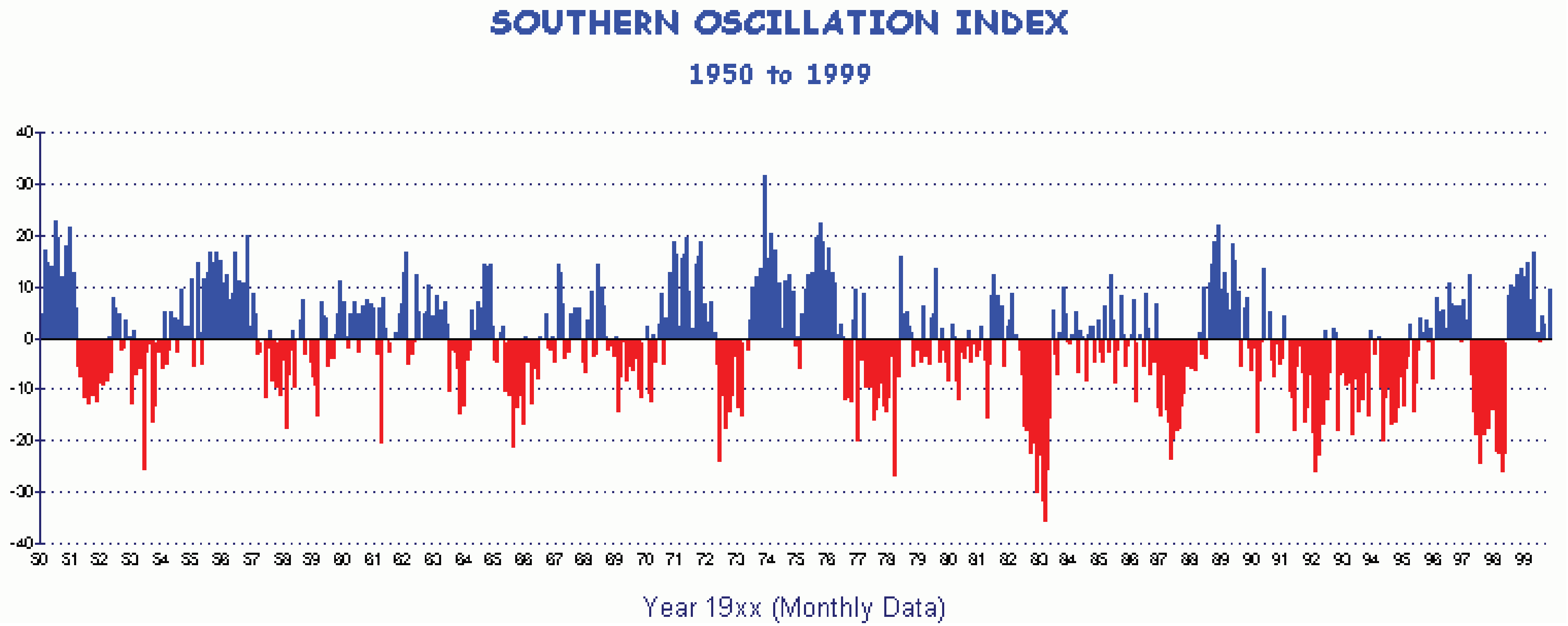


- The behavior of the **Walker cell** is a key factor giving rise to the ENSO.

- When the convective activity weakens or reverses, an **El Niño** phase takes place, causing the ocean surface to be warmer than average, reducing or terminating the upwelling of cold water.

- A particularly strong Walker circulation causes a **La Niña** event, resulting in cooler sea-surface temperature (SST) due to stronger upwelling.

- SOI gives a simple measure of the strength and phase of the Southern Oscillation: In the **El Niño** phase, the SOI is **negative or zero**, when in **La Niña** phase, the SOI is **strongly positive**, and in the **normal state** the SOI is **small and positive**.

