On Inviscid Limits for the Stochastic Navier-Stokes Equations and Related Questions

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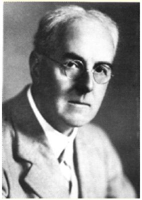
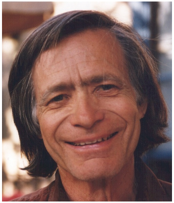
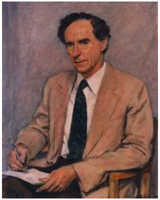
Recent Publications

- P. Constantin, N. Glatt-Holtz, V. Vicol *Unique Ergodicity for Fractionally Dissipative, Stochastically Forced 2D Euler Equations* (Communications in Mathematical Physics - 2013)
- N. Glatt-Holtz, V. Sverak, V. Vicol, *On Inviscid Limits for the Stochastic Navier-Stokes Equations and Related Models* (2013)
- S. Friedlander, N. Glatt-Holtz, V. Vicol, *Inviscid Limits for a Stochastically Forced Shell Model of Turbulent Flow* (to appear)
- J. Földes, N. Glatt-Holtz, G. Richards, E. Thomann, *Ergodic and Mixing Properties of The Boussinesq Equations with a Degenerate Random Forcing* (2013) (to appear)

Outline of the Talk

- Hypoellipticity and turbulence.
- The stochastic Boussinesq equations with degenerate forcing.
- Inviscid limits for 2D stochastic Navier Stokes: scaling and balance relations.
- Results on the L^∞ support of inviscid measures; Moser iteration for nonlinear SPDEs
- Some elements of the dynamical theory for 2D Euler on L^∞.
- Inviscid limits for the damped SNSE and other scalings.
- Unique ergodicity for the fractionally dissipative stochastic Euler.
- Shell models for turbulent flow and the anomalous dissipation of energy.

Statistical Theories of Turbulence

- Given the complexity observed in turbulent fluid flows it is unreasonable to try to predict ‘pathwise behavior’.
- Identifying universal statistical features may be more tractable.
- L.F. Richardson’s cascades: identify how energy in turbulent fluid is partitioned between different spatial (and temporal) scales.
- A. Kolmogorov (1941) based on dimensional analysis arguments predicted the energy-spectrum, two point correlations and smallest scales of motion, for fluids at high Reynolds number ($Re = UL/\nu$).
- A theory for 2D fluids has been developed in the late 1960’s by Kraichnan-Batchelor with analogous (but different) predictions.

Mathematical Challenges from Turbulence

... and the use of white noise driven forcings in the governing equations of fluid dynamics

- Identify 'universal' statistical features in the basic equations of fluid dynamics (i.e. uniqueness of statistically invariant solutions of the Navier-Stokes, Euler, damped Euler and related geophysical equations).
- Establish the 'observability' of these statistical features (i.e. ergodic and mixing properties of steady states)
- Establish rigorously various ansatz of turbulence (i.e. the anomalous dissipation of energy or enstrophy in the inviscid limit)

These problems are extremely difficult and remain (mostly) open even in 2D.

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- Starting in the 1960's one of the original motivations for studying stochastic NSE (and related systems) comes from the numerical simulation and theoretical study of turbulent fluid flow. White (in time) noise forcing was proposed going back to Novikov, to model a 'generic forcing'.
- More progress has been made for these questions in this stochastic setting.

$$du + (Au + N(u))dt = \sigma dW = \sum_{k=1}^{d} \sigma_k dW^k, \quad u(0) = u_0 \in H \tag{1}$$

- We define

$$P_t(u_0, A) = \mathbb{P}(u(t, u_0) \in A), \quad u_0 \in H, A \in \mathcal{B}(H) \quad \text{(Transition functions).}$$
$$P_t\phi(u_0) = \mathbb{E}\phi(u(t, u_0)); \quad \phi \in M_b(H) \quad \text{(Markov semigroup).}$$

- An *invariant measure* is an element in $\mu \in Pr(H)$ which is a fixed point of the dual semigroup P_t^*:

$$\int_H P_t(u_0, A) d\mu(u_0) = \mu(A), \text{ for all } A \in \mathcal{B}(H).$$

  As such invariant measures represent a *statistically steady states* of (1).
- We now have the mathematical goal of characterizing the existence, uniqueness and attraction properties of these invariant states. Of course this is all heavily dependent on the structure of σdW.

