Optimal Location Problems
with Routing Cost

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The problem we study originated from several discussions with some aeronautics engineers of University of Pisa (research group of Prof. Aldo Frediani). Their goal is to develop an efficient system to transport big amounts of resources at a large scale; to do that it is necessary to design:

- new types of airplanes, able to transport big loads at a low price;

- an efficient system of airports, on which the resources are collected and dispatched.
Created as a spin-off company of the University of Pisa, the SkyBox Engineering has been founded by Professor Aldo Frediani and the former researchers Emanuele Rizzo e Vittorio Cipolla, who have worked together for several years in the same research team at the Department of Aerospace Engineering “Lucio Lazzarino”.

Eng. Aldo Frediani
Full Professor of Aeroelasticity and Aeronautic Structures at the University of Pisa, with a wide experience on structural analysis and fatigue tests. Inventor and owner of patents in aeronautics and railway constructions.

Eng. Emanuele Rizzo
Ph.D. in Aerospace Engineering with experience in aircraft and UAV design, structures and FEM analysis, turbomachinery and optimisation methods.

Eng. Vittorio Cipolla
Ph.D. in Aerospace Engineering with experience in aircraft and UAV design, aerodynamics and CFD analysis, flight mechanics and solar powered flight.

The SkyBox Engineering founders have a common background on the design of boxplane aircraft, known as PrandtlPlane®, whose inventor is the co-founder Prof. Aldo Frediani. The studies around such configuration have lead up to the definition of new tools and methodologies for the preliminary design of unconventional aircraft. Today, these resources, together with a deep knowledge of the problems related to unconventional aircraft design, are a part of the know-how of the SkyBox Engineering.
We consider here a problem in which a given resources density \( \rho(x) \) has to be dispatched in an optimal way according to both a location cost and a routing cost.

A possible application could be the optimal work planning for a carrier company like DHL, FedEx, UPS or similar.

The data are a given region \( \Omega \), a known distribution of resources \( \rho(x) \) (that we normalize to a probability) and a given integer number \( N \).
The pure location problem consists in determining $N$ points $x_1, \ldots, x_N$ in $\Omega$ in order to make the cost

$$L(\Sigma) = \inf \left\{ A W_p^p(\rho, \nu) : \text{spt} \nu \subset \Sigma \right\}$$

as small as possible, where

- $W_p$ is the $p$-Wasserstein distance;
- $\Sigma = \{x_1, \ldots, x_N\}$ is the main unknown;
- the numbers $p > 0$ and $A > 0$ depend on the vehicles used to collect the resources (typically small trucks).
The pure location problem

$$\min \left\{ L(\Sigma) : \Sigma \subset \Omega, \#\Sigma = N \right\}$$

can also be written as a minimal average distance problem

$$\min \left\{ A \int_\Omega \left( \text{dist}(x, \Sigma) \right)^p \rho(x) \, dx : \Sigma \subset \Omega, \#\Sigma = N \right\}.$$  

While the existence of an optimal solution is trivial, the numerical computation of the optimal points $x_i$ is very difficult because the problem is \textit{NP-hard} due to the huge number of local minima.
To bypass this difficulty we consider, instead of the position of the points $x_i$, their density

$$\mu_N = \frac{1}{N} \sum_i \delta_{x_i}.$$  

The asymptotic analysis as $N \to \infty$ has been studied and the following result holds.

**Theorem** The $\Gamma$-limit of the rescaled costs

$$N^{p/d} \int_{\Omega} \left( \text{dist}(x, \Sigma) \right)^p \rho(x) \, dx,$$

in the weak* convergence $\mu_N \rightharpoonup \mu$, is

$$C_{p,d} \int_{\Omega} \frac{\rho(x)}{(\mu(x))^{p/d}} \, dx.$$
Here $C_{p,d}$ is a suitable constant and $\mu(x)$ denotes the density of the absolutely continuous part of $\mu$. Note that the limit optimal $\mu$ can be now **easily** computed and we find

$$
\mu_{opt} = c \rho^{1/(1+p/d)}.
$$

In the problem we are going to describe, when $N$ is large we may then **replace** the location cost $L(\Sigma)$ by its asymptotically equivalent cost

$$
AC_{p,d} N^{-p/d} \int_\Omega \frac{\rho(x)}{(\mu(x))^{p/d}} \, dx.
$$
Plot of the value of $C_{p,2}$ for $p \in [0, 2]$. 
The routing cost describes the cost for connecting all pairs of points \( x_i, x_j \). For simplicity we assume all connections of the type point-to-point (direct flights) and that the cost of a flight does not depend on the carried load (reasonable for carriers like DHL, FedEx or UPS). Then the routing cost is

\[
B \sum_{i,j} |x_i - x_j|^q = BN^2 \int_{\Omega \times \Omega} |x - y|^q \, d(\mu_N \otimes \mu_N)
\]

where \( B \) and \( q \) depend on the vehicles used to connect the points \( x_i \) (typically airplanes).
Ground = Ax, Air = B_0 + B_1 x^q \text{ with } q < 1.
The convolution integral in the routing cost is weakly* continuous, then if $\mu = \lim \mu_N$

$$\lim_{N \to \infty} \int_{\Omega \times \Omega} |x-y|^q \, d(\mu_N \otimes \mu_N) = \int_{\Omega \times \Omega} |x-y|^q \, d(\mu \otimes \mu).$$