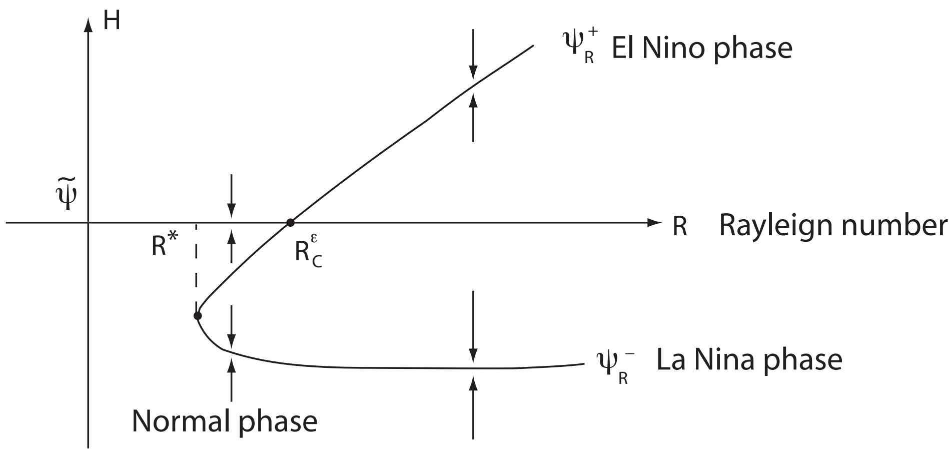


- There have been extensive studies in recent years, following the pioneering work of (Dijkstra 00, Ghil 00, Jin 96, Jin-Neelin-Ghil 96, Zebiak-Cane 87, ...).

- An interesting current debate is whether ENSO is best modeled as a stochastic or chaotic system - linear and noise-forced, or nonlinear oscillatory and unstable system (G. Philander and A. Fedorov 03)?

- A careful fundamental level examination of the problem is crucial.

Consider the basic atmospheric model over the tropics:



Mechanism of ENSO: ENSO is a self-organizing and self-excitation system, with two highly coupled oscillatory processes:

- the oscillation between the two metastable warm (El Nino phase) and cold events (La Nina phase), and
- the spatiotemporal oscillation of the sea surface temperature (SST) field.

The interplay between these two processes

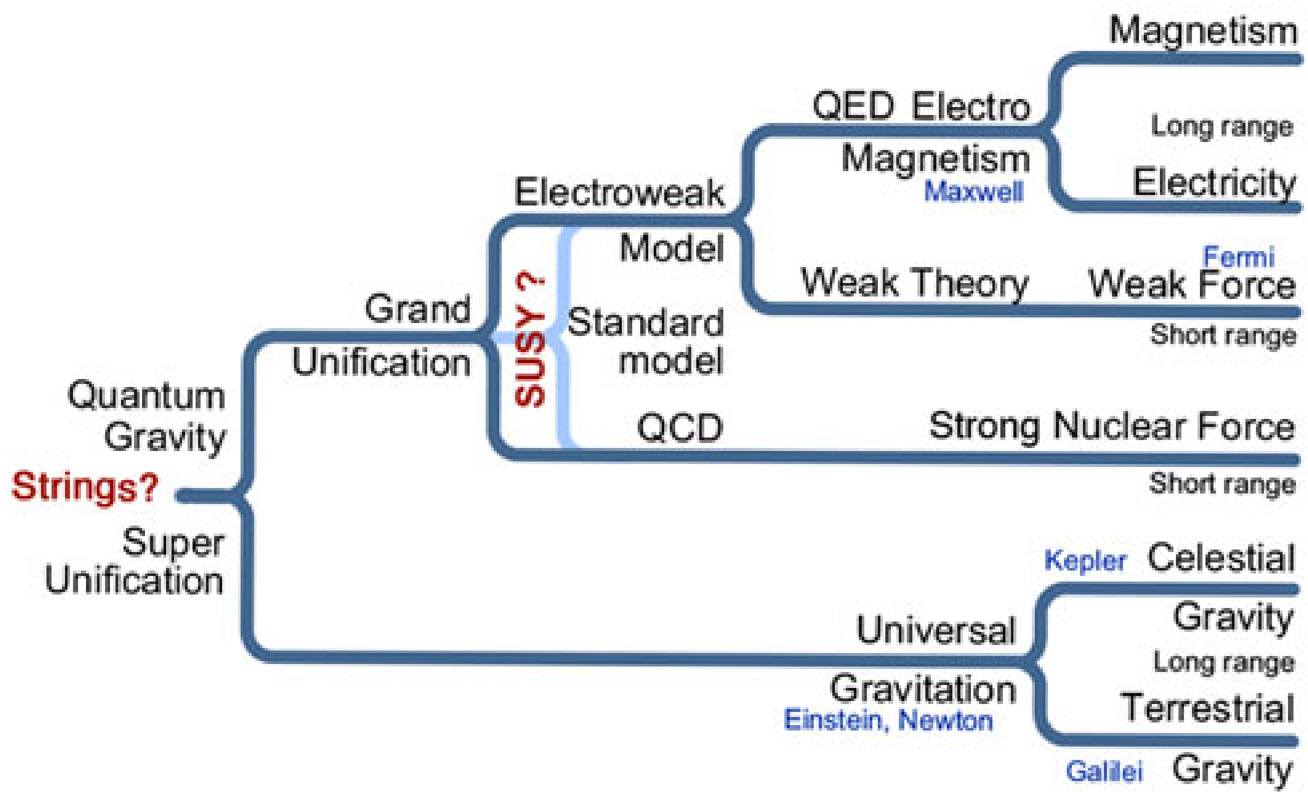
- gives rises the climate variability associated with ENSO,
- leads to both the random and deterministic features of ENSO, and
- defines a natural feedback mechanism, driving the sporadic oscillation of ENSO.