Invariant measures: Uniqueness

$$du + (Au + N(u))dt = \sigma dW = \sum_k^d \sigma_k dW^k, \quad u(0) = u_0 \in H$$

$$P_t\phi(u_0) := \mathbb{E}\phi(u(t,u_0)) \quad P_t^*\mu(A) := \int_H \mathbb{P}(u(t,u_0) \in A)d\mu(u_0)$$

Unlike existence, the uniqueness of invariant measures ($P_t^*\mu = \mu$) is a subtle issue. One needs to establish:

- Smoothing properties of P_t: This can be rephrased as a question of the Ellipticity or Hypoellipticity of the Kolmogorov Equation. Recall that $P_t\phi$ (formally) solves

$$\partial_t V = \frac{1}{2}Tr[(\sigma\sigma^*)D^2V] + \langle Au + N(u), DV\rangle, \quad V(0) := \phi \tag{2}$$

- A common state can be reached by the dynamics regardless of initial conditions (Irreducibility)

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[Flandoli & Maslowski, DaPrato & Zabczyk, E & Mattingly, Mattingly, Bricmont, Kupiainen & Lefevere, Kuksin & Shirikyan, Mattingly & Pardoux, Hairer & Mattingly, Kupiainen, Debussche . . .]

Hypoellipticity in Infinite Dimensional Systems

In a series of recent works Hairer & Mattingly ('06)–('11) developed a theory of Hypoellipticity for infinite dimensional systems of the general form

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They observe

(i) Time asymptotic smoothing is sufficient to produce unique ergodicity and mixing results. This is the 'asymptotic strong Feller' condition:

$$\|\nabla P_t \phi(u_0)\| \leq C(u_0) \left(\sup_{u \in H} |\phi(u)| + \delta(t) \sup_{u \in H} \|\nabla \phi(u)\| \right), \tag{4}$$

with $\delta(t) \to 0$ as $t \to \infty$.

(ii) An infinite dimensional analogue of the Hörmander bracket condition is sufficient to establish (4).

Of course verifying this Hörmander condition requires on a detailed understanding of the interaction between the nonlinear N and noise σ terms in (3).

The Stochastic Boussinesq Equation

$$d\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi - \nu_1\Delta\boldsymbol{u})dt = \mathbf{g}\theta dt, \quad \nabla\cdot\boldsymbol{u} = 0, \tag{5}$$

$$d\theta + (\boldsymbol{u}\cdot\nabla\theta - \nu_2\Delta\theta)dt = \sigma_\theta dW. \tag{6}$$

- Stochastic perturbation may be used to model thermal fluctuations in the fluid (for example in the earths mantle).
- A model of the 3D axis symmetric Navier-Stokes equation.
- Our main interest was to consider a model where the degeneracy of the stochastic perturbation is even more indirect in comparison to situation considered for the stochastic Navier-Stokes equations by Hairer and Mattingly.

$$\frac{d\omega_k}{dt} + \sum_{l+m=k}\langle l, m^\perp\rangle\left(\frac{1}{|l|^2} - \frac{1}{|m|^2}\right)\omega_l\omega_m + \nu|k|^2\omega_k = -ig\cdot k\theta_k,$$

$$d\theta_k + \left(-\sum_{l+m=k}\frac{\langle l, m^\perp\rangle}{|m|^2}\theta_l\omega_m + \eta|k|^2\theta_k\right)dt = \mathbf{1}_{k\in\mathcal{Z}}dW^k.$$

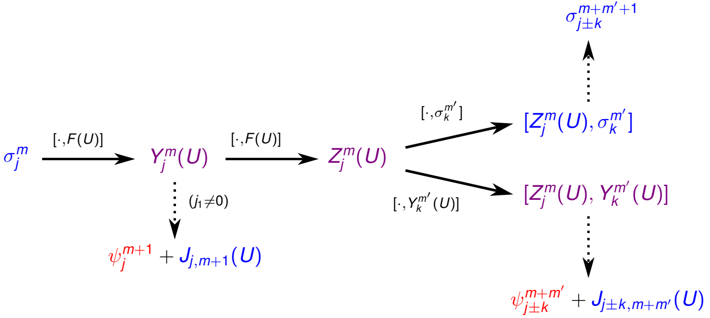
The Bracket Structure

$$d\omega + (\boldsymbol{u}\cdot\nabla\omega - \nu_1\Delta\omega)dt = g\partial_x\theta dt, \quad \boldsymbol{u} = K * \omega$$

$$d\theta + (\boldsymbol{u}\cdot\nabla\theta - \nu_2\Delta\theta)dt = \sigma_\theta dW = \sum_{k=1,2,l=0,1} \sigma_k^l dW^{k,l}.$$

$$\sigma_k^0(x) := (0, \cos(k\cdot x))^T, \quad \sigma_k^1(x) := (0, \sin(k\cdot x))^T$$

$$\psi_k^0(x) := (\cos(k\cdot x), 0)^T, \quad \psi_k^1(x) := (\sin(k\cdot x), 0)^T.$$



$$d\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi - \nu_1\Delta\boldsymbol{u})dt = \mathbf{g}\theta dt, \quad \nabla\cdot\boldsymbol{u} = 0,$$
$$d\theta + (\boldsymbol{u}\cdot\nabla\theta - \nu_2\Delta\theta)dt = \sigma_\theta dW.$$

Theorem (Földes, G-H, Richards, Thomann (2013))

Suppose that

$$\sigma_\theta dW = \cos x_1 dW^1 + \sin x_1 dW^2 + \cos x_2 dW^3 + \sin x_2 dW^4,$$

