HYDRODYNAMIC TURBULENCE AS A PROBLEM IN NON-EQUILIBRIUM STAT MECH.

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Turbulent fluid as a physical system

Physical assumptions for hydrodynamics:

- incompressibility
- Newtonian mech
- viscous dissipation

$\Rightarrow$ *Hamiltonian dynamics*

Physical description of turbulence in 3 dimensions:

- localized structures (eddies)
- (direct) energy cascade $\Leftarrow$ *why?*
- small scale dissipation

Remarks

- Hamiltonian nature of dynamics discussed by Arnold, present in inertial range, but generally ignored.
- Dissipation described in *NS* by 1-st order perturbation theory term: will fail at large velocity gradients.
- *NS* existence and uniqueness of questionable interest for turbulence theory.

Kolmogorov theory and intermittency

- If turbulence is assumed isotropic and homogeneous, the features of the energy cascade in inertial range are determined by dimensional argument.

- But experimentally, turbulence is intermittent (i.e., not homogeneous) $\Rightarrow$ multifractal approach: Mandelbrot, Frisch, Parisi, Sreenivasan, Yakhot, ...

$$\langle |\Delta_r v|^p \rangle \sim r^{\zeta_p} \qquad , \qquad \zeta_p = \frac{p}{3} - \frac{1}{\ln \kappa} \ln \Gamma\left(\frac{p}{3} + 1\right)$$

Physics of energy cascade and intermittency

Mechanical process:

folding and stretching of vorticity tubes

Formulation:

cubes C_{ni} ($n = 0, 1, \ldots$ and $i = 1, \ldots, \kappa^{3n}$)

side $\ell_n = \ell_0 \kappa^{-n}$

homothecy $\phi_{ni} : C_0 \to C_{ni}$

$2(\kappa^3 - 1)$ divergence-free real vector fields U_α with

$$\int_{\mathbf{R}^3} U_\alpha = 0$$

velocity field v with $\int v = 0$ has wavelet decomposition

$$v = \sum_{n=0}^{\infty} \sum_{i=1}^{\kappa^{3n}} \sum_{\alpha=1}^{2(\kappa^3-1)} c_{ni\alpha} U_\alpha \circ \phi_{ni}^{-1}$$

Claim: The equations of fluid mechanics can be rewritten as

Hamiltonian evolution equations for coupled systems (n, i) with $\kappa^3 - 1$ degrees of freedom

$+$ external forces at small n (large scales)

$+$ dissipation at large n (small scales)

with small coupling between systems (n, i) for different n

Problem of non-equilibrium statistical mechanics:

heat transport between the systems (n, i)

$\Rightarrow$ explains direction of energy cascade

Boltzmann, Kolmogorov $\Rightarrow$ temperature

$$T_n = \frac{1}{k} \frac{\epsilon^{2/3} \ell_n^{11/3}}{2(\kappa^3 - 1)}$$

very small $k \Rightarrow$ huge T_n and heat resistance
(small coupling)

general problem of looking for *SRB* measure replaced by
study of heat transport close to equilibrium

- Macroscopic approximation: Kolmogorov
- Microscopic fluctuations: intermittency.

Stat Mech fluctuations and turbulent intermittency

Warning: a rigorous microscopic theory of heat transport is still missing, what follows is "theoretical physics"

Claim: Given the energy V_{ni} in (n,i),the velocity $v = v_{(n+1)j}$ is fluctuating with Boltzmannian distribution:

$$\sim \exp\big(-\frac{|v|^3}{V_{ni}\kappa^{-1}}\big)\,d^3v$$

- Because of the large temperature gradient, the flow of energy is overwhelmingly from (n,i) to the systems $(n+1,j)$
- The energy V is proportional to the kinetic energy $\frac{1}{2}|v|^2$ with a weight $1/$time spent in a given spatial energy range, hence we take $V = |v|^3$
- $$|v_{ni}|^3/\ell_n = |v_{(n+1)j}|^3/\ell_{n+1} \quad \text{or} \quad |v|^3 = V_{ni}\kappa^{-1}$$
- Choice of κ discussed below.

Normalized distribution of the energy $V = V_{(n+1)j}$:

$$\frac{1}{V_n\kappa^{-1}} \exp\big(-\frac{V}{V_n\kappa^{-1}}\big)\, dV$$

Let $\tilde{V}_n = \kappa^n V_n$, then $\tilde{V} = \tilde{V}_{n+1}$ is distributed according to

$$\frac{1}{\tilde{V}_n} \exp\big(-\frac{\tilde{V}}{\tilde{V}_n}\big)\, d\tilde{V}$$

Structure functions, and exponents ζ_p

$$\langle |v_n|^p \rangle = \ell_n^{\zeta_p}$$

Estimating ζ_p

$$\zeta_p \ln \ell_n \sim \ln\langle V_n^{p/3}\rangle = -\frac{p}{3}\ln\kappa + \ln\langle \tilde{V}_n^{\frac{p}{3}}\rangle$$

where $\langle \tilde{V}_n^{\frac{p}{3}}\rangle$

$$= \int d\tilde{V}_1 \frac{e^{\tilde{V}_1/\tilde{V}_0}}{\tilde{V}_0} \int \cdots \int d\tilde{V}_{n-1} \frac{e^{-\tilde{V}_{n-1}/\tilde{V}_{n-2}}}{\tilde{V}_{n-2}} \int d\tilde{V}_n \frac{e^{-\tilde{V}_n/\tilde{V}_{n-1}}}{\tilde{V}_{n-1}} \cdot \tilde{V}_n^{p/3}$$

and

$$\int d\tilde{V}_n \frac{e^{-\tilde{V}_n/\tilde{V}_{n-1}}}{\tilde{V}_{n-1}} \cdot \tilde{V}_n^{p/3} = \tilde{V}_{n-1}^{p/3} \int_0^\infty d\xi\, e^{-\xi} \xi^{p/3} = \tilde{V}_{n-1}^{p/3} \Gamma(\frac{p}{3}+1)$$

hence by induction

$$\langle \tilde{V}_n^{\frac{p}{3}}\rangle = \Gamma(\frac{p}{3}+1)^n \tilde{V}_0^{p/3}$$

and

$$\zeta_p = \frac{p}{3} - \frac{1}{\ln\kappa} \ln\Gamma(\frac{p}{3}+1) \qquad (*)$$

Discussion

- $(*)$ is approximate: cannot hold for large p ($p \geq 50$: Yakhot)
- Experimentally $(\ln \kappa)^{-1} = .32 \pm .01$ (i.e. κ between 20 and 25)
- It is assumed that a single value of the energy V at a certain spatial scale ℓ_n thermalizes to a Boltzmann distribution at scale $\ell_{n+1} = \ell_n/\kappa$
- Attempted physical interpretation:

  change of behavior as one passes to small spatial scales analogous to onset of turbulence, namely:

  fast transport $\rightarrow$ thermalization

- Critical Reynolds number R_c corresponding to κ

$$R_c = \kappa^{4/3} \approx 60$$