The randomness is closely related to the uncertainty/fluctuations of the initial data between the narrow basins of attractions of the corresponding metastable events.

The deterministic feature is represented by a deterministic coupled atmospheric and oceanic model predicting the basins of attraction and the sea-surface temperature (SST).

III. Principle of Interaction Dynamics

Four fundamental forces/interactions in Nature:



Principle of Interaction Dynamics (PID) (Ma-Wang, 2012): Least action with energy-momentum conservation constraints.

With PID, we derive the following gravitational field equations:

$$L_{EH}(g_{\mu\nu}) = \int_M \left(R + \frac{8\pi G}{c^4} S \right) \sqrt{-g} dx \qquad \delta L_{EH} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$(\delta L_{EH}(g_{\mu\nu}), X) = 0 \quad \forall \ D^\mu X_{\mu\nu} = 0 \qquad \Longrightarrow \qquad \boxed{\begin{aligned} & R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} - D_\mu \Phi_\nu \\ & D^\mu \left(\frac{8\pi G}{c^4} T_{\mu\nu} + D_\mu \Phi_\nu \right) = 0 \end{aligned}}$$

Note: The new term $D_\mu \Phi_\nu$ cannot be derived 1) from any existing $f(R)$ theories, and 2) from any scalar field theories.

New gravitational vector bosonic field:

- The new vector particle field $\Phi_\mu = D_\mu \varphi$ is massless with spin $s = 1$:

$$\Box \Phi_\nu = \frac{e}{\hbar c} A_\mu D^\mu \Phi_\nu + \frac{8\pi G}{c^4} D^\mu T_{\mu\nu}$$

- The nonlinear interaction between this particle field Φ_μ and the graviton leads to a unified theory of dark matter and dark energy and explains the acceleration of expanding universe.

- Consider a spherically symmetric central matter field with mass M and radius r_0. The force exerted on an object with mass m is given by

$$F = mMG \left[-\frac{1}{r^2} - \frac{c^2}{2GM} \left(2 + \frac{2GM}{c^2} \frac{1}{r} \right) \frac{d\varphi}{dr} + \frac{c^2}{2GM} \Phi \; r \right], \qquad \text{for } r > r_0.$$

The first term is the classical Newton gravitation, the 2nd and 3rd terms are the coupling interaction between matter and the scalar potential φ.