Let $U = (\nabla^\perp\cdot\mathbf{u},\theta)$. *Then there exists a unique invariant ergodic probability measure* μ *on* $(L^2(\mathbb{T}^2))^2$. *Moreover, for any suitably regular observable* Φ *on* $(L^2(\mathbb{T}^2))^2$ *and any initial condition* U_0

$$\left|\mathbb{E}\Phi(U(t,U_0)) - \int_{(L^2(\mathbb{T}^2))^2}\Phi(U)d\mu(U)\right| \le C_{\Phi,U_0}\exp(-\eta t)$$
$$\frac{1}{T}\int_0^T\Phi(U(t,U_0)) \to \int_{(L^2(\mathbb{T}^2))^2}\Phi(U)d\mu(U) \quad \textit{in probability}$$

Inviscid Limits and Invariant Measures

$$d\omega + (\boldsymbol{u} \cdot \nabla\omega - \nu\Delta\omega)dt = \alpha\sigma dW \quad \text{(where } \boldsymbol{u} = \nabla^{\perp}\psi, \Delta\psi = \omega\text{)}.$$

$\{\mu_\nu\}_{\nu>0}$ with μ_ν the unique invariant measure for the 2D SNSE_ν

What happens when $\nu \downarrow 0$.

- The inviscid limit in the class of invariant measures should reflect the long time dynamics of 2D Euler and of (2D) turbulence.
- Scaling between the stochastic forcing the the kinematic viscosity is crucial.
- This limit does not exhibit an anomalous dissipation of enstrophy ($\lim_{\nu\to 0} \frac{\nu}{L}\mathbb{E}\|\nabla\omega\|^2 = 0$). We have to be careful in 2D due to the inverse energy cascade.

Scaling and Balance Relations

$$d\omega + (\boldsymbol{u} \cdot \nabla\omega - \nu\Delta\omega)dt = \alpha\sigma dW$$

- In the noise scaling $\alpha = \sqrt{\nu}$ the invariant measures μ_ν satisfy the *'Balance relations'*:

$$\int \|\omega_0\|_{L^2}^2 d\mu_\nu(\omega_0) = \frac{1}{2}\sum_k \|\rho_k\|_{L^2(\mathbb{T})}^2,$$

$$\int \|\omega_0\|_{H^1}^2 d\mu_\nu(\omega_0) = \frac{1}{2}\sum_k \|\sigma_k\|_{L^2(\mathbb{T})}^2, \tag{7}$$

  which follow from stationarity and the Itō lemma.
- Due to (7) $\{\mu_\nu\}_{\nu>0}$ is tight and thus possesses a subsequential limit μ_0.
- This scaling in unique for the stochastic NSE. More precisely, if we scale ν^γ for $\gamma > 1/2$ then $\mu_0 = \delta_0$ and if $\gamma < 1/2$ then $\{\mu_\nu\}_{\nu>0}$ does not have any convergent subsequences.

Theorem (Kuksin ('04)–('08))

Suppose that $\{\mu_\nu\}_{\nu>0}$ *is a sequence of invariant measures for*

$$d\omega + (\mathbf{u}\cdot\nabla\omega - \nu\Delta\omega)dt = \sqrt{\nu}\sigma dW.$$

Then $\{\mu_\nu\}_{\nu>0}$ *is tight on* $Pr(L^2)$*. For any sub-sequential limit* μ_0

(i) μ_0 *is supported on* H^1 *and is an invariant measure for the free Euler equation*

$$\partial_t\omega + \mathbf{u}\cdot\nabla\omega = 0. \tag{8}$$

More precisely let $\mathcal{X} := \{\omega(0) : \omega \in \mathcal{K}_E\}$ *where*

$$\mathcal{K}_E := \{\omega \in W^{1,1}(\mathbb{R}; H^0) \cap L^2_{loc}(\mathbb{R}, H^1) : \omega \textit{ solves } (8)\},$$

Then $\mu_0(S(t)^{-1}A) = \mu_0(A)$ *for* $A \in \mathcal{B}(\mathcal{X})$ *where* $S(t)\omega_0 = \omega(t, \omega_0)$*.*

(ii) *Under mild non-degeneracy assumptions on* σ*, the support for* μ_0 *is 'non-trivial': in particular* $\mu_0(\{\omega\}) = 0$ *for all* $\omega \in H$*.*

- We observe from (i) that μ_0 does not support vortex patch solutions.
- Natural Question: Are inviscid measures supported on L^∞?

Inviscid measures supported on L^∞

Theorem (G-H, Sverak, Vicol ('13))

Let $\{\mu_\nu\}_{\nu>0}$ *be a sequence of invariant measures for*

$$d\omega + (\mathbf{u}\cdot\nabla\omega - \nu\Delta\omega)dt = \sqrt{\nu}\sigma dW \quad \text{(where } \mathbf{u} = \nabla^\perp\psi, \Delta\psi = \omega\text{).}$$

There exists a subsequence and a measure μ_0 *such that* $\mu_{\nu_j} \rightharpoonup \mu_0$ *(weakly) in* $Pr(L^2)$ *as* $j \to \infty$ *and*

$$\int \|\omega_0\|_{L^\infty} d\mu_0(\omega_0) \le C < \infty \qquad (*)$$

and in particular $\mu_0(L^\infty) = 1$.