We then consider the asymptotic expression for the routing cost

$$BN^2 \int_{\Omega \times \Omega} |x-y|^q \, d(\mu \otimes \mu)$$

so that the total location-routing cost we consider is

$$\frac{AC_{p,d}}{Np/d} \int_{\Omega} \frac{\rho(x)}{(\mu(x))^{p/d}} \, dx + BN^2 \int_{\Omega \times \Omega} |x-y|^q \, d(\mu \otimes \mu).$$
Setting $\varepsilon = AC_{p,d} N^{-2-p/d}/B$ we end up with the minimization of the functionals

$$F_\varepsilon(\mu) = \varepsilon \int_\Omega \frac{\rho(x)}{(\mu(x))^{p/d}} \, dx + \int_{\Omega \times \Omega} |x-y|^q \, d(\mu \otimes \mu).$$

Our goal is to study the limit behavior of the minimizers $\mu_\varepsilon$ of the functionals $F_\varepsilon$ above. This will be done through the following steps:

1. **estimate** the order of vanishing $m_\varepsilon$ of the minima of $F_\varepsilon$;

2. **rescale** $F_\varepsilon$ and consider $G_\varepsilon = \frac{1}{m_\varepsilon} F_\varepsilon$;

3. **compute** the $\Gamma$-limit of $G_\varepsilon$. 
Proposition The order of vanishing for the minimal values of the functionals $F_\varepsilon$ above is

$$m_\varepsilon = \varepsilon^{1/(1+p/d)}.$$ 

The rescaled functionals are then

$$G_\varepsilon(\mu) = \varepsilon^{\alpha} \int_\Omega \frac{\rho(x)}{(\mu(x))^{p/d}} \, dx + \varepsilon^{-\beta} \int_{\Omega \times \Omega} |x-y|^q \, d(\mu \otimes \mu).$$

with $\alpha = \frac{p/d}{1+p/d}$ and $\beta = \frac{1}{1+p/d}$. Setting $V(x) = |x|^q$, the minimizers $\mu_\varepsilon$ verify the non-local equation

$$-\frac{\varepsilon \rho}{\mu^{1+p/d}} + \frac{2d}{p} V * \mu = c.$$
Due to the presence of $\varepsilon^{-\beta}$ in front of the routing term, we have that the $\Gamma$-limit $G$ of the rescaled functionals $G_\varepsilon$ satisfy

$$G(\mu) = +\infty \quad \text{whenever } \mu \neq \delta_{x_0} \text{ for some } x_0.$$  

It is then enough to compute the $\Gamma$-limit only for $\mu = \delta_{x_0}$.

**Theorem** We have

$$G(\delta_{x_0}) = K_{p,d} \int_\Omega |x - x_0|^{\alpha q} \left(\rho(x)\right)^\beta \, dx.$$
Γ-limsup strategy: use the approximating sequence

\[ \mu_\varepsilon = \varepsilon^\beta \phi + \left( 1 - \varepsilon^\beta \int \phi \right) \delta_{x_0} \]

and then optimize with respect to \( \phi \). The best choice is

\[ \phi(x) = \left( \frac{p}{2d} \frac{\rho(x)}{|x - x_0|^q} \right)^\beta. \]

Γ-liminf strategy: given \( \mu_\varepsilon \rightharpoonup \delta_{x_0} \) we split \( \mu_\varepsilon = \mu_\varepsilon^1 + \mu_\varepsilon^2 \) where \( \mu_\varepsilon^1 \) is the "concentrated part" (on a small set \( A_\varepsilon \)) and \( \mu_\varepsilon^2 \to 0. \)
Using the **Young** inequality

\[ X\varepsilon^\alpha + Y\varepsilon^{-\beta} \geq \frac{X^\beta Y^\alpha}{\alpha^\alpha \beta^\beta} \]

we obtain with some calculations \((u_\varepsilon)\) is the absolutely continuous part of \(\mu_\varepsilon\)

\[
G_\varepsilon(\mu_\varepsilon) \geq \int_\Omega \left[ \varepsilon^\alpha \frac{\rho}{u_\varepsilon^{p/d}} + \varepsilon^{-\beta} 2(V \ast \mu_\varepsilon) u_\varepsilon 1_{A_\varepsilon} \right] dx \\
\geq \int_\Omega \frac{1}{\alpha^\alpha \beta^\beta} \left( \frac{\rho}{u_\varepsilon^{p/d}} \right)^\beta (2(V \ast \mu_\varepsilon) u_\varepsilon 1_{A_\varepsilon})^\alpha dx \\
= K_{p,d} \int_{A_\varepsilon} \rho^\beta (V \ast \mu_\varepsilon)^\alpha dx \\
\rightarrow G(\delta x_0). \]
By the Γ-limit result above the minimizers \( \mu_\varepsilon \) of the location-routing total cost \( F_\varepsilon \) are close to

\[
\varepsilon^\beta \phi + \left( 1 - \varepsilon^\beta \int \phi \right) \delta_{x_0}
\]

where

\[
\phi(x) = \left( \frac{p}{2d} \frac{\rho(x)}{|x - x_0|^q} \right)^\beta
\]

and \( x_0 \) minimizes

\[
\int_\Omega |x - x_0|^\alpha q (\rho(x))^{\beta} \, dx.
\]

The point \( x_0 \) can be seen as the main hub of the system.
Determination of the Dzâƒ‹Š“dz:

A 1-d example with $\rho(x)$ a step function.
A 1-d example with $\rho(x)$ a function with peaks.
Real case study on US territory because of the availability of data.

‡ The points are reported in a Cartesian system (Mercator projection).

‡ For each point, socio-economic data has been extrapolated, and then the airfreight demand can be defined over the domain.

The $\rho(x)$ of the USA demand.
Results

The measure of probability $P_\gamma$ and the functional $F_G$ with the point of minimum.

A corresponding $\mu_\varepsilon$.

- 10 major freight airports in USA (airport council international, 2009)
Level set of the limit function $x_0 \mapsto G(\delta x_0)$. 
main hubs: DHL–Cincinnati (Ohio); UPS–Louisville (Kentucky); Fedex–Memphis (Tennessee)