This formula can be further simplified to derive the following approximate formula for $r_0 < r < r_1 \approx 10^{21} - 10^{22} km$:

(1) $$F = mMG\left[-\frac{1}{r^2} - \frac{k_0}{r} + k_1 r\right],$$

where $k_0 = 4 \times 10^{-18} km^{-1}$ and $k_1 = 10^{-57} km^{-3}$, which are estimated using rotation curves of galactic motion.

IV. Principle of Representation Invariance (PRI)

$SU(N)$ **gauge theory:** Certain physical properties of fermionic particles Ψ are not distinguishable under the $SU(N)$ transformations:

gauge fields: $A_\mu^a (a = 1, \cdots, N^2 - 1)$, Dirac fields: $\Psi = \left(\psi^1, \cdots, \psi^N\right)^t$

$$D_\mu = \partial_\mu + igA_\mu^a \tau_a, \quad F_{\mu\nu} = F_{\mu\nu}^a \tau_a = \frac{i}{g}[D_\mu, D_\mu] = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\lambda_{bc}^a A_\mu^b A_\nu^c)\tau_a$$

$$U(x) = e^{i\theta^a(x)\tau_a} \in SU(N),$$

$$\tilde{\Psi}(x) = U(x)\Psi(x), \qquad \tilde{A}_\mu^a \tau_a = \frac{i}{g}(\partial_\mu U)\Psi + UA_\mu^a \tau_a U^{-1}$$

where $\theta^a = \theta^a(x)$ $(1 \leq a \leq K)$ are real parameters, and the traceless and Hermitian matrices τ_a are generators of $SU(N)$ with $[\tau_a, \tau_b] = \tau_a\tau_b - \tau_b\tau_a = i\lambda_{ab}^c \tau_c$.

Principle of Representation Invariance (PRI) (Ma-Wang, 2012): Physical laws for an $SU(N)$ gauge theory should be independent of different representations of $SU(N)$:

- Transformation of the generators $\tau_a = \{\tau_1, \cdots, \tau_K\}$: $\tilde{\tau}_a = x_a^b \tau_b, \quad X = (x_a^b)$.
- θ^a, A_μ^a, and λ_{ab}^c are $SU(N)$-tensors under this transformation.
- $G_{ab} = \frac{1}{4N}\lambda_{ad}^c \lambda_{cb}^d$ is a symmetric positive definite 2nd-order covariant $SU(N)$-tensor, which can be regarded as a Riemannian metric on $SU(N)$.
- The representation invariant action and gauge field equations are

$$L = \int_M -\frac{1}{4} G_{ab} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^b + \bar{\Psi}\left[i\gamma^\mu(\partial_\mu + igA_\mu^a \tau_a) - m\right]\Psi,$$

$$\begin{cases} G_{ab}\left[\partial^\nu F_{\nu\mu}^b - g\lambda_{cd}^b g^{\alpha\beta} F_{\alpha\mu}^c A_\beta^d\right] - g\bar{\Psi}\gamma_\mu \tau_a \Psi = 0, \\ (i\gamma^\mu D_\mu - m)\Psi = 0 \qquad \text{Dirac eqs for fermions} \end{cases}$$

V. Unified Field Theory

The unified field model is derived based on the following principles:

- principles of general relativity and Lorentz invariance Einstein (1905, 1915)
- principle of gauge invariance, postulated by J. C. Maxwell (1861), O. Klein (1938), C. N. Yang & R. L. Mills (1954).
- spontaneous symmetry breaking by Y. Nambu 1960, Y. Nambu & G. Jona-Lasinio 1961
- principle of interaction dynamics (PID)
- principle of representation invariance (PRI)

Lagrangian action functional is the natural combination of the Einstein-Hilbert functional, the standard $U(1)$-QED, SU(2)-weak and $SU(3)$-strong interaction actions.