Why L^∞?

$$\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = 0, \quad \boldsymbol{u} = \nabla^\perp (-\Delta)^{-1} \omega \tag{2D Euler}$$

- Since $\nabla \cdot \boldsymbol{u} = 0$, on smooth solutions: $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$, $1 \leq p \leq \infty$.
- The initial value problem for 2D Euler is globally well-posed for vorticity in L^∞ [Yudovich ('63)].
- For $R > 0$, we equip with the weak* topology the set $X = X_R$

$$X_R = \left\{ \omega \in L^\infty, \int_{\mathbb{T}^2} \omega = 0, \|\omega\|_{L^\infty} \leq R \right\}$$

  which is a compact metric space.
- Yudovich's proof yields a stronger result: the Euler equation yield a well-defined dynamical system on (X, w^*). See [Majda-Bertozzi ('02)].
- We prove that inviscid (Kuksin) measures are supported on L^∞ and thus give *natural* invariant measures for the Euler dynamical system on $X \cap H^1$.

Notions for the long-term dynamics of 2D Euler

- The Euler equation has many additional conserved quantities: for smooth F, the *Casimirs* $\mathcal{I}_F$ defined by

$$\mathcal{I}_F(\omega) = \fint_{\mathbb{T}^2} F(\omega)dx.$$

- The invariance of these functionals need to be taken into account for the dynamics of Euler on (X, w^*). Recall:

$$X = \left\{\omega \in L^\infty, \int_{\mathbb{T}^2} \omega = 0, \|\omega\|_{L^\infty} \leq R\right\}$$

- Staring from $\omega_0 \in X$ all of the known conserved quantities are embodied in

$$\mathcal{O}_{\omega_0,E} = \mathcal{O}_{\omega_0} \cap \{\omega \in X \colon \mathcal{E}(\omega) = E\}$$

$$\mathcal{O}_{\omega_0} = \{\omega_0 \circ h, h \colon \mathbb{T}^2 \to \mathbb{T}^2 \text{ is a volume-preserving } C^1 \text{ diffeomorphism}\}.$$

- As such, the orbit $\Omega_+(\omega_0) = \overline{\cap_{t>0}\{\omega(s) \colon s \geq t, \omega(0) = \omega_0\}}^{w^*}$ obeys

$$\Omega_+(\omega_0) \subset \overline{\mathcal{O}_{\omega_0,E}}^{w^*}, \quad E = \mathcal{E}(\omega_0).$$

$$\mathcal{O}_{\omega_0,E} = \mathcal{O}_{\omega_0} \cap \{\omega \in X \colon \mathcal{E}(\omega) = E\} \tag{9}$$

$$\mathcal{O}_{\omega_0} = \{\omega_0 \circ h, h \colon \mathbb{T}^2 \to \mathbb{T}^2 \text{ is a volume-preserving } C^1 \text{ diffeomorphism}\}$$

- Inspired by equilibrium Statistical Mechanics, [Onsager ('49)] and more recently [Miller ('90), Robert ('91), Schnirelman ('93), Turkington ('99), Sverak ('12)...] explore the long term dynamics in the context of maximizing the *mixing* in the flow, subject to the constraint (9); e.g., enstrophy

$$\text{minimize } \mathcal{I}(\omega) = \int_{\mathbb{T}^2} |\omega|^2 dx \text{ subject to constraint } \omega \in \overline{\mathcal{O}_{\omega_0,E}}^{w^*}$$

- More generally: maximize $\mathcal{I}_F$ with F concave, subject to (9).
- These "entropies" $\mathcal{I}_F$ are usually not weakly* continuous and hence may increase on *end states*.

Mixing ansatz and end-states of Euler dynamics

$$\mathcal{O}_{\omega_0,E} = \mathcal{O}_{\omega_0} \cap \{\omega \in X \colon \mathcal{E}(\omega) = E\}$$

$$\mathcal{O}_{\omega_0} = \{\omega_0 \circ h, h \colon \mathbb{T}^2 \to \mathbb{T}^2 \text{ is a volume-preserving } C^1 \text{ diffeomorphism}\}$$

$$\text{maximize } \mathcal{I}_F(\omega) = \int_{\mathbb{T}^2} F(\omega)dx \text{ subject to constraint } \omega \in \overline{\mathcal{O}_{\omega_0,E}}^{w^*}$$

- These statistical mechanics considerations (based on maximizing the 'entropy' $\mathcal{I}_F$ subject to conservative constraints) suggest that the end-states of the evolution are solutions to

$$\Delta\psi = G(\psi)$$

  for suitable, often nonlinear, functions G which depends on F.
- It is however NOT known whether solutions of 2D Euler converge to these entropy-maximizing steady states. In fact we don't have a single example where the trajectory is not pre-compact in L^2.

