A statistical mechanics approach to stochastic parametrizations

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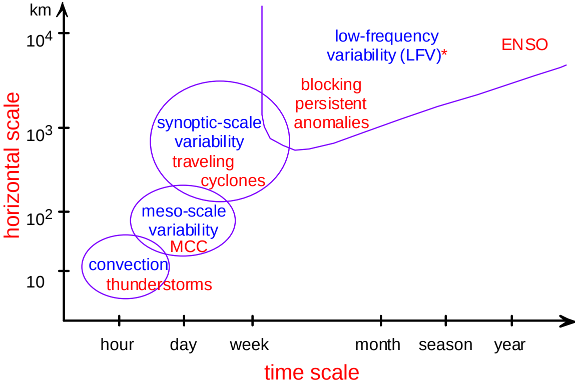
What is parametrization?

Parametrization = deriving reduced models

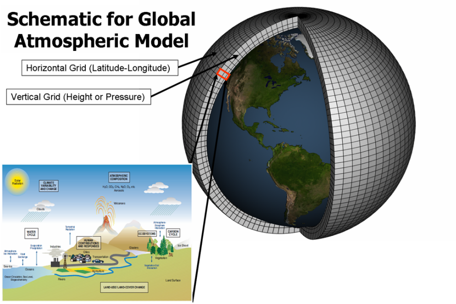
The climate system has variability over a large range of space and time scales[1]



[1]ECMWF 2005 workshop on stochastic-dynamic models, based on *Fraedrich* (1978) JAS

What is stochastic parametrization?

Climate models have a coarse resolution ($\sim$ 300 km)



Unresolved processes (clouds, convection, precip.) are effected by and effect resolved dynamics.

We need to represent unresolved processes in resolved dynamics

Similarities to statistical mechanics

Reduction of degrees of freedom is a central task of statistical mechanics, e.g. Brownian motion.

Hasselman (1976), "Stochastic climate models", Tellus

"The essential feature of stochastic climate models is that the non-averaged *"weather" components* are also retained. They appear formally as *random forcing* terms. The climate system, acting as an integrator of this short-period excitation, exhibits the same *random-walk* response characteristics as large particles interacting with an ensemble of much smaller particles in the analogous *Brownian motion* problem."

Multiscale methods

Applies to dynamical systems with a large time scale separation.

Averaging $\sim$ LLN

$$\begin{cases} \dot{x} &= f(x, y) \\ \dot{y} &= \frac{1}{\epsilon} g(x, y) + \frac{1}{\sqrt{\epsilon}} \beta(x, y) \dot{V} \end{cases}$$

For $\epsilon << 1$ and $t \in [0, T]$, $x(t) \to$ sol. of $\dot{X} = F(X)$, $F(X) = \langle f(X, y) \rangle_{\rho_{y|X}}$

Homogenisation $\sim$ CLT

$$\begin{cases} \dot{x} &= \frac{1}{\epsilon} f_0(x, y) + f_1(x, y) + \alpha(x, y) \dot{U} \\ \dot{y} &= \frac{1}{\epsilon^2} g(x, y) + \frac{1}{\epsilon} \beta(x, y) \dot{V} \end{cases}$$

For $\epsilon << 1$ and $t \in [0, T]$, weak convergence of x to the solutions of a reduced SDE with Gauss. white noise.[a]

[a] *Khasminskii, Kurtz, Papanicolaou,...* (60s, 70s), see also *Pavliotis & Stuart* (2008)

Stochastic averaging: applications

- Several low dimensional models:
  Monahan & Culina
  (2011) J. of Clim.
- From triad to barotropic flows:
  Majda, *Timofeyev*, *Vanden-Eijnden*, *Franzke*
  (1999) PNAS, (2001) Comm. Pure App. Math., (2003) J. Atmos. Sci., (2005) J. Atmos. Sci.
- 2D zonal jets:
  Bouchet, *Nardini*, *Tangarife*
  (2013) J Stat Phys
- Energy conserving multi-scale toy model of the atmosphere:
  Frank & Gottwald
  (2013) Physica D