Unified field model (Ma-Wang, 2012):

(2) $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{8\pi G}{c^4}T_{\mu\nu} = \left[\nabla_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_a^w}{\hbar c}W_\mu^a - \frac{g_s\alpha_k^s}{\hbar c}S_\mu^k\right]\Phi_\nu,$$

(3) $$\partial^\nu F_{\nu\mu} - e\bar{\psi}\gamma_\mu\psi = \left[\nabla_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_a^w}{\hbar c}W_\mu^a - \frac{g_s\alpha_k^s}{\hbar c}S_\mu^k\right]\phi^E,$$

(4) $$G_{ab}^w\left[\partial^\nu W_{\nu\mu}^b - \frac{g_w}{\hbar c}\lambda_{cd}^b g^{\alpha\beta}W_{\alpha\mu}^c W_\beta^d\right] - g_w\bar{L}\gamma_\mu\sigma_a L = \left[\nabla_\mu + \frac{1}{4}\left(\frac{m_H c}{\hbar}\right)^2 x_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_b^w}{\hbar c}W_\mu^b - \frac{g_s\alpha_k^s}{\hbar c}S_\mu^k\right]\phi_a^w,$$

(5) $$G_{kj}^s\left[\partial^\nu S_{\nu\mu}^j - \frac{g_s}{\hbar c}\Lambda_{cd}^j g^{\alpha\beta}S_{\alpha\mu}^c S_\beta^d\right] - g_s\bar{q}\gamma_\mu\tau_k q = \left[\nabla_\mu + \frac{1}{4}\left(\frac{m_\pi c}{\hbar}\right)^2 x_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_a^w}{\hbar c}W_\mu^a - \frac{g_s\alpha_j^s}{\hbar c}S_\mu^j\right]\phi_k^s,$$

(6) $$(i\gamma^\mu\tilde{D}_\mu - \tilde{m})\Psi = 0.$$

Conclusions and Predictions of the Unified Field Model

1. Duality: The unified field model induces a **natural duality**:

$$
(7) \qquad \begin{aligned}
&\{g_{\mu\nu}\} \text{ (massless graviton)} && \longleftrightarrow \Phi_\mu, \\
&A_\mu \quad \text{(photon)} && \longleftrightarrow \phi^E, \\
&W^a_\mu \quad \text{(massive bosons } W^\pm \text{ \& } Z) && \longleftrightarrow \phi^w_a \quad \text{for } a = 1, 2, 3, \\
&S^k_\mu \quad \text{(massless gluons)} && \longleftrightarrow \phi^s_k \quad \text{for } k = 1, \cdots, 8.
\end{aligned}
$$

2. Decoupling and Unification: An important characteristics is that the unified model can be easily decoupled. Namely, Both PID and PRI can be applied directly to individual interactions.

For gravity alone, we have derived modified Einstein equations, leading to a unified theory for dark matter and dark energy.

3. Spontaneous symmetry breaking and mass generation mechanism: We obtained a much simpler mechanism for mass generation and energy creation, completely different from the classical Higgs mechanism. This new mechanism offers new insights on the origin of mass.

4. The two $SU(2)$ and $SU(3)$ constant vectors $\{\alpha_a^w\}$ and $\{\alpha_k^s\}$, containing 11 parameters, represent the portions distributed to the gauge potentials by the weak charge g_w and strong charge g_s. We define, e.g., the total potential S_μ:

$$
\begin{aligned}
S_\mu = \alpha_k^s S_\mu^k &= \{S_0, \qquad\qquad S_1, S_2, S_3\} \\
&= \{\text{strong charge potential}, \qquad \text{strong rotational potential}\}.
\end{aligned}
$$