Possible consequences for Kuksin measures

Let us consider two *extreme* scenarios for the Inviscid measures μ_0:

- Scenario A: all solutions weakly* approach the entropy maximizers, which are shear flows.
  - Then one can expect the Kuksin measures to be supported on these shear flows. Indeed [Foldes-Sverak '13: $E << 1$ and F strictly concave it is shown rigorously that entropy maximizers are shear flows]. All the complexity of Euler escapes to infinity (in frequency) and never returns. There is no analogue of this scenario in finite dimensions or for completely integrable systems. See [Bouchet & Simonet '09] for some "numerical evidence".
- Scenario B: all solution orbits are pre-compact in L^2.
  - Here the weak and strong closures of the trajectories coincide, and hence the $\mathcal{I}_F$'s are conserved on the end-states and the mixing predicted by Statistical Mechanics doesn't take place. [Sverak ('12): The omega limit set Ω_+ of every trajectory contains an element ω_0, whose trajectory is pre-compact in L^2.]

One can conjecture that the answer lies somewhere in the middle...

Main Ingredient: Stochastic Moser Iteration

Theorem (G-H, Sverak, Vicol ('13))

Consider the linear stochastic drift-diffusion equation

$$d\omega + (\boldsymbol{a} \cdot \nabla\omega - \Delta\omega)dt = \sigma dW, \quad \nabla \cdot \boldsymbol{a} = 0$$

where the drift velocity $\boldsymbol{a}$ *and the stochastic forcing* σ *are smooth. If* $\omega \in L^4(\Omega; L^4(0, 2T; L^2))$ *for some* $0 < T$*, then* ω *instantly becomes bounded in the spatial variable and we have the bound*

$$\mathbb{E} \sup_{t \in [T, 2T]} \|\omega(t)\|_{L^\infty} \leq C(1 + T^{-5/4})\mathbb{E}\left(\|\omega\|_{L^4(0,2T;L^2)} \vee \|\sigma\|_{L^\infty}\right)$$

for some $C < \infty$*, that is independent of the drift* $\boldsymbol{a}$*.*

- Moser iteration for SPDE: L^∞ maximum principle of Denis, Matoussi, Stoica ('09) assumes $\omega_0 \in L^p(\Omega; L^\infty)$.
- Nash/DeGiorgi estimates don't seem to be appropriate for SPDE.
- Our proof works with $-\Delta$ replaced by $(-\Delta)^\gamma$ for *any* $\gamma > 0$.
- No L^∞ drift independent bounds for $d\omega + \boldsymbol{a} \cdot \nabla\omega dt = \sigma dW$.

The Stochastic Moser Iteration Scheme

- For $p \geq 2$ we apply the L^p Itō Lemma to $d\omega + (\boldsymbol{a} \cdot \nabla \omega - \Delta \omega)dt = \sigma dW$. Using $\nabla \cdot \boldsymbol{a} = 0$ we obtain

$$d\|\omega\|_{L^p}^p + \tfrac{1}{C}\|\omega\|_{L^{2p}}^p \leq \frac{p(p-1)}{2} \sum_m \int_{\mathbb{T}^2} |\omega(t,x)|^{p-2} \sigma_m(x)^2 dxdt + p \int_{\mathbb{T}^2} \sigma_m(x)\omega(t,x)|\omega(t,x)|^{p-2} dxdW^m$$

- Fix $T > 0$, $T_0 = 0$, $T_k \to T$ as $k \to \infty$, and let $I_k = [T_k, 2T]$.
- Interpolation gives

$$\frac{1}{C}\left(\|\omega\|_{L^\infty(I_{k+1};L^p)}^p + \|\omega\|_{L^2(I_{k+1};L^{2p})}^p\right) \geq \frac{1}{C}\|\omega\|_{L^{2\lambda p}(I_{k+1};L^{\lambda p})}^p.$$

- After estimates which carefully track constant dependencies:

$$\mathbb{E}\left(\|\omega\|_{L^{2\lambda p}(I_{k+1};L^{\lambda p})} \vee \|\sigma\|_{L^\infty}\right) \leq \kappa(p,T)^{\frac{1}{p}} C_{BDG}(p^{-1})\mathbb{E}\left(\|\omega\|_{L^{2p}(I_k;L^p)} \vee \|\sigma\|_{L^\infty}\right)$$

Some (Hard) Open Questions for Inviscid Measures

- Are the measures μ_0 is supported on a space which consists of "smoother" functions, such as C^0 or even C^α for some $\alpha > 0$?
- Do the inviscid measures *select* certain *long-term dynamics* of Euler as a dynamical system?
  - Is μ_0 supported on time-independent solutions?
  - To what extent does the structure of σ matter?
- Uniqueness of μ_0 is of course unclear (and may not be expected).