Mori-Zwanzig: formal derivation

Non-linear ODE:

$$\dot{z} = f(z) \qquad z = (x, y)$$

Linear PDE (Liouville equation):

$$\dot{u}(t, z) = Lu(t, z) \qquad u(0, z) = A(z) \qquad L = f.\nabla$$

$$\dot{u}(t, z) = Le^{tL}u(0, z) = e^{tL}(\mathbb{P} + \mathbb{Q})LA(z)$$

$\mathbb{P}$ a projector on a space of x-observables, $\mathbb{Q} = 1 - \mathbb{P}$

Using a Duhamel-Dyson decomposition:

$$e^{tL} = e^{t\mathbb{Q}L} + \int_0^t d\tau\, e^{(t-\tau)L}\mathbb{P}Le^{\tau\mathbb{Q}L}$$

$$\dot{x}_i(t, x) = e^{tL}\mathbb{P}Lx_i + e^{t\mathbb{Q}L}\mathbb{Q}Lx_i + \int_0^t d\tau\, e^{(t-\tau)L}\mathbb{P}Le^{\tau\mathbb{Q}L}\mathbb{Q}Lx_i$$

Mori-Zwanzig projection operators

$$\dot{x}_i(t,x) = e^{tL}\mathbb{P}Lx_i + e^{t\mathbb{Q}L}\mathbb{Q}Lx_i + \int_0^t d\tau\, e^{(t-\tau)L}\mathbb{P}Le^{\tau\mathbb{Q}L}\mathbb{Q}Lx_i$$

- term 1: Markovian part
- term 2: fluctuating part: obeys orthogonal dynamics $\mathbb{Q}L$
- term 3: memory term

Note:

- Formal rewriting of the equations
- Demonstrates the presence of correlations and memory in reduced systems
- Further approximation is needed for application

Mori-Zwanzig: aproximations

- Short memory: white noise, no memory
- t-model: $e^{tQL} \rightarrow e^{tL}$ in the memory term[2]

Applications:

- Burgers equation: *Bernstein* (2007) Mult. Mod. Sim.
- Euler equation: *Hald & Stinis* (2007) PNAS
- Kuramoto-Sivashinski equation: *Stinis* (2004) Mult. Mod. Sim.

[2] *Chorin*, *Hald*, *Kupferman* (2002) Phys. D
Chorin & Hald "Stochastic Tools in Mathematics and Science" (2013) Springer

Weakly coupled systems and response

Dynamical system

$$\begin{cases} \dot{x} = f_x(x) + \epsilon\psi_x(y) \\ \dot{y} = f_y(y) + \epsilon\psi_y(x) \end{cases} \tag{1}$$

Linear response

$$\dot{z} = f(z) + \epsilon\delta f(z)$$

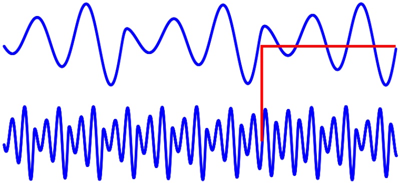
$$\delta\rho^{(1)}(A) = \frac{\partial\langle A\rangle_{\rho_\epsilon}}{\partial\epsilon} = \int_0^\infty d\tau \langle \delta f(x) \nabla e^{\tau L} A\rangle_{\rho_0} = \int_0^\infty d\tau R_{A,\delta f}(\tau)$$

Can one find a dyn. syst. $\tilde{x}$ such that

$$\langle A_x\rangle_{\tilde{\rho}} = \langle A_x\rangle_{\rho_\epsilon} + O(\epsilon^n)$$