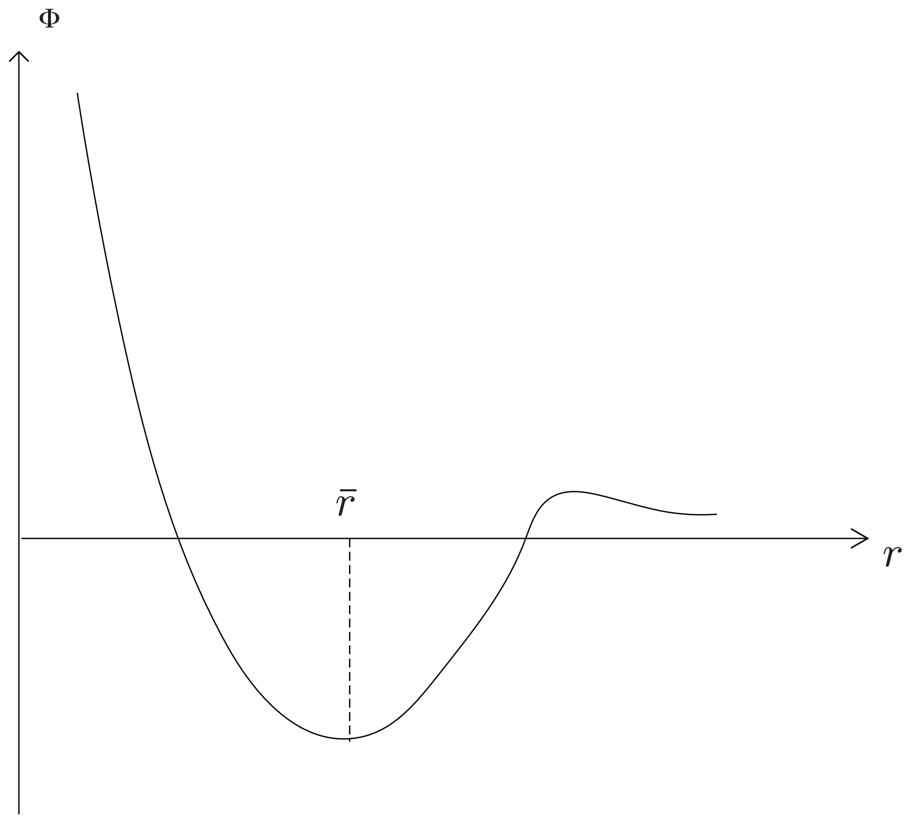
5. Strong and weak interaction potentials: For the first time, we derive three levels of strong interaction potentials: the quark potential S_q, the nucleon potential S_n and the atom/molecule potential S_a:

(8) $$S_q = g_s \left[\frac{1}{r} - \frac{A}{\rho_0}(1 + \frac{r}{r_0})e^{-r/r_0} \right],$$

(9) $$S_n = 3 \left(\frac{\rho_0}{\rho_1} \right)^3 g_s \left[\frac{1}{r} - \frac{A_n}{\rho_1}(1 + \frac{r}{r_1})e^{-r/r_1} \right],$$

(10) $$S_a = 3N \left(\frac{\rho_0}{\rho_2} \right)^3 g_s \left[\frac{1}{r} - \frac{A_n}{\rho_2}(1 + \frac{r}{r_1})e^{-r/r_1} \right],$$

where $g_s^2 = \frac{1}{12\sqrt{e}} 10^{16} \hbar c$, $A_n = 8\sqrt{e} 10^{-13}$ and A are constants, $r_0 = 10^{-16}$cm, $r_1 = 10^{-13}$cm, ρ_0 is the effective quark radius, ρ_1 is the radius of a nucleon, ρ_2 is the radius of an atom/molecule, and N is the number of nucleons in an atom/molecule.



These potentials match very well with experimental data, and offer an explanation of e.g. quark confinement, asymptotic freedom, and the short-range nature of strong and weak interactions.

Weak Interaction potential:

$$W = N g_w \left(\frac{\rho_w}{\rho}\right)^3 e^{-r/r_0} \left[\frac{1}{r} - \frac{B}{\rho}(1 + \frac{2r}{r_0})e^{-r/r_0}\right]$$

where $r_0 = 10^{-16}$, N is the number of weak charges, ρ is the radius of the particle, ρ_w is the radius of the weaktons, and B is a constant.