Damped-Driven Stochastic NSEs

Significance of Damping Terms in 2D Turbulences and Geophysical Applications

$$d\boldsymbol{u} + Y\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi - \nu\Delta\boldsymbol{u} = \rho dW, \ \nabla\cdot\boldsymbol{u} = 0,$$

- From the point of view of 2D Turbulence, on bounded domains there is no anomalous dissipation of enstrophy without a damping operator Y.
- One could also consider the regular 2D SNSEs on the *whole space* but here the existence/uniqueness of invariant measures is open.
- In geophysical applications often $Y\boldsymbol{u} = \tau\boldsymbol{u}$ (zeroth order) models friction with boundaries. For this case however one can show that there is no anomalous enstrophy dissipation rate: $\eta = 0$ [Constantin-Ramos ('07), Bessaih-Ferrario ('13)].
- Instead, for $Y\boldsymbol{u} = \tau(\chi_{|k|\leq\kappa_0}\hat{\boldsymbol{u}}(k))^{\vee}$, or $Y\boldsymbol{u} = \tau(-\Delta)^{-\gamma}\boldsymbol{u}$ and more. In these cases one so far cannot exclude that $\eta > 0$.

Theorem (G-H, Sverak, Vicol ('13))

Consider the damped-driven stochastic Navier-Stokes equations

$$d\boldsymbol{u} + Y\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi - \nu\Delta\boldsymbol{u} = \nu^{\alpha}\rho dW, \ \nabla\cdot\boldsymbol{u} = 0, \tag{10}$$

Let $Y = \tau(-\Delta)^{-\gamma/2}$ *and* $\tau > 0$, $\gamma \in [0, 1)$. *For* $\alpha \in \mathbb{R}$ *consider a collection of invariant measure* $\{\mu_\nu^\alpha\}_{\nu>0}$ *of* (10).

- *If* $\alpha > 0$, *then for any* $\nu_j \to 0$, $\mu_{\nu_j}^\alpha \rightharpoonup \delta_0$
- *If* $\alpha < 0$, *then for any* $\nu_n \to 0$ *such that* $\mu_{\nu_n}^\alpha \rightharpoonup \mu_0$ *then* $\int_{L^2} \|\boldsymbol{u}\|_{L^2}^2 d\mu_0(\boldsymbol{u}) = \infty$.
- *If* $\alpha = 0$, *there exists* $\nu_n \to 0$ *such that* $\mu_{\nu_n}^0 \rightharpoonup \mu_0$ *and with* μ_0 *a stationary martingale solution of* (10) *with* $\nu = 0$.

Ergodic Theory for Damped Stochastic Euler?

$$du + Yu + u \cdot \nabla u + \nabla \pi = \rho dW, \; \nabla \cdot u = 0, \tag{11}$$

where $Y = (-\Delta)^{-\gamma/2}$ and, $\gamma \in [0, 1)$.

- Existence of stationary states in the weak-martingale setting ($u \in L_t^\infty L_x^2$). Not clear how to obtain uniqueness or ergodic properties.
- Yudovich solutions $\omega_0 = \nabla^\perp \cdot u \in L^\infty$ can be obtained for (11). L^∞ with the weak* topology is not a Polish space.
- Not clear how to obtain decay for H^r norms $r > 2$.

We instead consider $Y = (-\Delta)^{\gamma/2}$ for $\gamma > 0$ and ask initially:

> *What is the weakest wave number dependence such that* (11) *possess a unique ergodic invariant measure?*

Unique Ergodicity for Fractionally Dissipative Euler

For any $\gamma > 0$, we consider the Fractionally dissipative model

$$d\omega + ((-\Delta)^{\gamma/2}\omega + \boldsymbol{u} \cdot \nabla\omega)dt = \sigma dW, \quad \boldsymbol{u} = K * \omega. \tag{12}$$

When $r > 2$ (12) has a unique pathwise solution $\omega(t, \omega_0) \in L^\infty_{loc}([0, \infty), H^r)$ for each $\omega_0 \in H^r$. We define the Markov semigroup on H^r

$$P_t\phi(\omega_0) = \mathbb{E}\phi(\omega(t, \omega_0)), \quad \phi \in M_b(H^r). \tag{13}$$

Theorem (Constantin, G-H, Vicol '13)

Fix any $\gamma > 0$ and suppose $H_N \subset range(\sigma)$ for $N = N(\gamma, \|\sigma\|_{H^r})$. Then, P_t is Feller and satisfies the gradient estimate

$$\|\nabla P_t\phi(\omega_0)\| \le C\left(1 + \|\omega_0\|^q_{H^r}\right)\exp(C\|\omega_0\|^2_{L^p})\left(\sup_{\omega\in H^r}|\phi(\omega)| + \delta(t)\sup_{\omega\in H^r}\|\nabla\phi(\omega)\|\right),$$

for every $\phi \in C^1_b(H)$ where $\delta(t) \to \infty$ as $t \to 0$. Here $C > 0$, $q, p \ge 2$ are universal constants depending only on $\gamma > 0$ and $r > 2$. Moreover (12) possesses a unique ergodic invariant measure.