Weak coupling: 1st order response

- Calculate response $\delta\rho^{(1)}(A_x)$ to coupling Ψ at first order



$$\delta^{(1)}\rho = \int_0^\infty d\tau \, \langle \langle \Psi_x(y) \rangle_{\rho_y} \nabla e^{\tau L} A_x \rangle_{\rho_x}$$
$$= \int_0^\infty d\tau \, \langle M \nabla e^{\tau L} A_x \rangle_{\rho_x}$$

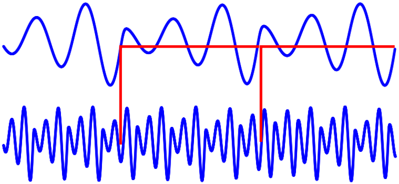
- Consider a reduced system

$$\dot{\tilde{x}} = f_x(\tilde{x}) + \epsilon M$$

then $\tilde{\rho}(A_x) = \rho(A_x) + O(\epsilon^2)$

Weak coupling: 2nd order (1/2)

- Second order response (1/2):



$$\left[\delta\rho^{(2)}\right]_1 \sim \langle \delta\Psi_x(y)\delta\Psi_x(f^s(y))\rangle_{\rho_y} = C_{\Psi_x;\Psi_x}(s)$$

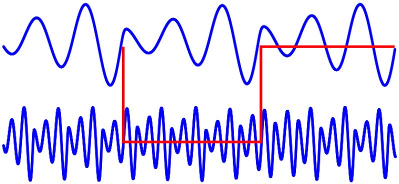
- parametrized by correlated noise

$$\dot{\tilde{x}} = f_x(\tilde{x}) + \epsilon M + \epsilon\sigma(t)$$
$$\langle\sigma(t)\rangle = 0$$
$$\langle\sigma(t)\sigma(t+s)\rangle = C_{\Psi_x;\Psi_x}(s)$$

Higher order moments are determined by higher order repsonse

Weak coupling: 2nd order (2/2)

- Second order response (2/2):



$$\left[\delta\rho^{(2)}\right]_2 \sim \langle \Psi_y(x).\nabla\Psi_x(f^s(y))\rangle_{\rho_Y} = R_{\Psi_x,\Psi_y(x)}(s)$$

- parametrized by a memory term

$$\dot{\tilde{x}} = f_x(\tilde{x}) + \epsilon M + \epsilon\sigma(t) + \epsilon^2 \int_0^\infty d\tau\, R_{\Psi_y,\Psi_x(\tilde{x}(t-\tau))}(s)$$

then $\tilde{\rho}(A_x) = \rho(A_x) + O(\epsilon^3)$

Numerical experiment (with S. Dolaptchiev & U. Achatz)

- Stochastically perturbed additive triad
- Fast Ornstein-Uhlenbeck process (y) coupled to a slow variable (x) by nonlinear coupling

$$\begin{cases} \frac{dx}{dt} = B_0 y_1 y_2 \\ \frac{dy_1}{dt} = B_1 y_2 x - \frac{\gamma_1}{\epsilon} y_1 + \frac{\sigma_1}{\sqrt{\epsilon}} \dot{U} \\ \frac{dy_2}{dt} = B_2 x y_1 - \frac{\gamma_2}{\epsilon} y_2 + \frac{\sigma_2}{\sqrt{\epsilon}} \dot{V} \end{cases} \rightarrow \begin{cases} \frac{dx}{d\tau} = \epsilon B_0 y_1 y_2 \\ \frac{dy_1}{d\tau} = \epsilon B_1 y_2 x - \gamma_1 y_1 + \sigma_1 \dot{U} \\ \frac{dy_2}{d\tau} = \epsilon B_2 x y_1 - \gamma_2 y_2 + \sigma_2 \dot{V} \end{cases}$$