Challenges for the Fractionally Dissipative Model

- Choosing the phase space is tricky. Well-posedness only in higher order Sobolev spaces H^r, $r > 2$. Due to a lack of cancelations in these spaces even the Feller property requires some technically involved arguments (stopping times and parabolic smoothing estimates).
- Moment in higher order Sobolev spaces require the use of bootstrapping, parabolic smoothing estimates and interpolation arguments. Such moments are needed even for the existence and regularity of invariant measures.
- Gradient estimates require delicate the use of interpolation arguments.

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Our main new idea is to explicitly use the parabolic smoothing inherent in the equations beyond the fact that it induces decay and contraction of the phase space

Parabolic Smoothing Estimates:

Theorem

Fix $r > 2$, $\gamma > 0$, and let $m \geq 0$ and $T_m > 0$ be arbitrary. Define

$$\alpha(t) = \begin{cases} mtT_m^{-1}, & t \in [0, T_m], \\ m, & t > T_m. \end{cases}$$

Then, for any $T > 0$ and any $q \geq 2$ we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\Lambda^{r+\alpha(t)}\omega(t)\|_{L^2}^q + \int_0^T \|\Lambda^{r+\gamma/2+\alpha(t)}\omega(t)\|_{L^2}^2 \|\Lambda^{r+\alpha(t)}\omega(t)\|_{L^2}^{q-2} dt \right)$$
$$\leq C(\mathbb{E}\|\omega_0\|_{H^r}^p) + CT(\|\sigma\|_{\mathbb{H}^{r+m}}^p),$$

where $C = C(q, T_m, m)$ $p = p(q, T_m, m) \geq 2$ are independent of T.

- First obtain moments in L^p where have cancellations. Here we need a fractional L^p Poincaré inequality.
- Use these to obtain high H^1 moments using commutator estimates.
- Climb in regularity from H^1 to $H^{r+\alpha(t)}$ since H^1 is already subcritical.

Gradient Estimates for the Markov Semigroup

Malliavin Calculus and the Control Problem

$$\|\nabla P_t \phi(\omega_0)\| \leq C(\omega_0) \left(\sup_{\omega \in H^r} |\phi(\omega)| + \delta(t) \sup_{\omega \in H^r} \|\nabla \phi(\omega)\| \right) \quad ?$$

- Using the machinery of Malliavin Calculus

$$\begin{aligned} \nabla P_t \phi(\omega_0) \cdot \xi =& \mathbb{E}\left(\nabla \phi(\omega(t,\omega_0)) \cdot (\nabla_{\omega_0} \omega(t,\omega_0) \cdot \xi)\right) \\ =& \mathbb{E}\left(\phi(\omega(t,\omega_0)) \int_0^t v dW \right) + \mathbb{E}\left(\nabla \phi(\omega(t,\omega_0)) \cdot \rho(t,\omega_0,\xi,v)\right) \end{aligned}$$

- We need to choose $v \in L^2_{loc}([0,\infty); L_2)$ to compensate for the perturbation in the initial condition. Here ρ satisfies the control problem

$$\partial_t \rho + \Lambda^\gamma \rho + \nabla B(\omega) \cdot \rho = \sigma v, \quad \rho(0) = \xi$$

Gradient Estimates for the Markov Semigroup:

The case of many forced modes.

- Using that $H_N \subset \text{range}(\sigma)$ we choose $\sigma v = \lambda_N^{\gamma/2} P_N \rho$. and obtain:

$$\partial_t \rho + \Lambda^\gamma \rho + B(\rho, \omega) + B(\omega, \rho) = \lambda_N^{\gamma/2} P_N \rho, \quad \rho(0) = \xi$$

- Using cancelations in H^{-1} and exponential moments for $\|\omega\|_{L^p}^2$ we find:

$$\frac{d}{dt}\|\rho\|_{H^{-1}}^2 \le \left(\kappa_\gamma \|\omega\|_{L^{6/\gamma}}^2 - \frac{\lambda_N^{\gamma/2}}{2} \right) \|\rho\|_{H^{-1}}^2$$

- On the other hand, smoothing estimates and interpolation imply

$$\frac{d}{dt}\|\Lambda^s \rho\|_{L^2} + \frac{\lambda_N^{\gamma/2}}{4}\|\Lambda^s \rho\|_{L^2} \le C \left(1 + \|\omega\|_{H^{s+1}}^{\frac{2(r+1+\gamma)}{\gamma}} \right) \|\Lambda^{-1}\rho\|_{L^2}$$

with

$$s(t) = \begin{cases} r - 1 + t\gamma, & t \in [0, T_\gamma], \\ r, & t > T_\gamma, \end{cases}$$

Shell Models in Turbulence

Recall that the Navier-Stokes equations can be expressed an infinite sequence of coupled ODEs on $\mathbb{Z}^3$:

$$d\boldsymbol{u}_{\mathbf{k}} + \left(\nu \frac{|\mathbf{k}|^2}{L^2}\boldsymbol{u}_{\mathbf{k}} + i\left(I - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\sum_{\mathbf{l}+\mathbf{m}=\mathbf{k}}\left(\boldsymbol{u}_{\mathbf{l}}\cdot\frac{\mathbf{m}}{L}\right)\boldsymbol{u}_{\mathbf{m}}\right)dt = \rho_{\mathbf{k}}dW. \qquad (14)$$

- Verifying the anomalous dissipation of energy, let alone deriving energy cascades from (14) is an outstanding (and seemingly impossibly difficult) open problem.
- Even in d =2 the analogous questions seem intractable.
- Shell models: Infinite sequences of coupled ODEs which greatly simplify the wave number interaction to provide a tractable framework for theoretical and numerical investigations of these questions.

The 'Dyadic' Model

$$\frac{d}{dt}u_0 + \nu u_0 + u_0 u_1 = f_0$$
$$\frac{d}{dt}u_j + \nu 2^{2j} u_j + (2^{cj} u_j u_{j+1} - 2^{c(j-1)} u_{j-1}^2) = 0 \quad j = 1, 2, \ldots$$

where c may take values in the (physically interesting) range $[1, 5/2]$.

- Derived for geophysical applications by Desnyansky & Novikov (1974).
- Independently discovered from a Littlewood-Paley analysis of the NSE by Katz & Pavlović (2005). See also Cheskidov, Constantin, Friedlander, and Shvydkoy.
- Subject of significant mathematical attention in the recent years: [see e.g. Friedlander & Pavlović, Cheskidov & Friedlander, Cheskidov & Friedlander & Pavlovic, D. Barbato & Flandoli & Morandin, Barbato & Morandin & Romito, Romito.]
- Other models have been extensively studied in the physics and mathematics literature: e.g. the GOY model (Gledzer-Okhitani-Yamada) and Sabra model (whose name is a play on words...) and Linear systems (Mattingly, Suidan, Vanden-Eijnden).

$$\begin{cases} du_0 + (\nu u_0 + u_0 u_1)dt = \alpha dW \\ \frac{d}{dt} u_j + \nu 2^{2j} u_j + (2^{cj} u_j u_{j+1} - 2^{c(j-1)} u_{j-1}^2) = 0 \quad j = 1, 2, \dots \end{cases} \tag{15}$$

Theorem (Friedlander, G-H, Vicol (2013))

Let $c \in [1, 5/2]$, $\alpha > 0$ be fixed parameters and for each $\nu > 0$ we let μ_ν be the law of a statistically stationary solution of (15)*.*

(i) *For any $c \in [1, 5/2]$ the sequence $\{\mu_\nu\}_{\nu>0}$ is tight (i.e. weakly compact) in ℓ^2. Moreover any weak limit μ_0 of $\{\mu_\nu\}_{\nu>0}$ gives the law of stationary (martingale) solution of* (15) *which exhibits a dissipation anomaly.*

(ii) *In the range $c \in [1, 2]$ and for all $\nu > 0$, μ_ν is unique, ergodic and mixing.*

(iii) *For such $c \in [1, 2]$ we have an anomalous dissipation of energy in the inviscid limit*

$$\lim_{\nu \to 0} \lim_{T \to \infty} \frac{\nu}{T} \int_0^T |u(t, u_0)|_{H^1}^2 dt = \frac{\alpha^2}{2},$$

Concluding Remarks

- Turbulence is a longstanding and ongoing motivation for the study of the stochastic Navier-Stokes Equations (and other related SPDEs equations).
- Invariant measures provide a mathematical setting to connect the equations to the statistical predictions of developed turbulence theories.
- Inviscid Limits for the 2D SNSEs in the class of invariant measures leads to non-trivial measures for Euler μ_0 but only in the $\sqrt{\nu}$ noise scaling.
- We establish that $\mu_0(L^\infty) = 1$ via Moser iteration techniques that illuminate a parabolic regularization property for a large class of SPDEs of drift-diffusion type.
- μ_0 can now be connected to the long term dynamics of the free Euler equations. Many open questions remain about the structure of μ_0 and its relationship to different scenarios about the Ω-limit sets of 2D Euler.
- Different scaling for the noise are apparent when damping terms are added $Y\omega$. These equations seem to be more relevant for 2D turbulence.
- Unique ergodicity for the stochastic damp Euler equations remains an outstanding open problem but we can address the case of arbitrary order fractional dissipation via 'bootstrapping techniques'.

Concluding Remarks

- Stochastic ‘shell models’ provide a mathematically and physically interesting setting to study properties of turbulent flows far from the reach of the analysis of the full fluids equations.
- Current work address inviscid limits and anomalous dissipation in a dyadic shell model with nearest neighbor wave number interactions.

Thank you!