$\tau = \frac{t}{\epsilon}$

- Homogenisation results in an Ornstein-Uhlenbeck process

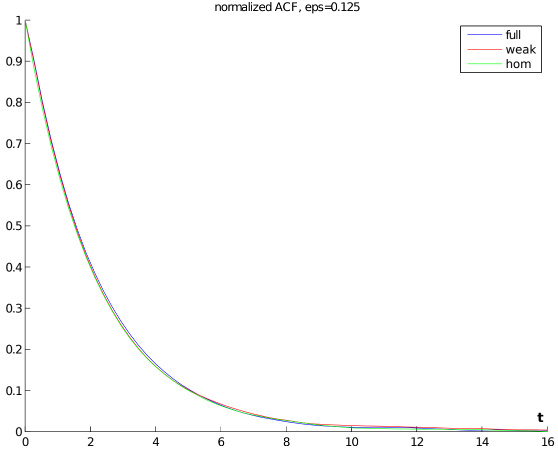
Numerical experiment: weak coupling

- The parametrization can be explicitly computed

$$
\begin{aligned}
C_{\Psi_x;\Psi_x}(s) &= \langle \Psi_x(y(t))\Psi_x(y(t+s))\rangle \\
&= (\epsilon B^{(0)})^2 \langle y_1(t) y_2(t) y_1(t+s) y_2(t+s)\rangle \\
&\sim e^{-(\gamma_1+\gamma_2)s} \\
R_{\Psi_x;\Psi_y}(s) &= \langle \Psi_y(x(t), y(t)) \nabla_y \Psi_x(y(t+s))\rangle \\
&= \langle \epsilon B^{(1)} y_2 x \partial_{y_1} (\epsilon B^{(0)} y_1(t+s) y_2(t+s))\rangle + (1 \leftrightarrow 2) \\
&\sim e^{-(\gamma_1+\gamma_2)s}
\end{aligned}
$$

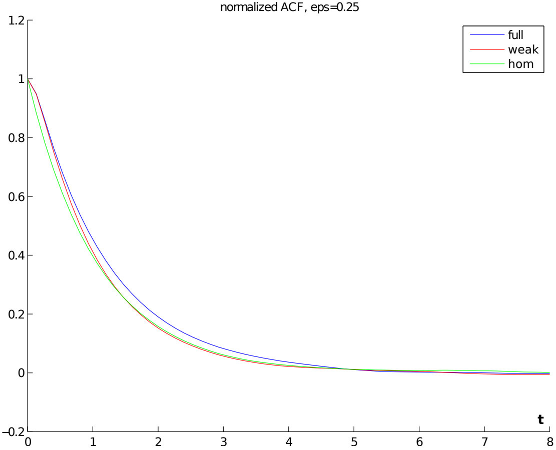
Numerical experiment: results

Autocorrelation of x for $\epsilon = 0.125$



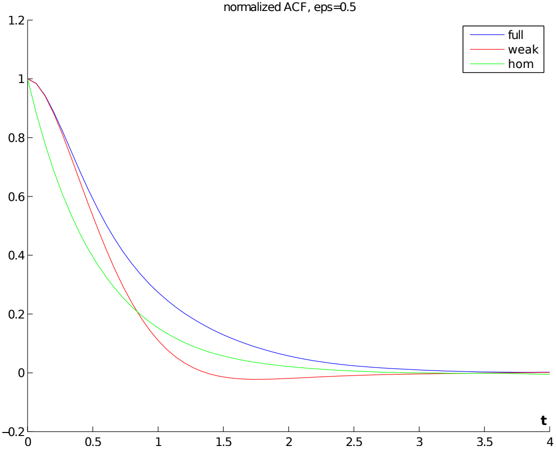
Numerical experiment: results

Autocorrelation of x for $\epsilon = 0.25$



Numerical experiment: results

Autocorrelation of x for $\epsilon = 0.5$



Conclusions

- In reduced models, *correlated noise* and *memory* appear
- Correlation function and memory kernel can be determined in the uncoupled Y system.
- Preliminary numerical experiments show encouraging results.

- Response: Wouters, Lucarini (2012) J. Stat. Mech.
- Mori-Zwanzig: Wouters, Lucarini (2013) J. Stat. Phys.

Thank you!